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Quantum Field theory

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ABSTRACT: QFT-II partial lecture notes.

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1. LSZ-reduction

The purpose of this section is to derive the connection between the S-matrix elements and the fundamental n-point Greens functions of the theory. As explained in the previous section, a generic physically observable $l \to m$ process may be related to a matrix element

$$S_{\rm fi}(l \to m) \equiv {}_{\rm out}\langle q_1, ..., q_m | p_1, ..., p_l \rangle_{\rm in} \,, \tag{1.1}$$

where $|p_i\rangle$ are interpreted as free one-particle states with the usual normalization (**??**). However, while we can prepare and observe intial and final state particles with a finite spatial and momentum resolution, these states are not truly free because (self-)interactions can never be turned off. Instead the observed particles – at best – correspond to stable, isolated one-particle excitations in the full interacting theory. Even this is eventually an idealization since all known particles are either unstable and/or couple to other massless states. Both these issues reduce the concept of an isolated particle excitation to an approximative quantity. Anyway, as experiments have shown, one-particle like states do appear in nature and they and look very much like free states, so this is a good starting point for our analysis.

1.1 Scalar fields

Let us first consider the simple interacting real scalar field. We begin by writing the free-particle creation operator in terms of the corresponding free field operators, by inverting the equation (**??**):

$$\hat{a}^\dagger_{\mathbf{p},\rm in} = -i \int {\rm d}^3\mathbf{x}\, e^{-ip\cdot x} \overleftrightarrow{\partial_t} \hat{\phi}_{\rm in}(x) \,, \tag{1.2}$$

where $A \overleftrightarrow{\partial_t} B \equiv B(\partial_t A) - (\partial_t A)B$ and $p_0 \equiv \omega_{\mathbf{p}}$. Analogous expression holds for the out-state creation operator. Using this formula we can now make explicit what we mean by the idealized in- and out-states:

$$\begin{aligned} |p\rangle_{\rm in} = a^\dagger_{\mathbf{p},\rm in}|0\rangle_{\rm in} &= -i \int {\rm d}^3\mathbf{x}\, e^{-ip\cdot x} \overleftrightarrow{\partial_t} \hat{\phi}_{\rm in}(x)|0\rangle_{\rm in} \\ &\equiv \lim_{t\to -\infty} -iR^{-1} \int {\rm d}^3\mathbf{x}\, e^{-ip\cdot x} \overleftrightarrow{\partial_t} \int \frac{{\rm d}^3 k}{(2\pi)^3 2\omega_{\mathbf{k}}} |\lambda_{\mathbf{k}1}\rangle\langle\lambda_{\mathbf{k}1}|\hat{\phi}(x)|\Omega\rangle \,. \end{aligned} \tag{1.3}$$

Here R is a constant yet to be defined and $|\lambda_{\mathbf{k}1}\rangle$ are the properly normalized interacting one-particle states with covariant dispersion relation $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m_P^2$, where m_P is the physical, observable mass of the state. That is, our "free" in-state in reality is a generic eigenstate of momentum created by the full field-operator $\hat{\phi}(x)$ from the actual interacting theory vacuum $|\Omega\rangle$, projected onto the subspace of interacting theory one particle states. This is the mathematical expression of the fact that our physical measuring apparatus in a scattering experiment projects out only correlations between on-shell excitations. Note that the projection onto 1-particle states is performed only in the infinite past; not at all times. The relation (1.3) is generic and the limit $t \to -\infty$ merely acts as a label for the particular free in-states. Indeed, similar equation (1.3) can be written also for $|p\rangle_{\rm out}$ if we just replace the time limit and the associate projection to one-particle states by $t \to +\infty$.

