The Scale of Inhomogeneity and its Implication for Cosmological Backreaction

Tejinder P. Singh

Tata Institute of Fundamental Research, Mumbai, India

Jyvaskyla, 15-19 August, 2011

Plan of the Talk

- I. Evidence for large scale homogeneity?
- II. If length scale for homogeneity much smaller than Hubble radius, then back-reaction is small.
- III. Possible importance of averaging on galactic scales? [arXiv 1108.2375]

I. Evidence for large scale homogeneity?

”We are currently unable to prove homogeneity of the Universe on large scales, even with the Copernican Principle. However, we can use observations of galaxies and clusters to test the Copernican Principle itself.”

- Roy Maartens, 2011; arXiv 1104.1300

I. Evidence for large scale homogeneity?

- Tests for homogeneity depend on testing for isotropy around us and testing the Copernican Principle [CP].
- Observational evidence for spacetime isotropy around our world-line can be established by investigating isotropy of CMB and galaxy distribution.
- Observational evidence from isotropy of matter distribution : can be established from isotropy of angular diameter distances, galaxy number counts, bulk velocities and lensing.
- Strictly speaking, establishing spacetime isotropy of CMB requires CP [spatial derivatives of higher multipoles must vanish].

I. Evidence for large scale homogeneity

- Given CP, and an isotropic matter and CMB, what can one say about spatial homogeneity?
- If all fundamental observers measure the same isotropic distribution of the four matter observables mentioned above, this implies the Universe is FLRW.
- Exact isotropy of CMB for all observerrs also implies isotropy.
- Almost isotropy of CMB implies almost FLRW Universe.

I. Evidence for large scale homogeneity : Testing CP

- Test consistency relations for FLRW cosmology.
- Check to rule out redshift dependence of FLRW curvature parameter.
- Time dependence of cosmological redshift
- Significant difference between radial and transverse BAO scales.
- Thermal or kinetic SZ temperature/polarization distortion of CMB
- No evidence against CP, nor clinching evidence for large -scale homogeneity, or for the scale of homogeneity. Back-reaction small if homogeneity scale significantly smaller than Hubble radius.

II. Homogeneity and back-reaction

- Averaging of tensors : Introduce a *bivector* $\mathcal{W}_b^{a'}(x', x)$ which transforms as a vector at event x' and as a co-vector at event x.

- Use bivector to define the "bilocal extension" of a general tensorial object

$$\widetilde{P}^a(x', x) = \mathcal{W}_{a'}^a(x, x') P^{a'}(x') \tag{1}$$

- "Average" of $P^a(x)$ over a 4-dimensional spacetime region $\boldsymbol{\Sigma}$ with a supporting point x :

$$\bar{P}^a(x) = \left\langle \widetilde{P}^a \right\rangle_{ST} = \frac{1}{V_\Sigma} \int_\Sigma d^4x' \sqrt{-g'}\, \widetilde{P}^a(x', x) \tag{2}$$

II. Homogeneity and backreaction

- In a volume preserving coordinate system ϕ^m, [VPC], i.e. one with $g(\phi^m) =$ constant, the coordination bivector takes its most simple form, namely

$$\mathcal{W}_j^{a'}(x', x) \mid_{\text{proper}} = \delta_j^{a'} \,. \tag{3}$$

- Averaged Geometry : The key idea of Macroscopic Gravity is that the average connection $\bar{\mathbf{\Omega}}^a{}_b(x)$

$$\bar{\mathbf{\Omega}}^a{}_b \equiv \langle \mathbf{\Omega^a{}_b} \rangle \,, \tag{4}$$

is defined as the connection 1-form on a new, averaged manifold $\bar{\mathcal{M}}$.

- Define a correlation 2-form

$$\mathbf{Z}^a{}_b{}^i{}_j = \left\langle \mathbf{\Omega^a{}_b} \wedge \mathbf{\Omega^i{}_j} \right\rangle_{ST} - \bar{\mathbf{\Omega}}^a{}_b \wedge \bar{\mathbf{\Omega}}^i{}_j \,. \tag{5}$$

II. Homogeneity and backreaction

- Denoting $\mathbf{R}^a{}_b \equiv \langle \widetilde{\mathbf{r}}^a{}_b \rangle_{ST}$, and the curvature 2-form on the averaged manifold $\bar{\mathcal{M}}$ as $\mathbf{M}^a{}_b$ gives

$$\mathbf{M}^a{}_b = \mathbf{R}^a{}_b - \mathbf{Z}^a{}_c{}^c{}_b \,. \tag{6}$$

- The inhomogeneous Einstein equations

$$g^{ak} r_{kb} - \frac{1}{2}\delta^a_b g^{ij} r_{ij} = -\kappa t^{a(\mathrm{mic})}_b \,, \tag{7}$$

average out to

-

$$E^a_b = -\kappa T^a_b + C^a_b \,, \tag{8}$$

$$C^a_b = \left(Z^a{}_{ijb} - \frac{1}{2}\delta^a_b Z^m{}_{ijm} \right) G^{ij}. \tag{9}$$

II. Homogeneity and backreaction

- Assume the averaged spacetime to be the FLRW spacetime. Then, spatial averaging limit of covariant averaging must be considered.

