Practicalities

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Homepage: users.jyu.fi/~kainulai/Siles/QFT-1_2023/index.html

Schedule: 4.9–1.12 2023

Mondays & Wednesdays 10–12 FYS 5

Exercises: Mondays 12.15–14.00 YFL 14

Exams: Midterm 1 27.10 2023

Midterm 2 1.12. 2023

Final: 19.1. 2024

Grading:

$$\text{Final points} = \text{ceil}\left\{30 \times \left(\frac{\text{exercise pts}}{\text{max exc pts}}\right) + 30 \times \left(\frac{\text{exam pts}}{\text{max exm pts}}\right)\right\} \leq 60$$

Final grade:

Grad	
1	30–35
2	36–41
3	42–47
4	48–53
5	54–60

0.1 Introduction

Quantum field theory ≙ quantum theory of fields

⊃ particle physics theories

$$
\begin{tikzcd}
\text{Classical mechanics} \arrow[r, "N\to\infty"'] \arrow[d, "{[q,p]=i\hbar}"'] \arrow[dr, "v\to c"] & \text{Classical field theory (Maxwell)} \arrow[d, "{[\phi,\pi]=i\hbar\,\delta^3(\vec{x}-\vec{y})}"] \\
\text{Quantum mechanics} \arrow[r, "N\to\infty"'] & \text{Quantum field theory (QED)}
\end{tikzcd}
$$

Simple idea: Generalize the usual quantum mechanics to the description of fields

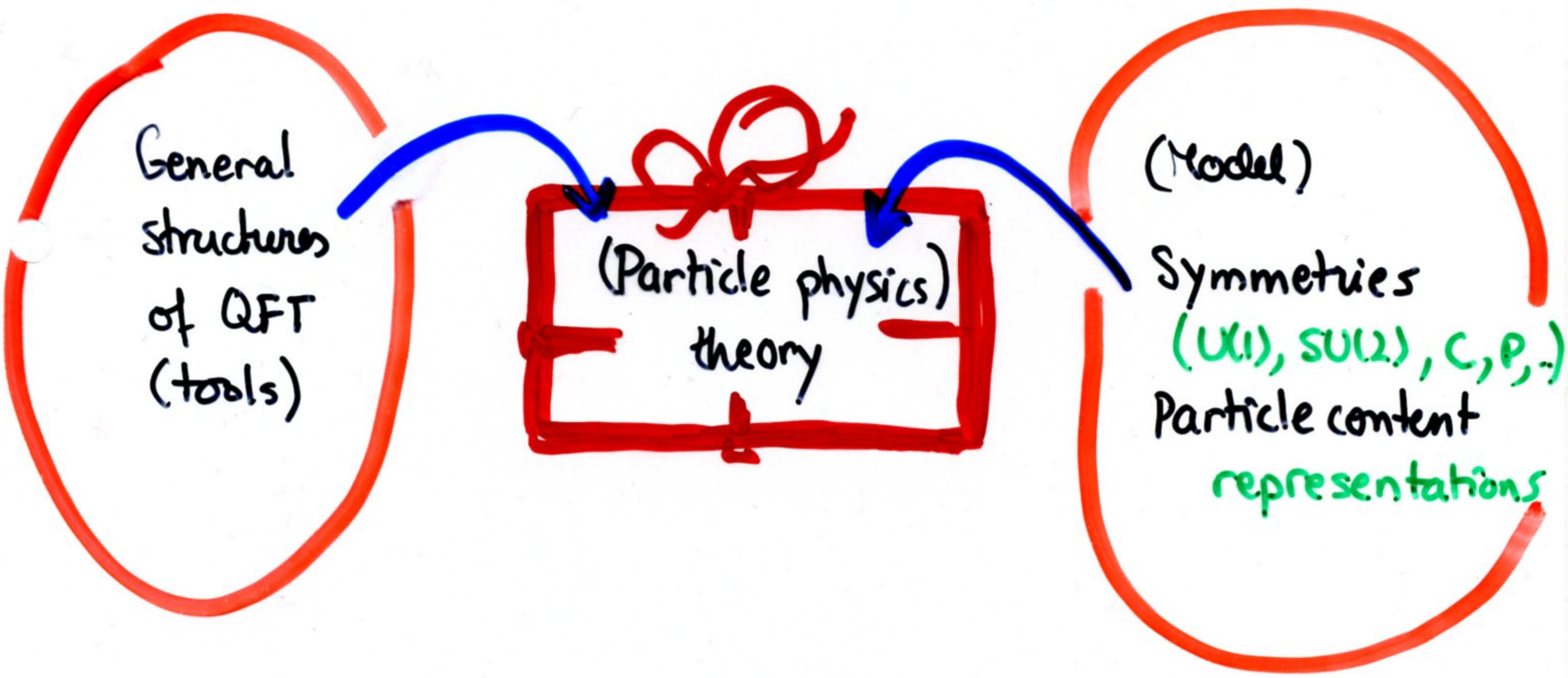
Great success: All particle physics theories, of electromagnetism, weak force and strong force are QFT's*. Only gravity has eluded all attempts of quantization so far.

* more accurately gauge field theories

QFT is fundamentally different kind of structure than eg. Einsteins General relativity, although they are often compared as achievements:

- GR $\hat{=}$ a single relativistic, classical theory that describes gravity.
- QFT $\hat{=}$ a collection of principles and tools

QFT tools are not specific to any given phenomenon. They can be used to construct and explore wide variety of different models.



The main emphasis of this course is on the left (tools).
We study the basic structures of the QFT with help of simple toy models ($\lambda\phi^4$, QED). (More of the rhs on spring course...)

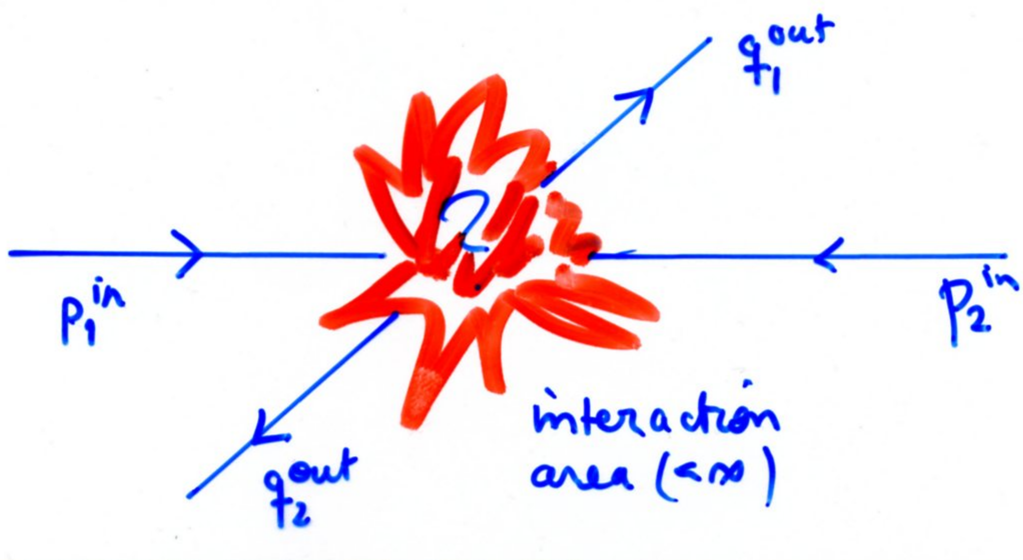
Is QFT difficult ?