The normalization factor R is necessary because interacting theory one-particle states do not span the entire Hilbert space of the full theory. In order to determine it we manipulate the $\hat{\phi}$-matrix element in Eq. (1.3) as follows:

$$\begin{aligned} \langle\lambda_{\mathbf{k}1}|\hat{\phi}(x)|\Omega\rangle &= \langle\lambda_{\mathbf{k}1}|e^{-i\hat{P}\cdot x}\hat{\phi}(0)e^{i\hat{P}\cdot x}|\Omega\rangle \\ &= \langle\lambda_{\mathbf{k}1}|\hat{\phi}(0)|\Omega\rangle e^{-ik\cdot x}\,|_{k_0=\omega_{\mathbf{k}}} \\ &= \langle\lambda_{01}|\hat{\phi}(0)|\Omega\rangle e^{-ik\cdot x}\,|_{k_0=\omega_{\mathbf{k}}}\,. \end{aligned} \tag{1.4}$$

Here we first used the fact that vacuum is invariant under space-time translations and second that both the vacuum and the the field operator $\hat{\phi}(0)$ are invariant under boosts, so that we can evaluate the matrix element in the frame $\mathbf{k} = 0$. We thus have

$$\begin{aligned} |p\rangle_{\rm in} &= \lim_{t\to-\infty} -i\int\frac{{\rm d}^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}\int{\rm d}^3\mathbf{x}\,e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}}\,[\,e^{-i\omega_{\mathbf{p}}t}\,\overleftrightarrow{\partial}_t\,e^{i\omega_{\mathbf{k}}t}\,]\,\langle\lambda_{01}|\hat{\phi}(0)|\Omega\rangle\;R^{1-}\;|\lambda_{\mathbf{k}1}\rangle \\ &= \langle\lambda_{01}|\hat{\phi}(0)|\Omega\rangle\;R^{-1}\;|\lambda_{\mathbf{p}1}\rangle\;\equiv\;|\lambda_{\mathbf{p}1}\rangle\,. \end{aligned} \tag{1.5}$$

That is, to have the correct normalization convention we need to set

$$R \equiv \langle\lambda_{01}|\hat{\phi}(0)|\Omega\rangle\,. \tag{1.6}$$

The matrix element R is the *field strength renormalization factor*. Physically R gives the weight of one-particle states in a general interacting state produced by $\hat{\phi}$, and it adjusts the normalization of the interacting theory on-shell amplitudes to match the free-particle normalization (**??**), which we used to relate S-matrix to observable cross sections in section **??**. Let us now define the projected field operator onto interacting theory 1-particle states:

$$\hat{\phi}_1(x) \equiv \int\frac{{\rm d}^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}|\lambda_{\mathbf{k}1}\rangle\langle\lambda_{\mathbf{k}1}|\hat{\phi}(x)\,, \tag{1.7}$$

and an auxiliary operator that interpolates between $\hat{\phi}$ and $\hat{\phi}_1$:

$$\hat{\phi}_1^\infty(x) \equiv [1 - f(t)]\hat{\phi}(x) + f(t)\hat{\phi}_1(x)\,, \tag{1.8}$$

where $f(t) \approx 0$ for any finite t and $f(t) \to 1$ as $t \to \pm\infty$. The precise form of $f(t)$ is not relevant as long as it may be assumed to be zero over the physcial time scales in which the interactions and observation is taking place[1]. Then, over the interaction region we have $\hat{\phi}_1^\infty(x) = \hat{\phi}(x)$, but asymptotically $\hat{\phi}_1^\infty(x)$ creates only the projected 1-particle states. Given (1.7-1.8) we can rewrite Eq. (1.3) as

$$|p\rangle_{\rm in} = \lim_{t\to-\infty} -iR^{-1}\int \mathrm{d}^3\mathbf{x}\, e^{-ip\cdot x}\, \overleftrightarrow{\partial}_t\, \hat{\phi}_1(x)|\Omega\rangle\,. \tag{1.9}$$