- Assume the averaged metric to be

$$^{(\bar{\mathcal{M}})}ds^2 = -d\tau^2 + a^2(\tau)\delta_{AB}dy^A dy^B \tag{10}$$

- and the inhomogeneous metric to be

$$^{(\mathcal{M})}ds^2 = -\frac{d\bar{t}^2}{h(\bar{t},\mathbf{x})} + h_{AB}(\bar{t},\mathbf{x})dx^A dx^B \,. \tag{11}$$

- Define the expansion tensor as

$$\Theta^A_B \equiv \frac{1}{2N}h^{AC}\partial_{\bar{t}}h_{CB} \,. \tag{12}$$

II. Homogeneity and backreaction

- The modified Friedmann equations are

$$H^2_{\mathrm{FLRW}} = \frac{8\pi G_N}{3}\rho - \frac{1}{6}\left[\mathcal{Q}^{(1)} + \mathcal{S}^{(1)}\right], \tag{13a}$$

$$\frac{1}{a}\frac{d^2 a}{d\tau^2} = -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{1}{3}\mathcal{Q}^{(1)}. \tag{13b}$$

- where

►

$$\mathcal{Q}^{(1)} = \bar{a}^6 \left[\frac{2}{3} \left(\langle \frac{1}{h} \Theta^2 \rangle - \frac{1}{\bar{a}^6} ({}^{\mathrm{F}}\Theta^2) \right) - 2 \langle \frac{1}{h} \sigma^2 \rangle \right] \;\; ; \;\; \frac{1}{\bar{a}^6} ({}^{\mathrm{F}}\Theta^2) = (3H) \tag{14a}$$

$$\mathcal{S}^{(1)} = \frac{1}{\bar{a}^2} \delta^{AB} \left[\langle {}^{(3)}\Gamma^J_{AC} \, {}^{(3)}\Gamma^C_{BJ} \rangle - \langle \partial_A (\ln \sqrt{h}) \partial_B (\ln \sqrt{h}) \rangle \right] , \tag{14b}$$

►

$$\tau = \int^{\bar{t}} \frac{dt}{\bar{a}^3(t)} \;\; ; \;\; y^A = x^A \, . \tag{15}$$

$$a(\tau) = \bar{a}(\bar{t}(\tau)) \tag{16}$$

►

$$\left(\partial_\tau \mathcal{Q}^{(1)} + 6 H_{\mathrm{FLRW}} \mathcal{Q}^{(1)} \right) + \left(\partial_\tau \mathcal{S}^{(1)} + 2 H_{\mathrm{FLRW}} \mathcal{S}^{(1)} \right) = 0 \, . \tag{17}$$

II. Homogeneity and backreaction : perturbation theory

- Is cosmological perturbation theory stable against growth of backreaction?
- Need to check this iteratively :

$$a^{(0)} \to \phi^{(0)} \to C^{(0)} \to a^{(1)} \to \phi^{(1)} \to \ldots \qquad (18)$$

- Let the perturbed FLRW metric be $ds^2 =$

$$a^2 \left[-(1+2\phi)d\eta^2 + 2\omega_A dx^A d\eta + \left((1-2\psi)\gamma_{AB} + \chi_{AB}\right) dx^A dx^B \right] . \qquad (19)$$

- We work with a VPC which has no residual degrees of freedom. Further, this VPC is constructed by starting from the conformal Newtonian gauge, and by making a steady coordinate transformation. This ensures that all averaged quantities are gauge invariant.

II. Homogeneity and backreaction : perturbation theory

- Evaluate the correlation scalars for a given initial power spectrum - standard CDM.
- For a constant nonevolving potential $\phi(\vec{x})$, and with a power spectrum

$$\frac{k^3 P_{\phi i}(k)}{2\pi^2} = A(k/H_0)^{n_s - 1}\,, \tag{20}$$

- the back reaction is

$$\frac{\mathcal{S}^{(1)}}{H_0^2} \sim -\frac{1}{a^2}(10^{-4})\,. \tag{21}$$

- The smallness of backreaction holds also for the exact sCDM model.
- This analysis ignores contribution of scales that have become fully nonlinear in matter density at late times.