- Some of the problems are conceptual. These tend to be drowned by technical issues however. The really hard ones, such as the infinite vacuum energy, can be neglected as they do not have any practical effects....

- In learning QFT, the toughest problems seem to be practical. When you try to calculate something using QFT, you will often find that formalism is complicated. The most important tool for QFT is perturbation theory. (PT).

PT is used because interacting QFT's typically prove too complicated to be solved even in the simplest cases (apart some mainly academic examples)

Typical application of QFT in the scattering problem:



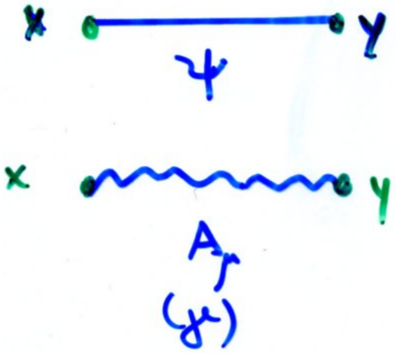
Even this simple problem, to predict $q_{1,2}^{out}$ from known input $p_{1,2}^{in}$, is typically overwhelmingly difficult to be solved exactly, even in the simplest of theories.

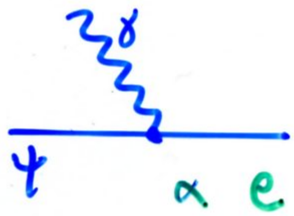
In PT we divide (f.ex.)

$$\boxed{\mathcal{L}_{QED} = \mathcal{L}_{\text{Free theory}} + \mathcal{L}_{\text{Interactions}}}$$

Evolution of the free theory (propagators)

Interactions





$\propto e$

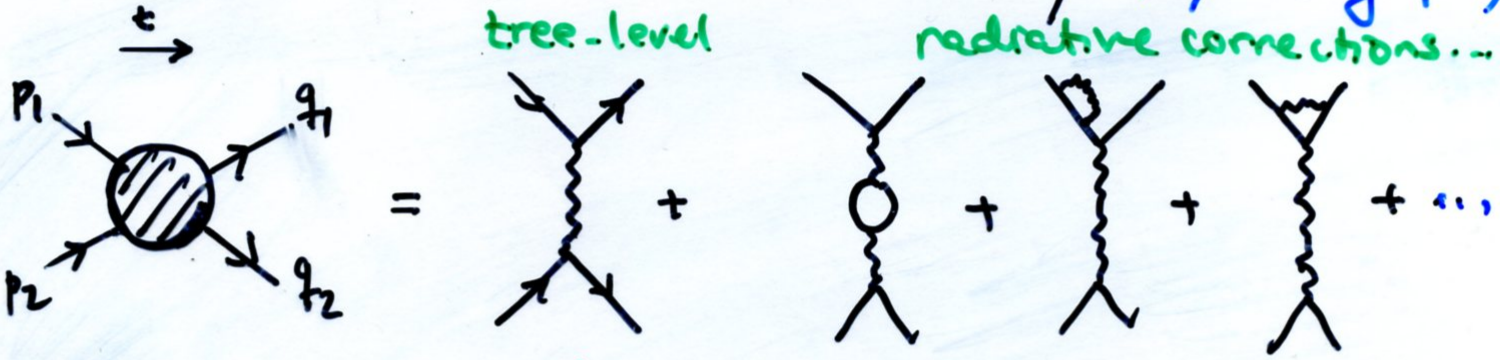
↑

coupling constant

Interactions are then treated hierarchically: (Feynman graphs)

tree-level

radiative corrections...



$$\underbrace{\sim e^2 \qquad \sim e^4 \qquad \sim e^4 \qquad \sim e^4}_{\text{quantum mechanical superposition}}$$

- expansion converges if $e^2 < 1$.

Already deriving the PT (Feynman-) rules is a lengthy task (that we will go through with several examples), but interpreting its results is even harder. For example the radiative (loop-) corrections of the previous example, while formally smaller ($\sim e^4$) than the leading term ($\sim e^2$), are in practice found to be numerically infinite! To get rid of this problem one needs to develop the renormalization theory for PT in QFT.

Typically, for each development mentioned, there exists several different possible methods.

Quatization:

- Canonical quantization
- Path integral method
- ...

Renormalization $<$ "bare" PT (conventional) / "renormalized" PT

- Regularization (MAKING FINITE)

  Pauli-Villairs, Cut-off, Dimensional reg...

- Scheme dependence MS, $\overline{\text{MS}}$, MOM ... (SUBTRACTING INFINITIES)

Different choices lead to different intermediate results in calculations but physical quantities come out the same to the order in coupling constant considered.

Course contents

We will be mainly following the book

- Peskin & Schröder : Introd. to. QFT
  Westview press, 1995.

covering roughly the sections 1–9. This will include

Quantization of free fields.

- Klein-Gordon
- Dirac
  - propagators, causality

Interacting theories

- Perturbation theory
- S-matrix
- Feynman rules

Examples of scattering processes.

- $e^+e^- \to \mu^+\mu^-$
- ...

Radiative corrections

- UV- and IR-divergences
- Renormalization & Regularization
- Examples.

- Path integral methods
  - Functional quantization
  - generating functions, Perturbation th.
  - quantizing the gauge-fields : Faddeev–Popov.
  - ...

This is only a very rough outline, and it may change depending on how we progress.

Other good books (PS is clear choice 1 though)

- M. Kaku ; QFT
  Oxford Univ. Press 1993 (ok)
- G. Sterman, An introd. to QFT
  Cambridge Univ. press 1993 (technical)
- C. Itzykson & J.B. Zuber, QFT
  McGraw & Hill 1980 (a bit obscure)
- J.D. Björken & S. Drell, Relativistic quantum mech.
  McGraw-Hill 1964 (classic. old)

There are also several net-books available by now. Addresses can be found from the course home page.

