General Relativity (FYSS7320), 2022 **Excercise 9.**

Return by 12.00, Wednesday 16.3.2022 (electronically to pyry.m.rahkila@jyu.fi).

1. **Action for perturbations around Minkowski.** Expanding the GR action to quadratic order in perturbations around the Minkowski solution yields

$$S = \int d^4x \frac{1}{2}\left(\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\rho\sigma}\partial_\rho h^\mu{}_\sigma + \frac{1}{2}\eta^{\mu\nu}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h \partial_\nu h\right) .$$

Show that by varying this with respect to $h_{\mu\nu}$ you recover the linearised Einstein equations for the vacuum case $T_{\mu\nu} = 0$ given in the lectures notes.

2. **Plane wave with circular polarisation.** Consider a monochromatic plane wave travelling in the x^3 direction, using the TT gauge. We say that the wave is circularly polarised if $A_{\mu\nu} = \alpha(e^+_{\mu\nu} \pm i e^\times_{\mu\nu})$. Consider two test particles infinitesimally close to each other with the coordinate separation vector $s^i = (s^1, s^2, 0) =$ constant.

   a) Show that the particles move in a circle in relation to each other.

   b) The polarisation is right handed if the particles move counterclockwise as seen from the side towards which the wave is travelling. Does this correspond to + or − above?

3. **Is gravity always attractive:** The relative acceleration between adjacent geodesics (i.e. relative acceleration between neighbouring freely falling test bodies) is given by

$$\frac{Dv^\mu}{d\tau} \equiv u^\nu \nabla_\nu v^\mu = (\nabla_\nu u^\mu) v^\nu ,$$

where v^μ measures the deviation between two adjacent geodesics, $u^\mu = dx^\mu/d\tau$ is the geodesic tangent and $[v, u] = 0$. Using the projection operator $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ we can decompose

$$\nabla_\mu u_\nu = \frac{1}{3}P_{\mu\nu}\nabla^\lambda u_\lambda + \left(\nabla_{(\mu}u_{\nu)} - \frac{1}{3}P_{\mu\nu}\nabla^\lambda u_\lambda\right) + \nabla_{[\mu}u_{\nu]} \equiv \frac{1}{3}P_{\mu\nu}\theta + \sigma_{\mu\nu} + \omega_{\mu\nu} ,$$

where θ describes the volume expansion, $\sigma_{\mu\nu}$ the shear deformations and $\omega_{\mu\nu}$ the vorticity.

   a) Show that the volume expansion θ obeys the Raychaudhuri equation

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu .$$

   Hint: Recall that $[\nabla_\mu, \nabla_\nu]u^\lambda = R^\lambda{}_{\sigma\mu\nu}u^\sigma$.

   b) Use the Einstein equation to express $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ and T. Consider two test particles initially at rest at so that $\theta = \sigma_{\mu\nu}\sigma^{\mu\nu} = \omega_{\mu\nu}\omega^{\mu\nu} = 0$ at the initial time τ_0. Write the Raychaudhuri equation at τ_0 using the local Lorentz frame where $g_{\hat\mu\hat\nu} = \eta_{\hat\mu\hat\nu}$ and $u^{\hat\mu} = (1, 0, 0, 0)$. Gravity acts attractively if θ decreases and repulsively θ increases. For what kind of matter gravity is repulsive?