POST-NEWTONIAN COSMOLOGICAL DYNAMICS OF PLANE-PARALLEL PERTURBATIONS AND BACK-REACTION

Eleonora Villa

Università degli Studi di Milano
Dipartimento di Fisica

in collaboration with

Sabino Matarrese Università degli Studi di Padova
Davide Maino Università degli Studi di Milano

Jyväskylä August 15, 2011

OUTLINE

1. Gravitational instability in cosmology
2. Post-Newtonian plane-parallel dynamics
3. Post-Newtonian estimation of cosmological back-reaction

Approximation schemes for gravitational instability

Large-scale structures grew by gravitational instability around primordial seed perturbations generated during inflation.

- Standard perturbation theory
  Expansion in powers of the amplitude of the perturbations around an homogeneous FRW background
  Applicability: small fluctuations

- Newtonian approximation of general relativity involve weak gravitational fileds and slow motion of particles
  Applicability: scales λ such that Schwarzschild radius $\ll \lambda \ll$ Hubble horizon

- Post-Newtonian approximation is suitable for a system of slowing moving particles bound together by gravitational forces but gravitational fields are not assumed to be weak
  Applicability: the mildly non-linear stage of the evolution of matter fluctuations.

- Globally plane-parallel configuration
  $\Rightarrow$ initial inflationary seed and evolution of perturbations involve one spatial direction

- Evolution during matter-dominated epoch
  $\Rightarrow$ Universe filled of pressure-less and irrotational fluid of Cold Dark Matter

- Synchronous and comoving gauge
  $\Rightarrow$ Lagrangian description

- Conformal rescaling of the metric

$$ds^2 = a^2(\tau)\left(-c^2 d\tau^2 + \gamma_{\alpha\beta}(\tau, \mathbf{q}) dq^\alpha dq^\beta\right)$$

  where $a(\tau) \propto \tau^2$ for the Einstein-de Sitter model

- Einstein-de Sitter subtraction
  $\Rightarrow$ dynamics in terms of $\vartheta^\alpha_\beta = ac\Theta^\alpha_\beta - \frac{2}{\tau}\delta^\alpha_\beta = \frac{1}{2}\gamma^{\alpha\sigma}\partial_\tau \gamma_{\sigma\beta}$

Einstein equations

- Dynamics in terms of $\vartheta^{\alpha}_{\beta} = ac\Theta^{\alpha}_{\beta} - \frac{2}{\tau}\delta^{\alpha}_{\beta} = \frac{1}{2}\gamma^{\alpha\sigma}\partial_{\tau}\gamma_{\sigma\beta}$

energy constraint $\vartheta^2 - \vartheta^{\alpha}_{\beta}\vartheta^{\beta}_{\alpha} + \frac{8}{\tau}\vartheta + c^2\,{}^{(3)}\mathcal{R} = \frac{24}{\tau^2}\delta$

momentum constraint $\mathcal{D}_{\alpha}\vartheta^{\alpha}_{\beta} = \partial_{\beta}\vartheta$

evolution equations $\partial_{\tau}\vartheta^{\alpha}_{\beta} + \frac{4}{\tau}\vartheta^{\alpha}_{\beta} + \vartheta\vartheta^{\alpha}_{\beta} + \frac{1}{4}\left(\vartheta^{\mu}_{\nu}\vartheta^{\nu}_{\mu} - \vartheta^2\right)\delta^{\alpha}_{\beta} + \frac{c^2}{4}\left[4\,{}^{(3)}\mathcal{R}^{\alpha}_{\beta} - {}^{(3)}\mathcal{R}\delta^{\alpha}_{\beta}\right] = 0$

Raychaudhuri equation $\partial_{\tau}\vartheta + \frac{2}{\tau}\vartheta + \vartheta^{\mu}_{\nu}\vartheta^{\nu}_{\mu} + \frac{6}{\tau^2}\delta = 0$

solution of continuity equation $\delta = (1+\delta_{in})\sqrt{\frac{\gamma_{in}}{\gamma}} - 1 \qquad \delta := \frac{\varrho - \varrho_{EdS}}{\varrho_{EdS}}$

- $\gamma_{\alpha\beta}$ is the only one (tensor) variable

$$\gamma_{\alpha\beta} = \overline{\gamma}_{\alpha\beta} + \frac{1}{c^2}w_{\alpha\beta} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