0.3 NOTATIONS AND CONVENTIONS

UNITS:

$$\hbar = c = 1$$

$$\Rightarrow [\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$$

for example:

$$m_e \simeq 9{,}11 \times 10^{-28}\,g \simeq 0.511 \text{ MeV}$$

VECTORS & TENSORS

Flat Minkowski space $\hat{=}$ no gravity.

$$\boxed{g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1; -1, -1, -1)}$$

Greek indices : 0, 1, 2, 3

Roman —"— 1, 2, 3

- Contravariant ↔ covariant components

$$x^\mu = (x^0, \vec{x}) \quad \Rightarrow \quad x_\mu = (x^0, -\vec{x}) \equiv g_{\mu\nu} x^\nu \qquad (0,2)$$

- Dot-product:

$$p \cdot x = g_{\mu\nu} p^\mu x^\nu = p^0 x^0 - \vec{p} \cdot \vec{x} \qquad (0,3)$$

$$\Rightarrow \quad p^2 = g_{\mu\nu} p^\mu p^\nu = p_\mu p^\nu = p^{0\,2} - \vec{p}^{\,2} = m^2 \qquad (0,4)$$

on-shell
$p_0 = E$

- Four-derivative

!

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}; + \vec{\nabla} \right) \qquad (0,5)$$

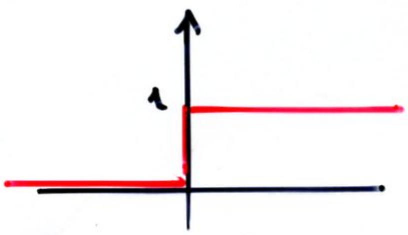
In this way naturally

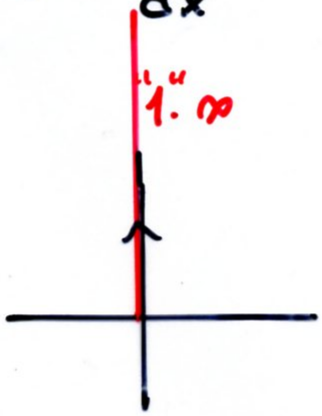
$$\boxed{p^\mu = i\partial^\mu} \quad \Leftrightarrow \quad \begin{cases} E = i \dfrac{\partial}{\partial x^0} \\ \vec{p} = -i \vec{\nabla} \end{cases} \qquad (0,6)$$

FOURIER-TRANSFORMATIONS, DISTRIBUTIONS

- θ-function $\rightarrow$ δ-function

$$\theta(x) = \begin{cases} 0 \;;\; x<0 \\ 1 \;;\; x>0 \end{cases} \qquad \delta(x) = \frac{d\theta}{dx}$$





Definition (normalization)

$$\int d^n x \; \delta^n(x) \equiv 1$$

- $(2\pi)^n$-positions (convention)

$$f(x) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(k)$$

$$\tilde{f}(k) = \int d^4x \; e^{ik\cdot x} f(x) \qquad (0,7)$$

Furthermore

$$\int d^4x \; e^{ik\cdot x} = (2\pi)^4 \delta^4(k) \qquad (0,8)$$

ELECTRODYNAMICS

- $\Phi = \frac{Q}{4\pi r}$ (Coulomb)
- $\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} = \frac{e^2}{4\pi} \simeq \frac{1}{137} \Rightarrow e^2 \approx 0.092$

fine structure constant

Maxwell's equations

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \qquad (0.9a)$$

$$\partial_\mu F^{\mu\nu} = e j^\nu \qquad (0.9b)$$

↑ 4-current $j^\mu = (\rho, \vec{j})$

Where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (0.10)$$

$$\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad \text{(dual)} \qquad (0.11)$$

↑ $\epsilon^{0123} = -\epsilon_{0123} = 1$

totally antisymmetric tensor

and

$$A^\mu \equiv (\Phi, \vec{A}) \qquad (0.12)$$

is the gauge field describing the photon.

Coulomb

0.3 Classical Field Theory

In classical physics

Action $\Leftrightarrow$ Equation of motion

Extremal of action $\hat{=}$ path of the particle

In quantum mechanics the deterministic path is replaced by a probabilistic wave function, or equivalently (path integral) to a set of quantum paths deviating from the classical one by corrections of order $\hbar$ (uncertainty relation).

For example for a scalar field $\phi(x)$:

$$\boxed{S = \int L\,dt = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)} \qquad (0.13)$$

Classical requirements $\left\{ \right.$
- Causal, bounded
- Local
- Lorenz-covariant
- Dimensionless

Quantum: $\left\{ \right.$ Renormalizable !

Minimum of Action ⇒ Equation of motion

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$\delta\phi(\infty) = 0$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\} \delta\phi$$

holds for all $\delta\phi$

$$\Rightarrow \quad \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0} \qquad (0.14)$$

Euler-Lagrange equation.

- Explicitly Lorenz covariant (if $\mathcal{L}$ is)
- Basis for path integral quantization

Equivalently one may use Hamilton formalism

basis for canonical quantization

Define

$$\boxed{H = \int d^3x \left[\pi(x) \dot{\phi}(x) - \mathcal{L} \right]} \qquad (0.15)$$

where

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \qquad (0.16)$$

Hamiltonian density : $H = \int d^3x \, \mathcal{H}$

$$\Leftrightarrow \quad \mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

Now a variation $\delta H = 0$ leads to e.o.m:

$$\boxed{\dot{\phi}(x) = \frac{\partial H}{\partial \pi(x)} \quad ; \quad \dot{\pi}(x) = -\frac{\partial H}{\partial \phi(x)}} \qquad (0.17)$$

It is easy to show that these are equivalent with the E-L-equations.

<u>Example</u>: free field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\overset{\text{E.L.}}{\Rightarrow} \qquad (\partial_\mu \partial^\mu + m^2)\phi = 0$$

(Klein-Gordon eq.)

$$\Rightarrow \pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}$$

$$\Rightarrow \mathcal{H} = \underbrace{\frac{1}{2}\dot{\phi}^2}_{\text{kinetic}} + \underbrace{\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2}_{\text{potential}}$$

NOETHER'S THEOREM – Conservation laws.

> $\mathcal{L}$ invariant under a continuous transformation
>
> $\Leftrightarrow$ $\exists$ conserved current & charge

Example 1 (global U(1)-symmetry : $e^{i\alpha} \in U(1)$ 

$$\mathcal{L} \equiv |\partial_\mu \phi|^2 - m^2 |\phi|^2 \qquad (0,21)$$

where now $\phi \in \mathbb{C}$. This theory is clearly invariant under the transformation

$$\phi \to \phi' \equiv e^{i\alpha}\phi \overset{\alpha \ll 1}{\simeq} (1+i\alpha)\phi \equiv \phi + \delta\varphi$$

$$\phi^* \to \phi^{*\prime} \equiv e^{-i\alpha}\phi \simeq (1-i\alpha)\phi^* \equiv \phi^* + \delta\varphi^*$$

Now

$$\Delta\mathcal{L} = \mathcal{L}(\phi', \partial_\mu\phi') - \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\approx \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\varphi + h.c + \mathcal{O}((\delta\varphi)^2)$$

$$= i\alpha\, \partial_\mu \Big[\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi - h.c}_{J^\mu} \Big] + \mathcal{O}(\alpha^2) \equiv 0$$

$$\Rightarrow \partial_\mu J^\mu = \partial_\mu \big(\underbrace{(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*}_{} \big) = 0 \qquad (0,22)$$

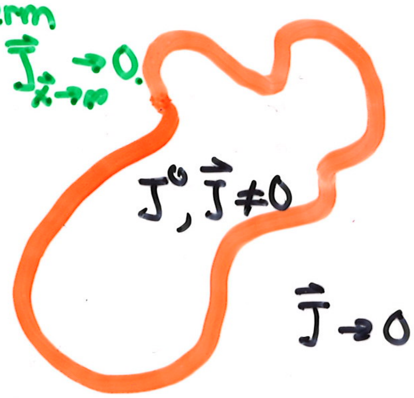
J^μ is conserved 4-current.