It is obvious that an exactly analogous formula can be written for the out-states, corresponding just to a relabeling of the time limit:

$$|q\rangle_{\rm out} = \lim_{t\to\infty} -iR^{-1}\int \mathrm{d}^3\mathbf{y}\, e^{-iq\cdot y}\, \overleftrightarrow{\partial}_t\, \hat{\phi}_1(y)|\Omega\rangle\,. \tag{1.10}$$

Armed with relations (1.9-1.10) we can evaluate the overlaps between the in- and out-states in (1.1). First note that all fields are really created out from the true vacuum by the full field operators so that $S_{\rm fi}$ must be proportional to a vacuum expectation value of some operator, although we do not not write the vacuum state explicitly before all in- and out-momenta are exhausted. Let us begin by writing one of the in-states in terms of the field operator. Using Eq. (1.9), where we can obviously replace $\hat{\phi}_1$ by $\hat{\phi}_1^\infty$, and the following the simple trick:

$$\lim_{t\to\infty} \Big|_{-t}^{t} A(t') = \int_{-\infty}^{\infty} \mathrm{d}t\, \left[\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right] \tag{1.11}$$

we find

$$\begin{aligned} S_{\rm fi} &= \lim_{t\to-\infty} -iR^{-1}\int \mathrm{d}^3\mathbf{x}_1\, e^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial}_{t_1}\, {}_{\rm out}\langle q_1,...,q_m|\, \hat{\phi}_1^\infty(x_1)\, |p_2,...,p_l\rangle_{\rm in} \\ &= iR^{-1}\int \mathrm{d}^4x_1\, \partial_{t_1}\left[e^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial}_{t_1}\, {}_{\rm out}\langle q_1,...,q_m|\, \hat{\phi}_1^\infty(x_1)\, |p_2,...,p_l\rangle_{\rm in}\right] \\ &+ \lim_{t\to\infty} -iR^{-1}\int \mathrm{d}^3\mathbf{x}_1\, e^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial}_{t_1}\, {}_{\rm out}\langle q_1,...,q_m|\, \hat{\phi}_1^\infty(x_1)\, |p_2,...,p_l\rangle_{\rm in}\,. \end{aligned} \tag{1.12}$$

In the second term we can replace $\hat{\phi}_1^\infty$ by $\hat{\phi}_1$ and then use the formula (1.10) in reverse sense:

$$\begin{aligned} &\lim_{t\to\infty} iR^{-1}\int \mathrm{d}^3\mathbf{x}_1\, e^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial}_{t_1}\, {}_{\rm out}\langle q_1,...,q_m|\, \hat{\phi}_1(x_1)\, |p_2,...,p_l\rangle_{\rm in} \\ &= {}_{\rm out}\langle q_1,...,q_m|\Big(\, |p_1\rangle_{\rm out}\, |p_2,...,p_l\rangle_{\rm in}\Big) = \sum_{j=1}^{m}\langle q_j|p_1\rangle S_{\rm fi}(l-1\to m-1)\,, \end{aligned} \tag{1.13}$$

where $\langle q|p\rangle = {}_{\rm out}\langle q_j|p_1\rangle_{\rm out} = {}_{\rm in}\langle q_j|p_1\rangle_{\rm in}$ is the usual free particle normalization factor (**??**). Clearly this term corresponds to a process where one particle went from initial to final state without interacting. These *disconnected* processes may contribute to T-matrix if the $l-1 \to m-1$ subprocess is nontrivial. However, we do not need to include them here

[1]Mathematically, one also assumes that the change can be done adiabatically so that $f'(t) \approx 0$ at all times. This might not be always possible, for example for unstable particles. It does make sense however, when the life-time of the particle is large in comparison with the time-scales related to the observation process.

because such processes can always be reconstructed from the nontrivial *connected* scattering processes as long as the initial states are uncorrelated, which we are assuming is the case here. That is, for example for $l = 2$ and $m = 3$ the physical observable described by the disconnected term (1.13) is nothing but a decay $1 \to 2$ of either of the in-states.