The zeroth-order: Newtonian approximation

Newtonian limit ($c \to \infty$) of Einstein equation:

$\left\{ \begin{array}{l} \text{energy constraint} \\ \text{evoluton equations} \end{array} \right.$ $\Rightarrow$ spatial curvature vanishes   Ellis 1971

$$\overline{\gamma}_{\alpha\beta} = \delta_{\mu\nu} \mathcal{J}^{\mu}_{\alpha} \mathcal{J}^{\nu}_{\beta}$$

where $\mathcal{J}^{\mu}_{\alpha} = \delta^{\mu}_{\alpha} + \partial_\alpha \mathcal{S}^{\mu}$ is the Jacobian matrix of the coordinate transformation from Lagrangian to Eulerian (comoving) observers

$$\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} + \mathcal{S}(\mathbf{q}, \tau)$$

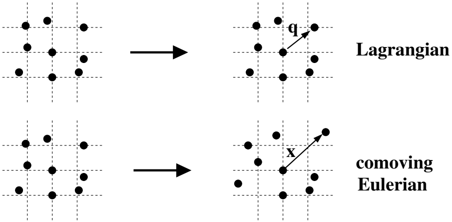
Raychaudhuri equation

$$\mathcal{J}^{\alpha}_{\mu} \partial^2_\tau \mathcal{J}^{\mu}_{\alpha} + \frac{2}{\tau} \frac{\partial_\tau \mathcal{J}}{\mathcal{J}} = \frac{6}{\tau^2} \left(1 - \frac{1}{\mathcal{J}} \right)$$

Momentum constraint

$$\varepsilon^{\alpha\beta\gamma} \mathcal{J}^{\mu}_{\beta} \partial_\tau \mathcal{J}_{\mu\gamma} = 0$$

Matarrese & Terranova   MNRAS 1997



from Peacock 1999

$\rightarrow$ Lagrangian perturbation theory

The Newtonian Zel'dovich solution

Lagrangian perturbation theory is an expansion in powers of the Jacobian matrix of $\mathbf{x}(\mathbf{q},\tau) = \mathbf{q} + \mathcal{S}(\mathbf{q},\tau)$. The background is represented by the FRW models
$\rightarrow$ Lagrangian and Eulerian (comoving) observers coincide

Zel'dovich approximation

The displacement vector and the Jacobian matrix are calculated from the equations at first order

$$\mathbf{x}(\mathbf{q},\tau) = \mathbf{q} - \frac{\tau^2}{6}\nabla\varphi \qquad \text{where} \quad \nabla^2\varphi = 4\pi G a^2 \delta$$

BUT all dynamical variables are calculate exactly from their non-perturbative definition
$\rightarrow$ ZA mimics the true non-linear behaviour

For plane-parallel perturbations Zel'dovich approximation yields an exact solution of the Newtonian equations

Newtonian Zel'dovich solution

$$\bar{\gamma}^{Zel}_{\alpha\beta} = \begin{pmatrix} \left(1 - \frac{\tau^2}{6}\partial_1^2\varphi\right)^2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

The post-Newtonian expansion

Post-Newtonian expansion of the metric

$$\gamma^{Zel}_{\alpha\beta} = \overline{\gamma}_{\alpha\beta} + \frac{1}{c^2} w_{\alpha\beta}$$

- initial conditions at the end of inflation $w^{\alpha}_{\beta \mathrm{in}} = -\frac{10}{3}\varphi\delta^{\alpha}_{\beta}$
- evolution does not switch on the dependence on q^2 and q^3 nor the off-diagonal components of the metric
- $w^2_2 = w^3_3$ from Einstein equations if $\varphi = \varphi\left(\tau, q^1\right)$

Therefore the post-Newtonian expansion is performed according to

$$\gamma_{11} = \left(1 - \frac{\tau^2}{6}\partial_1^2\varphi\right)^2 + \frac{1}{c^2}\left(1 - \frac{\tau^2}{6}\partial_1^2\varphi\right)^2 w^1_1$$

$$\gamma_{22} = 1 + \frac{1}{c^2} w^2_2$$

$$\gamma_{33} = 1 + \frac{1}{c^2} w^2_2$$

with initial conditions $w^1_{1,\mathrm{in}} = w^2_{2,\mathrm{in}} = -\frac{10}{3}\varphi$