Moreover, each conserved current is accompanied by a conserved charge:

surface term $\to 0$, when $\vec{J}_{\vec{x}\to\infty} \to 0$.

$$0 = \int d^3x \, \partial_\mu J^\mu = \int d^3x \left(\partial_0 J^0 + \nabla \cdot \vec{J}\right)$$

$$= \partial_t \int d^3x \, J^0 = \partial_t Q \qquad (0.23)$$



[* One example of a conservation law associated with an internal symmetry, similar to the above example, is the conservation of the electric charge, as a result of the U(1)-symmetry behind electromagnetic interactions. *]

Example 2. (space-time translations)

Let us make a transform

$$x^\mu \longrightarrow x^\mu + a^\mu \qquad (0.24)$$

Then because $\mathcal{L}$ is a scalar

$$\mathcal{L} \longrightarrow \mathcal{L} + \underbrace{a^\mu \partial_\mu \mathcal{L}}_{\neq 0} + \mathcal{O}(a^2)$$

$\neq 0$ in general, since $\mathcal{L}$ does depend on x. However a total divergence integrates out in S and does not affect E-L eqs.

However, x-dependence only comes through $\mathcal{L}$'s dependence on fields. Eg. in the scalar case $\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu\phi(x))$

and $\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi = \phi + \delta\phi$

$= \underbrace{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}} \; : \; \text{E-L.}; \; \delta\phi \equiv a^\nu \partial_\nu \phi$

$$\Delta \mathcal{L} = a^\nu \partial_\nu \mathcal{L} = a^\nu \left(\frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \partial_\nu \phi \right)$$

$$= a^\nu \partial_\mu \left(\frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right)$$

Because this holds for all possible $\overset{(\text{small})}{a^\nu}$, we get

$$\boxed{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L} \right] = 0} \qquad (0,25)$$

$\equiv \underline{\underline{T^\mu{}_\nu}}$; The <u>conserved energy-momentum tensor.</u>

The corresponding <u>conserved charges</u> form a four-vector:
$(\partial_\mu T^{\mu\nu} = 0 \Rightarrow)$

momentum

$$P^\mu = \int d^3x \, T^{0\mu} \equiv (H, \vec{P}) \qquad (0,26)$$

Energy (Hamiltonian)

So we have seen:

Invariance in time-translations $\longleftrightarrow$ E is conserved

— " — space- — " — $\longleftrightarrow$ $\vec{P}$ — " —

Furthermore it holds that:

Invariance under rotations $\longleftrightarrow$ $\vec{L}$ (angular mom.) is conserved

1. QUANTIZATION OF A FREE FIELD

1.1 BOSONIC FIELD (Klein-Gordon)

We follow canonical quantization method, which is a straightforward generalization of quantum mechanics to the case of continuous fields. In usual qm, we have

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$$
$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \qquad (1.1)$$

where

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \qquad \text{conjugate momentum} \qquad (1.2)$$

The formal connection between $L(q_i, \dot{q}_i)$ and $\mathcal{L}(\phi(\vec{x}), \partial_\mu\phi(\vec{x}))$ is $q_i \to \phi(\vec{x})$, so we shall postulate analogously:

$$\boxed{\begin{aligned} &[\hat{\phi}_i(\vec{x},t), \hat{\pi}_j(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})\,\delta_{ij} \\ &[\hat{\phi}_i(\vec{x},t), \hat{\phi}_j(\vec{y},t)] = [\pi_i(\vec{x},t), \pi_j(\vec{y},t)] = 0 \end{aligned}}$$

Note! Equal time! (1.3)

where

$$\boxed{\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)}} \qquad (1.4) \qquad = \dot{\phi}(x) \text{ for Klein-Gordon field.}$$

momentum density.

- Conjugate variables of the theory follow the Heisenberg uncertainty relation. All quantities on this page are operators.

[*

If you are familiar with the formal methods of classical mechanics, you might recognize that quantization can be expressed as a rule (Dirac)

$$\underbrace{\{\underbrace{A,B}_{\text{Classical variables}}\}_{PB}}_{\text{Poisson Bracket}} \rightarrow \frac{i}{\hbar}\underbrace{[\underbrace{\hat{A},\hat{B}}_{\text{QM-operators}}]}_{\text{Commutator}} \qquad (1.5)$$

*]

Let us now study the structure of the theory under the conditions (1.3-1.4). Take the free scalar theory (0.18):

$$\mathcal{L}_{K-G} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \qquad (1.6)$$

Make the Fourier decomposition

$$\boxed{\hat{\phi}(x) \equiv \int \frac{d^3p}{(2\pi)^3} N_{\vec{p}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}\, e^{ip\cdot x} \right)} \qquad (1.7)$$

FIELD OPERATOR

creation operator

$a^{\dagger}_{\vec{p}}|0\rangle \propto |\vec{p}\rangle$

annihilation operator

$a_{\vec{p}}|0\rangle = 0 \quad \forall\, \vec{p}$

This defines the vacuum of the theory

Normalization of operators

$\hat{=}$

phase space density

$\hat{=}$

normalization of states

- Indeed, the relation $a_{\vec{p}}|0\rangle = 0 \ \forall \vec{p}$ defines the unique <u>vacuum state</u> $|0\rangle$, which is empty. We can normalize

$$\langle 0|0\rangle \equiv 1. \qquad (1.8)$$

- Creation operators define the particle states (with a suitable normalization)

$$a^\dagger_{p_1} a^\dagger_{p_2} \cdots a^\dagger_{p_N} |0\rangle = |\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N\rangle$$

Creation and annihilation operators satisfy simple commutation relations. We can derive them from (1.3). From (1.4) we see that:

$$\boxed{\hat{\pi}(x) = \int \frac{d^3p}{(2\pi)^3} (-i\omega_p N_{\vec{p}}) \left(a_{\vec{p}} e^{-ip\cdot x} - a^\dagger_{\vec{p}} e^{ip\cdot x} \right)} \qquad (1.10)$$