Now consider the remaining term with the full space-time integral in Eq. (1.12). Before evaluating it, let us replace the sharp momentum eigenstate (1.9) with a smooth but sharp wave packet that better describes the preparation of the initial state, just as we did earlier when deriving the cross-section formulae:

$$\begin{aligned}|p\rangle_{\rm in} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hat{\varphi}_{\mathbf{p}}(\mathbf{k}) |k\rangle_{\rm in} \\ &= \lim_{t\to -\infty} -iR^{-1} \int d^3\mathbf{x} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \hat{\varphi}_{\mathbf{p}}(\mathbf{k}) e^{-ik\cdot x}\ \overleftrightarrow{\partial}_t\, \hat{\phi}_1^{\infty}(x)|\Omega\rangle\, . \\ &= \lim_{t\to -\infty} -iR^{-1} \int d^3\mathbf{x}\, \varphi_{\mathbf{p}}(\mathbf{x}) e^{-i\omega_{\mathbf{p}} t}\ \overleftrightarrow{\partial}_t\, \hat{\phi}_1^{\infty}(x)|\Omega\rangle\, . \end{aligned} \tag{1.14}$$

where $k = (\omega_{\mathbf{p}}, \mathbf{k})$. We can assume that the wave packet $\hat{\varphi}_{\mathbf{p}}(\mathbf{k})$ is strongly peaked around $\mathbf{k} = \mathbf{p}$, so that

$$-i\nabla\varphi_{\mathbf{p}}(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \mathbf{k}\, \hat{\varphi}_{\mathbf{p}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \approx \mathbf{p}\varphi_{\mathbf{p}}(\mathbf{x})\, , \tag{1.15}$$

all the while $\varphi_{\mathbf{p}}(\mathbf{x})$ is well localized in the scale of the experimental setup. In general these conditions are met to an extremely good accuracy in the scattering experiments. We can now write the connected part of the scattering matrix as

$$\begin{aligned} S_{\rm fi} &= iR^{-1} \int d^4x_1\, \varphi_{\mathbf{p}1}(\mathbf{x}_1) e^{-i\omega_{\mathbf{p}_1} t_1} (\partial_{t_1}^2 + \omega_{\mathbf{p}1}^2)\, {}_{\rm out}\langle q_1, ..., q_m|\, \hat{\phi}_1^{\infty}(x_1)\, |p_2, ..., p_l\rangle_{\rm in} \\ &\approx iR^{-1} \int d^4x_1\, \varphi_{\mathbf{p}1}(\mathbf{x}_1) e^{-i\omega_{\mathbf{p}_1} t_1} (\Box_1 + m_{\rm P}^2)\, {}_{\rm out}\langle q_1, ..., q_m|\, \hat{\phi}_1^{\infty}(x_1)\, |p_2, ..., p_l\rangle_{\rm in} \end{aligned} \tag{1.16}$$

In the last step we first used the fact that by definition $\omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m_{\rm P}^2$, and then used Eq. (1.15) to replace $\mathbf{p}_1^2 \approx -\nabla_1^2$ under the integral. For this to work it was essential that $\varphi_{\mathbf{p}}(\mathbf{x})$ has a finite spatial extension so that we could perform spatial partial integrations neglecting surface terms. To show that we can consistently perform this step was the only use we have for the wave packet, and we shall replace $\varphi_{\mathbf{p}}(\mathbf{x}) \to e^{i\mathbf{p}\cdot\mathbf{x}}$ from now on. Hence we have:

$$S_{\rm fi} = iR^{-1} \int d^4x_1\, e^{-ip_1\cdot x_1} (\Box_1 + m_{\rm P}^2)\, {}_{\rm out}\langle q_1, ..., q_m|\, \hat{\phi}(x_1)\, |p_2, ..., p_l\rangle_{\rm in} + \cdots \tag{1.17}$$

where $\cdots$ refer to the disconnected terms. What we have accomplished here is to replace an asymptotic one-particle state, defined only in the initial state, with an interacting field operator capable of describing correlations over the entire volume of the spacetime.