The post-Newtonian equations

momentum constraint

$$\frac{2\tau\partial_1^2\varphi}{\tau^2\partial_1^2\varphi-6}\partial_1 w_2^2=\partial_\tau\partial_1 w_2^2$$

Raychaudhuri equation

$$\partial_\tau^2\left(w_1^1+2w_2^2\right)+\frac{2}{\tau}\left(\partial_\tau\left(w_1^1+2w_2^2\right)+\frac{4\tau\partial_1^2\varphi}{\tau^2\partial_1^2\varphi-6}\right)=\frac{36}{\tau^2\left(\tau^2\partial_1^2\varphi-6\right)}\left(w_1^1+2w_2^2+10\varphi\right)$$

energy constraint
$$\frac{16\partial_1^2\varphi}{\tau^2\partial_1^2\varphi-6}+\frac{72\left(\tau^2\partial_1^3\varphi\partial_1 w_2^2+\partial_1^2 w_2^2\left(6-\tau^2\partial_1^2\varphi\right)\right)}{\left(\tau^2\partial_1^2\varphi-6\right)^3}=\frac{24\partial_1^2\varphi}{6-\tau^2\partial_1^2\varphi}$$

evolution equation
$$-\frac{2}{\tau^2}+\frac{8\partial_1^2\varphi}{\tau^2\partial_1^2\varphi-6}+\frac{18\left(\tau^2\partial_1^3\varphi\partial_1 w_2^2+\partial_1^2 w_2^2\left(6-\tau^2\partial_1^2\varphi\right)\right)}{\left(\tau^2\partial_1^2\varphi-6\right)^3}=0$$

initial conditions
$$w_{1,\mathrm{in}}^1=w_{2,\mathrm{in}}^2=-\frac{10}{3}\varphi$$

The post-Newtonian solution

$$\gamma_{11} = \left(1 - \frac{\tau^2}{6}\partial_1^2\varphi\right)^2 + \frac{1}{c^2}\left(-6 + \tau^2\partial_1^2\varphi\right)\left(\frac{21\tau^2 C - 25\tau^4\partial_1^2\varphi(\partial_1\varphi)^2 - 350\varphi\left(-6 + \tau^2\partial_1^2\varphi\right)}{3780}\right)$$

$$\gamma_{22} = 1 + \frac{1}{c^2}\left(-\frac{10}{3}\varphi + \frac{5}{18}\tau^2(\partial_1\varphi)^2\right)$$

$$\gamma_{33} = 1 + \frac{1}{c^2}\left(-\frac{10}{3}\varphi + \frac{5}{18}\tau^2(\partial_1\varphi)^2\right)$$

The initial condition C of the post-Newtonian growing mode is still undetermined..

- it must be a second-order term $\Rightarrow$ involve the primordial non-Gaussianity

By comparison with the second-order metric in `Bartolo, Matarrese & Riotto JCAP 2005`
C is found to be

$$C = \frac{25}{3}\left[\left(1 - 4\left(a_{NL} - 1\right)\right)\left(\partial_1\varphi\right)^2 + \left(4 - 4\left(a_{NL} - 1\right)\right)\varphi(\partial_1^2\varphi)\right]$$

where a_{NL} is the primordial non-Gaussianity strength parameter

The post-Newtonian solution

The post-Newtonian solution

$$\gamma_{11} = \left(1 - \frac{\tau^2}{6}\partial_1^2\varphi\right)^2 + \frac{1}{c^2}\left\{\left[\frac{5}{108}\tau^2\left(\left(4\left(a_{NL}-1\right)-1\right)\left(\partial_1\varphi\right)^2 + \left(4\left(a_{NL}-1\right)-4\right)\varphi\partial_1^2\varphi\right) + \right.\right.$$
$$\left.\left. + \frac{5}{576}\tau^4\partial_1^2\varphi\left(\partial_1\varphi\right)^2\right]\left(6-\tau^2\partial_1^2\varphi\right) - \frac{5}{54}\varphi\left(6-\tau^2\partial_1^2\varphi\right)^2\right\}$$

$$\gamma_{22} = 1 + \frac{1}{c^2}\left(-\frac{10}{3}\varphi + \frac{5}{18}\tau^2(\partial_1\varphi)^2\right)$$

$$\gamma_{33} = 1 + \frac{1}{c^2}\left(-\frac{10}{3}\varphi + \frac{5}{18}\tau^2(\partial_1\varphi)^2\right)$$