Now:

$$[\hat{\phi}(x,t), \hat{\pi}(y,t)]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} (-i\omega_{\vec{p}'}) e^{i\vec{p}\cdot\vec{x} + i\vec{p}'\cdot\vec{y}} \times$$

$$\times \left[a_{\vec{p}} N_{\vec{p}} e^{-i\omega_p t} + a^\dagger_{-\vec{p}} N_{-\vec{p}} e^{+i\omega_p t}, \ a_{\vec{p}'} N_{p'} e^{-i\omega_{p'} t} - a^\dagger_{-\vec{p}'} N_{-\vec{p}'} e^{+i\omega_{p'} t} \right]$$

I changed $\vec{p}\ (\vec{p}\,') \to -\vec{p}\ (-\vec{p}\,')$ in the integral for creation operators

Here

$$[\ldots] = [a_{\vec p}, a_{\vec p'}]\, N_{\vec p} N_{\vec p'}\, e^{-i\overbrace{(\omega_{\vec p}+\omega_{\vec p'})}^{>0}t}$$

$$- [a^+_{-\vec p}, a^+_{-\vec p'}]\, N_{-\vec p} N_{\vec p'}\, e^{i(\omega_{\vec p}+\omega_{\vec p'})t}$$

time dependent unless $[\ ]\equiv 0$

$$- [a_{\vec p}, a^+_{-\vec p'}]\, N_{\vec p} N_{-\vec p'}\, e^{-i(\omega_{\vec p}-\omega_{\vec p'})t}$$

$$+ [a^+_{-\vec p}, a_{\vec p'}]\, N_{\vec p} N_{\vec p'}\, e^{i(\omega_{\vec p}-\omega_{\vec p'})t}$$

time independent only when $\omega_{\vec p} = \omega_{\vec p'}$, i.e. when $\vec p = \pm \vec p'$

So we see that

$$[a_{\vec p}, a_{\vec p'}] = [a^+_{\vec p}, a^+_{\vec p'}] = 0 \qquad (1.11a)$$

And we can set :

$$[a_{\vec p}, a^+_{\vec p'}] = C_{\vec p}\, \delta^3(\vec p - \vec p') \qquad (1.11b)$$

To fix $C_{\vec p}$ put these back to (1.3) :

$$[\phi(x,t), \pi(\vec y,t)] = i\int \frac{d^3p}{(2\pi)^3}\left[\frac{N^2_{\vec p} C_{\vec p}}{(2\pi)^3}\cdot 2\omega_p\right] e^{i\vec p\cdot(\vec x-\vec y)}$$

$$\equiv i\,\delta^3(\vec x - \vec y)$$

$$\Rightarrow \underline{N^2_{\vec p} C_{\vec p} = \frac{(2\pi)^3}{2\omega_p}} \qquad (1.12)$$

Normalization is obviously not unique. In this course we will use $C_{\vec{p}} \equiv 2\omega_p \cdot (2\pi)^3$, so that

$$[a_{\vec{p}}, a_{\vec{p}'}^{+}] \equiv 2\omega_p \cdot (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \qquad (1.13)$$

We can now compute Hamiltonian for our model:

$$\hat{H} = \int d^3x \, \frac{1}{2}\left(\dot{\hat{\phi}}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2 \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \, 2\omega_p^2 N_{\vec{p}}^2 \left(a_{\vec{p}}^{+} a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a_{\vec{p}}^{+}] \right) \qquad (1.14)$$

The commutator term here is infinite. Using (1.11) and (1.12) we can see that it is independent of normalization:

$$\frac{1}{V} H_{vac} = \frac{1}{V}\int \frac{d^3p}{(2\pi)^3} \, \omega_p^2 N_p^2 \overbrace{[a_{\vec{p}}, a_{\vec{p}}^{+}]}^{= C_p \delta_p^3(0) \, = \, C_p \frac{V}{(2\pi)^3}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} = \infty \qquad (1.15)$$

This is the vacuum energy of the theory(*). H_{vac} is a problem at deeper level, in connection with gravity, but it is irrelevant for practical problems in particle physics, since these are only sensitive to energy-differences.

(*) Since $\langle 0|H|0\rangle = \langle 0|H_{vac}|0\rangle$.

Another normalization

In the normalization used above, the counting of states is not very obvious. Indeed, we find that

$$\langle p|p\rangle = (2\pi)^3\, 2\omega_{\vec{p}}\, \underbrace{\delta^3(0)}_{\equiv \frac{1}{(2\pi)^3}\int_V d^3x\, e^{-ip\cdot x}\big|_{p=0} = \frac{V}{(2\pi)^3}} = 2\omega_{\vec{p}} V$$

Thus also $[a_{\vec{p}}, a_{\vec{p}}^\dagger] = 2\omega_{\vec{p}} V$.

To count states "normally", we'd rather have $\langle p|p\rangle \equiv 1$, which means choosing

$$C_p \equiv \frac{(2\pi)^3}{V} \Rightarrow N_p^2 = \frac{(2\pi)^3}{2\omega_p C_p} = \frac{V}{2\omega_p}$$

In this normalization

$$\frac{1}{V} H = \frac{1}{V}\int \frac{d^3p}{(2\pi)^3}\, 2\omega_p^2 \frac{V}{2\omega_p}\left(\tilde{a}_{\vec{p}}^\dagger \tilde{a}_{\vec{p}} + \frac{1}{2}[\tilde{a}_p, \tilde{a}_p^\dagger]\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(\tilde{a}_{\vec{p}}^\dagger \tilde{a}_p + \frac{1}{2}\right),$$

which should look much more familiar. The covariant normalization is still much more practical for our purposes.

A similar infinity will be encountered whenever an operator has creation operators appearing to the right from the annihilation operators. A formal procedure to get rid of these (vacuum-) contributions is to define normal-ordering

$$: \mathcal{O}(a a^\dagger a^\dagger a \ldots) : = \mathcal{O}(a^\dagger a^\dagger \ldots a a) \qquad (1.16)$$

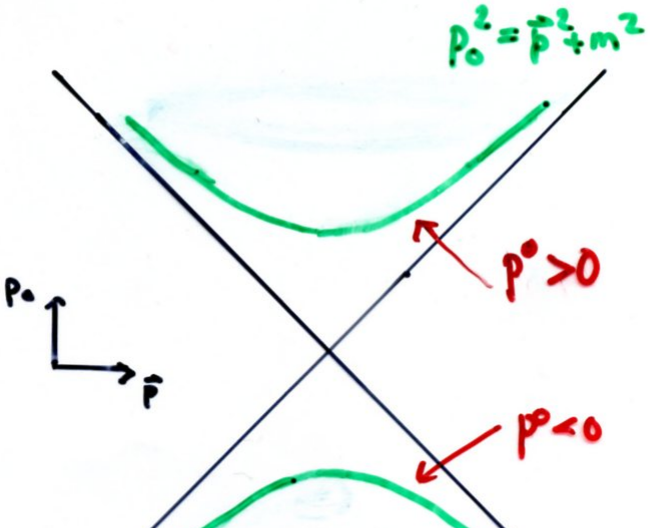
move, or (anti)commute all (fermions)
a:s to the right of all $a^\dagger$:s.
Throw out all [..]- terms.