II. Homogeneity and back-reaction : structure formation

- We study backreaction in a toy model of spherical collapse, using the LTB solution.
- The initial density is chosen to be

$$\rho(t_i, r) = \rho_{bi} \begin{cases} (1+\delta_*), & r < r_* \\ (1-\delta_v), & r_* < r < r_v \\ 1, & r > r_v \, , \end{cases} \tag{22}$$

- We match the initial velocity and coordinate scaling to the global background solution, by requiring

$$R(t_i, r) = a_i r \, , \tag{23}$$

$$\dot{R}(t_i, r) = a_i H_i r \, , \tag{24}$$

II. Homogeneity and backreaction : structure formation

- For the FLRW background we consider an Einstein-deSitter (EdS) solution with scale factor and Hubble parameter given by

$$a(t) = (t/t_0)^{2/3} \;\; ; \;\; t_0 = 2/(3H_0)\,, \tag{25}$$

$$H(t) \equiv \dot{a}/a = 2/(3t)\,, \tag{26}$$

  with t_0 denoting the present epoch. a_i fixes the initial time as

$$t_i = 2/(3H_0)a_i^{3/2}\,. \tag{27}$$

- We use $a_i = 10^{-3}$, so that the initial conditions are being set around the CMB last scattering epoch.

II. Homogeneity and backreaction : structure formation

- Mass function $M(r)$ and curvature function $k(r)$:

$$GM(r) = \frac{1}{2}H_0^2 r^3 \begin{cases} 1+\delta_*, & 0 < r < r_* \\ 1+\delta_v\left((r_c/r)^3 - 1\right), & r_* < r < r_v \\ 1+(\delta_v/r^3)\left(r_c^3 - r_v^3\right), & r > r_v\,, \end{cases} \tag{28}$$

- where we have defined a "critical" radius r_c by the equation

$$\left(\frac{r_c}{r_*}\right)^3 = 1 + \frac{\delta_*}{\delta_v}\,. \tag{29}$$

Homogeneity and backreaction : structure formation

- The significance of r_c :

$$k(r) = \frac{H_0^2}{a_i} \begin{cases} \delta_*, & r < r_* \\ \delta_v \left((r_c/r)^3 - 1 \right), & r_* < r < r_v \\ (\delta_v/r^3) \left(r_c^3 - r_v^3 \right), & r > r_v \, . \end{cases} \tag{30}$$

- Since $\delta_*, \delta_v > 0$, we have $r_c > r_*$ by definition. The following possibilities arise :
- If $r_c > r_v$, then $k(r) > 0$ for all r, and every shell will ultimately collapse, including the "void" region $r_* < r < r_v$.
- If $r_c < r_v$, then $k(r) > 0$ for $r < r_c$ and changes sign at $r = r_c$. Hence, the region $r_* < r < r_c$ will collapse even though it is underdense, while the region $r > r_c$ will expand forever.
- If $r_c = r_v$, then the "void" exactly compensates for the overdensity, and the universe is exactly EdS for $r > r_v$.

Parameter name	Parameter value
a_i	0.001
H_0	$1/13.59\,\mathrm{Gyr}^{-1} \quad (= 72\,\mathrm{km/s/Mpc})$
t_0	$2/(3H_0) = 9.06\mathrm{Gyr}$
c	$306.6\,\mathrm{MpcGyr}^{-1}$
δ_*	$1.25a_i(3\pi/4)^{2/3} = 2.21 \times 10^{-3}$
δ_v	0.005
r_*	$0.004c/H_0 = 16.7\,\mathrm{Mpc}$
t_{turn}/t_0	0.72
r_c	$r_*\,(1+\delta_*/\delta_v)^{1/3} = 18.8\,\mathrm{Mpc}$
r_v	$1.25r_c = 23.5\,\mathrm{Mpc}$
$R(t_0, r_*)$	$6.8\,\mathrm{Mpc}$
$R(t_0, r_v)$	$33.3\,\mathrm{Mpc}$

Table: Values of various parameters used in evolution

- Transforming to the perturbed FLRW form : We want a coordinate transformation $(t, r) \to (\tau, \tilde{r})$ such that the metric in the new coordinates is

$$ds^2 = -(1+2\phi)d\tau^2 + a^2(\tau)(1-2\psi)\left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right) , \quad (31)$$

with at least the conditions

$$\mid \phi \mid \ll 1 \ ; \quad \mid \psi \mid \ll 1 \, , \quad (32)$$

being satisfied.