Let us now try to treat similarly one of the out-state particles. Using the complex conjugate form of (1.10):

$${}_{\rm out}\langle q| = \lim_{t\to\infty} \langle\Omega| iR^{-1} \int d^3\mathbf{y}\, e^{iq\cdot y}\, \overleftrightarrow{\partial}_t\, \hat{\phi}_1^{\infty}(y)\, , \tag{1.18}$$

we can immediately write

$$
\begin{aligned}
&{}_{\rm out}\langle q_1, ..., q_m|\, \hat\phi_1^\infty(x)\, |p_2, ..., p_l\rangle_{\rm in} \\
&= \lim_{y_1^0\to\infty} iR^{-1}\int {\rm d}^3\mathbf{y}_1\, e^{iq_1\cdot y_1}\, \overleftrightarrow{\partial}_{y_1^0}\, {}_{\rm out}\langle q_2, ..., q_m|\, \hat\phi_1^\infty(y_1)\hat\phi_1^\infty(x)\, |p_2, ..., p_l\rangle_{\rm in}\,. \qquad (1.19)
\end{aligned}
$$

We would now like to rewrite this again as an integral over the entire space of a derivative term and a disconnected term. To this end we would like to use (the conjugate of) Eq. (1.9) in the reverse sense at the in-boundary to pull out the ${}_{\rm in}\langle q_1|p_i\rangle_{\rm in}$-overlaps analogously to Eq. (1.13). However, the rule (1.11) does not work anymore, because it would leave $\hat\phi_1^\infty(y)$ to the left of $\hat\phi_1^\infty(x)$ while we need to have it acting on the vacuum, and the field operators do not commute. The way out is to introduce a new identity with time-ordering:

$$
\lim_{y^0\to\infty} \Big|_{-y^0}^{y^0} T(\hat\phi(y)\hat\phi(x)) = \int_{-\infty}^{\infty} {\rm d}y^0 \left[\frac{\rm d}{{\rm d}y^0} T(\hat\phi(y)\hat\phi(x))\right] \qquad (1.20)
$$

Using (1.20) $\hat\phi(y)$ does act directly onto the vacuum in the in-limit and we can apply (1.9) to extract the disconnected term. Having properly identified the disconnected part we can drop it with a same reasoning as before. Now, we can apply the reasoning (1.14-1.17) to the out state ${}_{\rm out}\langle q_1|$ without essential modifications, so that after extracting one particle from the in- and another from the out-state, we have

$$
\begin{aligned}
S_{\rm fi} = (iR^{-1})^2 \int & {\rm d}^4y_1{\rm d}^4x_1\, e^{iq_1\cdot y_1}e^{-ip_1\cdot x_1}(\Box_{y_1}+m_{\rm P}^2)(\Box_{x_1}+m_{\rm P}^2)\,\times \\
& \times\; {}_{\rm out}\langle q_2, ..., q_m|\, T(\hat\phi_1^\infty(y_1)\hat\phi_1^\infty(x_1))\, |p_2, ..., p_l\rangle_{\rm in} \;+\; \cdots \qquad (1.21)
\end{aligned}
$$

After this all remaining states can be treated analogously. In each extracton step one can use the rule (1.20) and the identity $T(\hat\phi(x_1)T(\hat\phi(x_2)\hat\phi(x_3)...)) = T(\hat\phi(x_1)\hat\phi(x_2)\hat\phi(x_3)..)$, until all in- and out-states are exhausted. In this end we eventually find:

$$
\begin{aligned}
S_{\rm fi} = \prod_{i=1}^{l}\int & {\rm d}^4y_j\, e^{iq_j\cdot y_j}\frac{(\Box_j+m_{\rm P}^2)}{-iR}\prod_{j=1}^{m}\int{\rm d}^4x_i\, e^{-ip_i\cdot x_i}\frac{(\Box_i+m_{\rm P}^2)}{-iR}\,\times \\
& \times\; \langle\Omega|\, T(\hat\phi_1(y_1), ..., \hat\phi_1(y_m)\hat\phi_1(x_1), ..., \hat\phi_1(x_l))\, |\Omega\rangle \;+\; \cdots \qquad (1.22)
\end{aligned}
$$