In this way we get the normal ordered Hamiltonian:

$$\boxed{\begin{aligned} :H: &= \int \frac{d^3p}{2\omega_p (2\pi)^3}\, \omega_p\, a_p^\dagger a_p \\ &= \int \frac{d^4p}{(2\pi)^4}\, \delta(p^2 - m^2)\, \Theta(p_0)\, p_0\, a_{\vec p}^\dagger a_{\vec p} \end{aligned}} \qquad (1.17)$$

Where we used (1.14) together with the normalization $N_{\vec p} = 1/2\omega_p$ as implied by (1.12) & (1.13).

Note that in (1.17) the integral is over the Lorentz invariant 3-D phase space, or equivalently over the positive energy $p^2 = m^2$ hyperboloid in 4D - phase space.



The advantage of our normalization (1.13) is that now the 1-particle states are normalized in a Lorentz-invariant way:

$$\langle \vec{p} | \vec{q} \rangle = \langle 0 | a_{\vec{p}} a^{\dagger}_{\vec{q}} | 0 \rangle$$
$$= \langle 0 | [a_{\vec{p}}, a^{\dagger}_{\vec{q}}] + a^{\dagger}_{\vec{q}} a_{\vec{p}} | 0 \rangle$$
$$\overset{1.13}{=} \underline{2\omega_p (2\pi)^3 \delta^3(\vec{q}-\vec{p})} . \qquad (1.18)$$

As mentioned already before, we can write an arbitrary (non-interacting) many particle state as

$$|p_1, \ldots, p_N\rangle = \prod_{i=1}^{N} a^{\dagger}_{\vec{p}_i} |0\rangle \equiv |\psi_N\rangle \qquad (1.19)$$

It is easy to show that

$$\langle \psi_N | :H: | \psi_N \rangle = \sum_{i=1}^{N} \omega_{p_i} \langle \psi_N | \psi_N \rangle \qquad (1.20)$$

Statistics: Because $[a^{\dagger}_p, a^{\dagger}_q] = 0$, the order of operators on eqn. (1.19) is irrelevant. That is states corresponding to different orderings are identical. There can also be an arbitrary number of identical operators

$\Rightarrow$ K-G particles are bosons.

Let us write down a few useful relations:

- $$\underbrace{[H, a^\dagger_{\vec p}]}_{} = \omega a^\dagger_{\vec p} \qquad (1.21)$$

  adds energy ω_p

- $$[H, a_{\vec p}] = -\omega_p a_{\vec p} \qquad (1.22)$$

  removes energy ω_p

- $$:\vec P: = -\int d^3x :\pi(x)\nabla\phi(x): = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \vec p\, a^\dagger_{\vec p} a_{\vec p} \qquad (1.23)$$

  Total momentum

- $$\underbrace{\langle 0 | \hat\phi(x) | p \rangle}_{=\text{ position space representation of the wave function}} = e^{-ip\cdot x} \qquad (1.24)$$

- $$\mathbb{1}_{1\text{-particle}} = \int \frac{d^3p}{(2\pi)^3 2\omega_p} |\vec p\rangle\langle \vec p| \qquad (1.25)$$

  (1-particle completeness relation)

Heisenberg picture evolution:

$$\hat\phi(x) = e^{i\hat P\cdot x}\,\hat\phi(0)\,e^{-i\hat P\cdot x} \qquad (1.26)$$

where $\hat P^\mu \equiv (H, \vec P)$

(1.17) (1.23)

1.2 CHARGED SCALAR FIELDS, ANTIPARTICLES

Let us now quantize the theory (0.21) where

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2). \qquad (1.27)$$

Commutation relations can be imposed to both fields ϕ_1 and ϕ_2 independently,(*) so that we will simply get a copy of the preceeding chapter with $a \to a_i$, $a^\dagger \to a_i^\dagger$ with

$$[a_{\vec{p},i}, a^\dagger_{\vec{p}',j}] = 2\omega_p (2\pi)^3 \delta^3(\vec{p}-\vec{p}')\,\delta_{ij} \qquad (1.28)$$

Alternatively, we can write a complex field operator

$$\hat{\phi}(x) \equiv \int \frac{d^3p}{(2\pi)^3 2\omega_p}\left(\underset{\substack{\uparrow\\ \text{destroys an}\\ \text{'a-state'}}}{a_{\vec{p}}\, e^{-ip\cdot x}} + \underset{\substack{\uparrow\\ \text{creates a}\\ \text{'b-state'.}}}{b^\dagger_{\vec{p}}\, e^{ip\cdot x}} \right) \qquad (1.29)$$

and its conjugate

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger$$

$$\Rightarrow \quad \hat{\pi}(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} (i\omega_p) \left(a^\dagger_{\vec{p}}\, e^{+ip\cdot x} - b_{\vec{p}}\, e^{-ip\cdot x} \right) \qquad (1.30)$$

(*) Because we can rewrite:

$$|\partial_\mu \phi|^2 - m^2 |\phi|^2 = \frac{1}{2}\sum_{i=1}^{2}\left[(\partial_\mu \phi_i)^2 - m^2 \phi_i^2\right]$$

Because $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ we have:

$$a_{\vec{p}} = \frac{1}{\sqrt{2}}(a_{1\vec{p}} + i a_{2\vec{p}})$$
$$b_{\vec{p}} = \frac{1}{\sqrt{2}}(a_{1\vec{p}} - i a_{2\vec{p}}) \qquad (1.31)$$

and

$$[a_{\vec{p}}, a^{+}_{\vec{p}'}] = [b_{\vec{p}}, b^{+}_{\vec{p}'}] = 2\omega_p (2\pi)^3 \delta^3(\vec{p}-\vec{p}') \qquad (1.32)$$

- We can now compute the charge corresponding to the Noether current (Q,22): $(j^\mu \propto (\partial_\mu \phi^*)\phi - \phi^*(\partial_\mu \phi))$

$$:\hat{Q}: = \frac{i}{2} q :\int d^3x \, (\dot{\hat{\phi}}^{+}\hat{\phi} - \hat{\phi}^{+}\dot{\hat{\phi}}):$$
$$= \frac{i}{2} q :\int d^3x \, (\hat{\phi}^{+}\hat{\pi}^{+} - \hat{\pi}\hat{\phi}):$$
$$= q \int \frac{d^3p}{(2\pi)^3 2\omega_p} (a^{+}_p a_p - b^{+}_p b_p)$$