Convergence of perturbative series

for γ_{11} $\qquad \mathcal{O}\left(\frac{1}{(1+\bar{\delta})^2}\right) + \mathcal{O}\left(\frac{\varphi/c^2}{1+\bar{\delta}}\right) + \mathcal{O}\left(\frac{\varphi/c^2}{(1+\bar{\delta})^2}\right)$

for γ_{22} $\qquad \mathcal{O}(1) + \mathcal{O}\left(\frac{\varphi}{c^2}\right)$

where from cosmological Poisson equation $\frac{\varphi}{c^2} \sim \left(\frac{\lambda_{proper}}{cH^{-1}}\right)^2\delta$

Application: cosmological back-reaction

It has been proposed that the observed increase in the expansion rate of the Universe could be due to the back-reaction of the non-linear sub-horizon cosmic structures on the background Universe expansion.

How to quantify this effect?

- standard perturbative theory (even at second or high order) is inadequate to evaluate back-reaction in the average Einstein equations

  Kolb, Matarrese & Riotto, New Jour.Phys. 2006

- in the Newtonian approximation back-reaction terms are negligible

  Buchert & Ehlers, Astron.Astrophys. 1997

Averaging Einstein equations

Spatial average of a scalar field

$$\langle \Psi(t) \rangle_{\mathcal{D}} := \frac{1}{\mathcal{V}_{\mathcal{D}}(t)} \int_{\mathcal{D}} \Psi(\mathbf{q}, t) \sqrt{h(\mathbf{q}, t)} d^3 q$$

- $\sqrt{h}$ is the determinant of spatial metric in synchronous and comoving gauge
- $\mathcal{V}_{\mathcal{D}}(t) := \int_{\mathcal{D}} \sqrt{h(\mathbf{q}, t)} d^3 q$ is the volume of the coarse-graining domain $\approx$ Hubble volume

By smoothing the scalar Einstein equations, effective Friedmann equations for the average scale factor $a_{\mathcal{D}} = \left(\frac{\mathcal{V}_{\mathcal{D}}}{\mathcal{V}_{\mathcal{D}_0}} \right)^{1/3}$ are obtained

Effective Friedmann equations

Effective Friedmann equations

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = \frac{8}{3}\pi G \varrho_{eff}$$

$$\left(\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right) = -\frac{4}{3}\pi G \left(\varrho_{eff} + \frac{3P_{eff}}{c^2}\right)$$

$$\dot{\varrho}^{eff}_{\mathcal{D}} + 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\left(\varrho_{eff} + \frac{P_{eff}}{c^2}\right) = 0$$

$$\left(a^6_{\mathcal{D}}\mathcal{Q}_{\mathcal{D}}\right)^{\cdot} + c^2 a^4_{\mathcal{D}}\left(a^2_{\mathcal{D}}\langle\mathcal{R}\rangle_{\mathcal{D}}\right)^{\cdot} = 0$$

Buchert Gen.Rel.Grav. 2001

Effective fluid

$$\varrho^{eff}_{\mathcal{D}} = \langle\varrho\rangle_{\mathcal{D}} - \frac{\mathcal{Q}_{\mathcal{D}}}{16\pi G} - \frac{c^2\langle\mathcal{R}\rangle_{\mathcal{D}}}{16\pi G}$$

$$P^{eff}_{\mathcal{D}} = -\frac{c^2\mathcal{Q}_{\mathcal{D}}}{16\pi G} + \frac{c^4\langle\mathcal{R}\rangle_{\mathcal{D}}}{48\pi G}$$

$$w_{eff} = \frac{P_{eff}}{\varrho_{eff}} = \frac{\mathcal{Q}_{\mathcal{D}} - c^2/3\langle\mathcal{R}\rangle_{\mathcal{D}}}{\mathcal{Q}_{\mathcal{D}} + c^2\langle\mathcal{R}\rangle_{\mathcal{D}}}$$

Buchert Gen.Rel.Grav. 2001

The effects of the cosmic structures are encoded in:

Kinematical back-reaction

$$\mathcal{Q}_{\mathcal{D}} := \tfrac{2}{3}\langle(\Theta - \langle\Theta\rangle_{\mathcal{D}})^2\rangle_{\mathcal{D}} - 2\langle\Sigma^2\rangle_{\mathcal{D}}$$