We have thus managed to realate the S-matrix element to an interacting theory Greens function projected on-shell. Note that we replaced $\hat\phi_1^\infty$ by the full operator $\hat\phi$ in the final expression. This is correct, because we can take the time $|t|$ at which $\hat\phi_1^\infty$ starts to deviate from $\hat\phi$ arbitrarily large. Indeed, the fact that the operators differ asymptotically, is now taken care for by the operators $\sim(\Box+m_{\rm P}^2)$, which perform an on-shell projection explicitly on the Greens function. Indeed, we can write the expression (1.22) in momentum space by formally performing two partial integrations. Being careful with the on-shell limits $p_i^2\to m_{\rm P}^2$ and $q_j^2\to m_{\rm P}^2$ we get:

$$
S_{\rm fi} = \lim_{k_i^2\to m_{\rm P}^2}\prod_{i=1}^{n}\Big[\frac{k_i^2-m_{\rm P}^2}{iR}\Big]\, G^{(n)}(k_1, ..., k_n)\,, \qquad (1.23)
$$

where

$$G^{(n)}(k_1,...,k_n) = \prod_{i=1}^{n} \int \mathrm{d}^4 x_i \, e^{-ik_i\cdot x_i} \, \langle\Omega|\, T(\hat{\phi}(x_1),...,\hat{\phi}(x_n))\, |\Omega\rangle \tag{1.24}$$

and we have treated all momenta on the same footing introducing an $n = l + m$ length vector k, which is defined to always point towards the interaction region. That is, for in-states $k_i = p_i$ and for out states $k_j = -q_j$, as shown in Fig. **??**.

To summarize, we have shown that the S-matrix for an arbitrary scattering process with n external legs is given by the residue of a multiple pole of the interacting n-point Greens function in momentum space, normalized by the field strength renormalization factor $(R^{-1})^n$. We shall soon learn how to compute these Greens functions and the pole in particular in the perturbation theory, but let us already borrow the result here (at this point you can take this as a definition):

$$G^{(n)}(k_1,...,k_n) = \prod_{i=1}^{n} \Big[\frac{iR^2}{k_i^2 - m_P^2} + \cdots\Big]\Gamma^{(n)}(k_1,...,k_n) \tag{1.25}$$

where $\Gamma^{(n)}$ is a special type of "amputated" Greens function, R is the field strength renormalization factor defined in (1.6) and $\cdots$ define terms that vanish on-shell. (These vanishing terms are exactly the difference between the full Greens function defined with field operators $\hat{\phi}$ and the projected Greens function defined with operators $\hat{\phi}_1$.) So after all the trouble we get the following very simple result:

$$S_{\mathrm{fi}} = R^n \Gamma^{(n)}(k_1,...,k_n)\,|_{k_i^2 = m_{\mathrm{P}}^2} \; . \tag{1.26}$$

There is one issue that one should be wary of with the previous derivation. From the outset we implicitly assumed that we can unambiguously define and isolate the free-particle like pole states in the interacting theory. This idea is borne out from experience in dealing for example with electrons in various observational setups (say, tracks seen in bubble chamber photographs). Under this assumption equations (1.7-1.9) in fact constitute a *definition* of an effective one-particle creation operator for the pole states of the interacting theory. Whether this construction is really valid is not guranteed however - or to tell the truth, in is known *not* not bo be so in any cases we are aware of! Indeed as a result of instabilities and/or interactions with massless states (photons), none of the field theories describing observed particles create truly isolated real poles in their Greens functions. Yet, the above construction is very useful in a well defined approximative sense.