$|N\rangle \equiv \prod_{i=1}^{N} a^{+}_i \prod_{j=1}^{\bar{N}} b^{+}_j |0\rangle$

$\Rightarrow \langle N, \bar{N} | \hat{Q} | N, \bar{N} \rangle / \langle N\bar{N} | N\bar{N} \rangle$

$= q(N - \bar{N}) \qquad (1.33)$

$= qN + (-q\bar{N})$

– sign follows directly from the computation. It is OK when we interpret Q as a charge and not as a particle number. States $b^{+}_p|0\rangle$ are antiparticles. This makes sense because

$$\langle 0 | \phi(x) Q \begin{Bmatrix} a^{+}_p \\ b^{+}_p \end{Bmatrix} |0\rangle = \begin{Bmatrix} q\, e^{-ip\cdot x} \\ 0 \end{Bmatrix}$$

; particle state $\underline{\omega_p > 0}$ (1.34)

$$\langle 0 | \phi^{+}(x) Q \begin{Bmatrix} a^{+}_p \\ b^{+}_p \end{Bmatrix} |0\rangle = \begin{Bmatrix} 0 \\ -q\, e^{-ip\cdot x} \end{Bmatrix}$$

antiparticle state $\underline{\omega_p > 0}$! (1.35)

1.3. PROPAGATOR (FREE)

Let us now generalize K-G-theory (1.6) by adding a source $j(x)$:

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + j(x)\phi(x) \qquad (1.36)$$

The E.O.M now takes the form: $(\partial^2 \equiv \partial_\mu \partial^\mu)$

$$(\partial^2 + m^2)\phi = j(x) \qquad (1.37)$$

We now want to solve (1.37) by use of the Greens function method. To this end we define the propagator $\Delta(x,y)$ as a solution to eqn:

$$(\partial_x^2 + m^2)\Delta(x,y) \equiv -i\delta^4(x-y) \qquad (1.38)$$

Given (1.38), we can obviously write a solution to (1.37) in the form:

$$\phi(x) = \phi_0(x) + i\int d^4y\, \Delta(x,y)\, J(y) \qquad (1.39)$$

where $\phi_0(x)$ is an arbitrary solution to K-G-equation (i.e. to (1.37) with $j=0$). Assuming that the system is translation-invariant, we have:

$$\Delta(x,y) = \Delta(x-y) \qquad (1.40)$$

Let us now Fourier transform Δ:

$$i\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \, i\Delta(p) \qquad (1.41)$$

Operating by $\partial^2 + m^2$ from left and requiring (1.38) we get a formal solution

$$\Delta(p) = \frac{i}{p^2 - m^2} \qquad (1.42)$$

We can also now rewrite (1.39) as:

$$\phi(x) = \phi_0(x) - \int \frac{d^4p}{(2\pi)^4} \, i\Delta(p) \, J(p) \, e^{-ip\cdot x} \qquad (1.43)$$

This looks good, but the problem is that $\Delta(p)$ has a pole at $p^2 = m^2$, and so the integral expressions (1.41) and (1.43) are not well defined. Problem is of physical origin!

BOUNDARY CONDITIONS → 4 GREENS FUNCTIONS

Depening on how we avoid the problematic singularities, we get four different solutions for (1.43), as 4 different solutions for the propagator $\Delta(p)$.

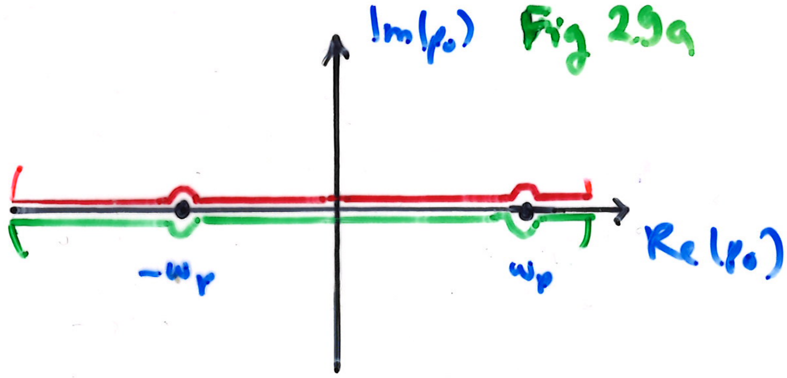


Fig 2.9a

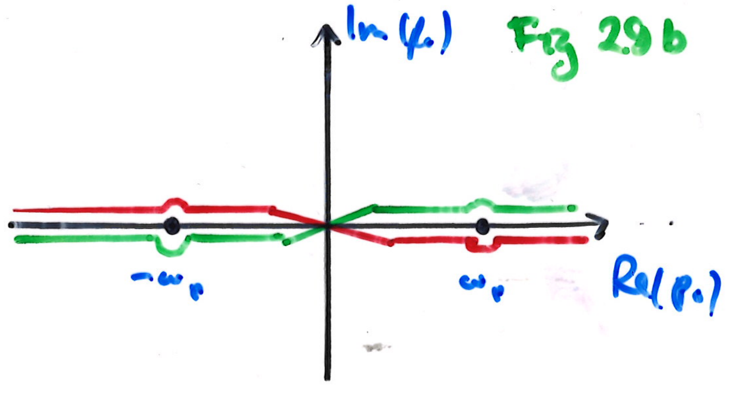


Fig 2.9b

Indeed, deforming the integration path according to figs. 29a and 29b, can equally well be accouted by giving p_0 a small complex component in $\Delta(p)$. To be precise the 4 different propagators are:

1: $\Delta(p) \longrightarrow \Delta_R(p) \equiv \dfrac{i}{(p_0+i\epsilon)^2-\vec{p}^2-m^2}$ $\qquad \delta p_0 = i\epsilon$ $\qquad$ (1.44a)

2: $\Delta(p) \longrightarrow \Delta_A(p) \equiv \dfrac{i}{(p_0-i\epsilon)^2-\vec{p}^2-m^2}$ $\qquad \delta p_0 = -i\epsilon$ $\qquad$ (1.44 b)

3: $\Delta(p) \longrightarrow \Delta_F(p) \equiv \dfrac{i}{p^2-m^2+i\epsilon}$ $\qquad \delta p_0 = i\epsilon\, \mathrm{sgn}(p_0)$ $\qquad$ (1.44c)

4: $\Delta(p) \longrightarrow \Delta_{\bar{F}}(p) \equiv \dfrac{i}{p^2-m^2-i\epsilon}$ $\qquad \delta p_0 = -i\epsilon\, \mathrm{sgn}(p_0)$ $\qquad$ (1.44d)