- variance in the mean expansion rate
- average shear

$$\Sigma^\alpha_\beta = \Theta^\alpha_\beta - (1/3)\Theta\delta^\alpha_\beta$$

Mean spatial curvature

$$\langle{}^{(3)}R\rangle_{\mathcal{D}} = \frac{1}{\mathcal{V}_{\mathcal{D}}}\int_{\mathcal{D}} {}^{(3)}R\sqrt{h}\, d^3q$$

- average curvature of inhomogeneous space-time

Post-Newtonian results for back-reaction

Post-Newtonian expression for kinematical back-reaction and mean spatial curvature:

Kinematical Backreaction and mean spatial curvature

$$\mathcal{Q}_{\mathcal{D}} = -\frac{2a^4\tau^2}{27}\left(\frac{1}{\overline{\mathcal{V}}_{\mathcal{D}}}\int_{\mathcal{D}} \partial_1^2\varphi \; d^3q\right)^2 - \frac{10a\tau^2}{81c^2\overline{\mathcal{V}}_{\mathcal{D}}}\int_{\mathcal{D}} \partial_1(\partial_1\varphi)^3 \; d^3q$$
$$= -\frac{2}{3}\left(\langle \partial_{r_1}\overline{v}_1\rangle_{\mathcal{D}}\right)^2 + \frac{10}{3c^2 a\tau}\langle \partial_{r_1}\overline{v}_1^3\rangle_{\mathcal{D}}$$

$$\langle {}^{(3)}R\rangle_{\mathcal{D}} = \frac{20a}{3c^2\overline{\mathcal{V}}_{\mathcal{D}}}\int_{\mathcal{D}} \partial_1^2\varphi \; d^3q$$

Post-Newtonian expression for average expansion rate:

Average expansion rate

$$\langle \Theta\rangle_{\mathcal{D}} = 3H\left(1 - \frac{5}{12c^2}\langle\left(1+\overline{\delta}\right)\overline{v}_1^2\rangle_{\mathcal{D}}\right)$$

⇒ Negligible post-Newtonian correction
for plane-parallel dynamics

Caustic formation

Density contrast and spatial curvature diverge

Density contrast and spatial curvature

$$\delta = \frac{\tau^2 \partial_1^2 \varphi}{6 - \tau^2 \partial_1^2 \varphi} + \frac{5\tau^2 \left[(\partial_1 \varphi)^2 \left(5\partial_1^2 \varphi \tau^2 + 42(3 - 4a_{NL}) \right) - 168(a_{NL} - 2)\varphi \partial_1^2 \varphi \right]}{42c^2 \left(\tau^2 \partial_1^2 \varphi - 6 \right)^2}$$

$${}^{(3)}R = \frac{(20/3)\partial_1^2 \varphi}{c^2 a^2 \left(1 - \tau^2 \partial_1^2 \varphi / 6 \right)}$$

but the average quantities are finite

Average density and mean spatial curvature

$$\langle \rho \rangle_{\mathcal{D}} = \varrho_{EdS} \left(1 + \frac{5a^3}{36c^2 \overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \tau^2 \left(\partial_1 \varphi \right) \, d^3 q \right) = \varrho_{EdS} \left(1 + \frac{5}{4c^2} \langle \left(1 + \overline{\delta} \right) \overline{v}_1^2 \rangle_{\mathcal{D}} \right)$$

$$\langle {}^{(3)}R \rangle_{\mathcal{D}} = \frac{20a}{3c^2 \overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \partial_1^2 \varphi \, d^3 q$$

Conclusions

- Post-Newtonian extension of the Zel'dovich Newtonian solution for plane-parallel dynamics has been provided
- It has been explicitly shown that Lagrangian approach allows to obtain a quantitative estimate of back-reaction
- The divergences due to the caustic formation is completely eliminated by the spatial average
- No relevant back-reaction effect is found for post-Newtonian plane-parallel dynamics

Villa, Matarrese & Maino accepted for publication in JCAP

Future work

- Analysis of the photon geodesics to study e.g. the luminosity-distance relation
- Extension of post-Newtonian Zel'dovich expansion to a (more complex!) full 3D calculation

Villa, Matarrese & Maino in preparation