- Since t is the proper time of each matter shell, the quantity $\partial_t \tilde{r}$ is simply the velocity of matter in the $(\tau, \tilde{r})$ frame (which is comoving with the Hubble flow) :

$$\tilde{v} \equiv \frac{\partial \tilde{r}}{\partial t} \, , \quad (33)$$

is the radial comoving peculiar velocity of the matter shells in the $(\tau, \tilde{r})$ frame.

- We show that the required transformation exists, provided matter peculiar velocities remain small.

Backreaction during nonlinear growth of structure

- We already have in place the formalism for calculating the backreaction when the metric is of the perturbed FLRW form.
- From here it follows that the backreaction is very small, in the nonlinear structure formation regime, provided matter peculiar velocities are small.

III. The possible relevance of averaging for galaxy rotation curves

- The Scalar-Tensor-Vector gravity theory modifies general relativity by including a massive vector field with coupling ω and mass μ and turning G into a dynamical filed.
- Newton's law of gravitational acceleration gets modified by the inclusion of a Yukawa term :

$$a(r) = -\frac{G_\infty M}{r^2} + K\frac{\exp(-r/r_0)}{r^2}\left(1 + \frac{r}{r_0}\right), \qquad (34)$$

where G_∞ is defined to be the effective gravitational constant at infinity

$$G_\infty = G\left(1 + \sqrt{\frac{M_0}{M}}\right) \qquad (35)$$

and $r_0 = 1/\mu$. Here, M_0 denotes a parameter that vanishes when $\omega = 0$. The constant K is assumed to equal

$$K = G\sqrt{MM_0}. \qquad (36)$$

III. Averaging and rotation curves

- One can rewrite the acceleration in the form

$$a(r) = -\frac{GM}{r^2}\left\{1+\sqrt{\frac{M_0}{M}}\left[1-\exp(-r/r_0)\left(1+\frac{r}{r_0}\right)\right]\right\}. \quad (37)$$

and generalize this to the case of a mass distribution by replacing the factor GM/r^2 in (37) by $G\mathcal{M}(r)/r^2$. The rotational velocity of a star v_c is obtained from $v_c^2(r)/r = a(r)$ and is given by

$$v_c = \sqrt{\frac{G\mathcal{M}(r)}{r}}\left\{1+\sqrt{\frac{M_0}{M}}\left[1-\exp(-r/r_0)\left(1+\frac{r}{r_0}\right)\right]\right\}^{1/2}. \quad (38)$$

A good fit to a large number of galaxies has been achieved with the parameters:

$$M_0 = 9.60 \times 10^{11}\, M_\odot, \quad r_0 = 13.92\,\mathrm{kpc} = 4.30 \times 10^{22}\,\mathrm{cm}. \quad (39)$$

III. Quadrupole gravitational polarization and the biharmonic equation

- Thinking of galaxies as ‘molecules’ made of atoms [the stars], one would like to analyze how the averaged gravitational field inside a galaxy, is modified by the polarization of the molecules, due to the external pull of other galaxies. Indeed one has at the back of the mind the polarization induced modification of electromagnetic fields in a an electrically charged medium.

Quadrupole gravitational polarization and the biharmonic equation

- Averaging of Einstein equations using Kauffman's method of molecular moments gives

$$R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(0)} = -\kappa (T_{\mu\nu}^{(free)} + T_{\mu\nu}^{(GW)} + \frac{1}{2} c^2 Q_{\mu\rho\nu\sigma}^{;\rho\sigma}), \quad (40)$$

- Static weak-field averaged gravitating medium with quadrupole gravitational polarization : the averaged Einstein equations reduce to

$$\nabla^2 \phi - \frac{1}{k^2} \nabla^4 \phi = 4\pi G \rho \quad (41)$$

with

$$\frac{1}{k^2} = \frac{2\pi G}{c^2} \epsilon_g \quad (42)$$

III. Solution of the biharmonic equation

- In vacuum :

$$a = C_0 \frac{e^{-kr}}{kr} + C_0 \frac{e^{-kr}}{k^2 r^2} - \frac{C_2}{r^2} \tag{43}$$

The constants C_0 and C_2 can be related by the following reasoning : For $kr \ll 1$ we assume Newton's law of gravitation to hold, so that $C_2 = GM + C_0/k^2 \equiv G_\infty M$ where $G_\infty = G[1 + C_0/k^2 MG]$. For $kr \gg 1$ the exponential term can be ignored, and G_∞ represents the effective gravitational constant at large distances.

III. Galaxy rotation curves

- We could show that for observed luminous matter density profiles the interior solution of the biharmonic equation matches with that coming from the STVG theory.

- Furthermore, the parameter $k = 1/r_0$ can be estimated from theory, and matches well with what is used in STVG.

  However one parameter is still free. Is averaging playing an important role in galactic dynamics?