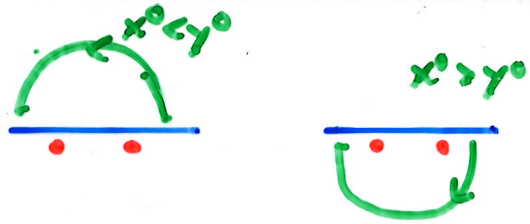
What is the physical content of these definitions? Consider the expression (1.41). It can be computed unambiguously when the path (the form of the propagator 1.44 a-d) is given, by use of the Cauchy theorem. The exponent term $e^{-ip^0(x^0-y^0)}$ determines how the contour must be closed in the complex p^0-plane (to get rid of the arc-integral):

upper plane when $x^0-y^0<0$ $\quad (\propto \Theta(y^0-x^0)$)

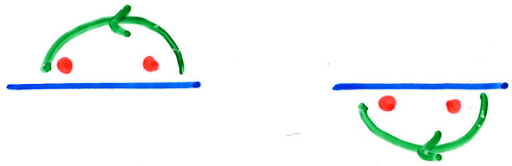
lower plane when $x^0-y^0>0$ $\quad (\propto \Theta(x^0-y^0)$)

Quite clearly then

1. $\Delta_R(x-y) \propto \Theta(x_0-y_0)$



2. $\Delta_A(x-y) \propto \Theta(y_0-x_0)$



Since $\Delta_R(x-y)$ is only nonzero if the the time y^0 is smaller than time x^0, we see that this choice allows that only past values of $J(y)$ affect the solution at x^0 in the expression (1.39). With $\Delta_A(x-y)$ the case is the opposite. Thus we say that

- $\Delta_R(x-y)$ propagates influence only to the future
- $\Delta_A(x-y)$ —"— —"— only to the past

$\Rightarrow$ nomenclature $\Delta_R \triangleq$ retarded and $\Delta_A \triangleq$ advanced propagator.

We can now compute:

$$\Delta_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p_0+i\epsilon)^2-\omega_p^2} e^{-ip\cdot(x-y)} \qquad ; \omega_p^2 = \vec{p}^{\,2}+m^2$$

$$= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \int \frac{dp^0}{2\pi} \left(\frac{i}{p_0-\omega_p+i\epsilon} - \frac{i}{p_0+\omega_p+i\epsilon} \right) e^{-ip\cdot(x-y)}$$

$$= \theta(x^0-y^0) \cdot \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \qquad (1.45)$$

I changed $\vec{p} \to -\vec{p}$ here.

$$\equiv \theta(x^0-y^0)\,\big(D(x-y) - D(y-x) \big)$$

It is easy to show that $D(x-y)$ is actually nothing but the vacuum expectation value

$$D(x-y) = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \qquad (1.46)$$

So we get:

$$\Delta_R(x-y) = \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle \qquad (1.47)$$

This is an example of a more general connection:

$$\boxed{\text{Greens functions} \triangleq \text{vacuum expectation values of field-operators}}$$

Similarly one can show that

$$\Delta_A(x-y) = -\theta(y^0-x^0)\langle 0|[\phi(x),\phi(y)]|0\rangle \qquad (1.48)$$

and

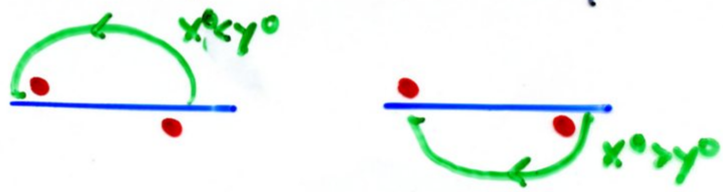
$$\Delta_F(x-y) = \theta(x^0-y^0)\,D(x-y) + \theta(y^0-x^0)\,D(y-x)$$

$$\equiv \langle 0|T[\phi(x)\phi(y)]|0\rangle \qquad (1.49)$$

Time ordered product.

This is the famous Feynman propagator. It will have a central role in the development of the perturbation theory below.

From the structure:
you realize that



Δ_F propagates positive frequency states forward – and negative frequency states backwards in time. The latter corresponds to propagating positive energy antiparticles forward in time (Feynman-Stueckelberg; see particle physics I-course).

1.4 Causality

Non-relativistic quantum theory in non-causal. For example the propagation of a free particle from $\vec{x}_0$ to $\vec{x}$ can be expressed as:

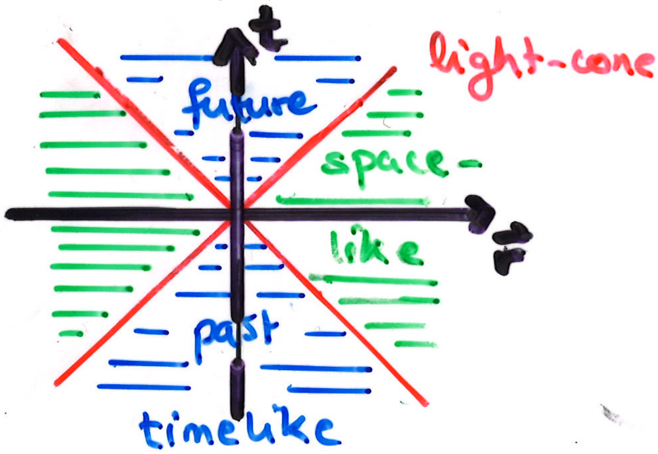
$$U(t) \equiv \langle x | e^{-iHt} | x_0 \rangle \qquad ; \quad H = \frac{p^2}{2m} = -\frac{\nabla^2}{2m}$$

$$= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\vec{x}-\vec{x}_0)^2/2t} \neq 0 \qquad \forall\ \vec{x}-\vec{x}_0 \text{ and } t.$$

"Relativistic theory, on the other hand, must respect causality.

Causality does not require that particles cannot propagate over space-like intervals; they do. Indeed, one can show that $D(x-y) \sim e^{-m|\vec{x}-\vec{y}|}$ over large spatial separations (equal times). However, measurements performed over spatial distances must not influence each others. This is guaranteed if the commutator $\langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle$ vanishes when $(x-y)^2 < 0$.

ideed, for a free field we can readily show that



$$\langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle = D(x-y) - D(y-x)$$

$$= \frac{i}{4\pi r}\frac{\partial}{\partial r}\begin{cases} \mathrm{sign}(t)\, J_0(m\sqrt{t^2-r^2}) & ; |t| > r \\ 0 & ; |t| < r \end{cases} \qquad \begin{aligned} t &\equiv x^0 - y^0 \\ r &\equiv |\vec{x}-\vec{y}| \end{aligned}$$

Generalizing this to complex scalar field, one would see that vanishing of $\langle 0 | [\hat{\phi}(x), \hat{\phi}(y)^\dagger] | 0 \rangle$ only results from cancellation between particle and antiparticle contributions, iff $m_{\text{antiparticle}} = m_{\text{particle}}$!