General Relativity (FYSS7320), 2022 **Excercise 8.**

Return by 12.00, Wednesday 9.3.2022 (electronically to pyry.m.rahkila@jyu.fi).

1. **Dropping beacons into a black hole.** Consider a comoving observer A sitting at constant spatial coordinates (r_A, θ_A, ϕ_A), around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole (straight down along a radial directory). The beacon emits radiation at a constant wavelength λ_{em} (in the beacon rest frame).

   a) Calculate the coordinate speed dr/dt of the beacon, as a function of r.

   b) Calculate the proper speed of the beacon. That is, imagine there is another comoving observer B at fixed r_B, with a locally inertial coordinate system set up as the beacon passes by, and calculate the speed measured by B. What is it as $r_B \to 2GM$?

   c) Calculate the wavelength λ_{obs}, measured by the observer A, as a function of the radius r_{em} at which the signal was emitted.

   d) Calculate the time t_{obs} at which a beam emitted by the beacon at radius r_{em} will be observed at r_A.

   e) Show that at late times, the redshift grows exponentially, $\lambda_{\mathrm{obs}}/\lambda_{\mathrm{em}} \propto e^{t_{\mathrm{obs}}/T}$. Give an expression for the time constant T in terms of the black hole mass M.

2. **How much time do you have left after crossing the event horizon?** Inside the Schwarzschild radius $r < 2GM$ the metric can be written as

$$ds^2 = -\left(\frac{2GM}{r} - 1\right)^{-1} dr^2 + \left(\frac{2GM}{r} - 1\right) dt^2 + r^2 d\Omega^2 \ ,$$

   where r is a temporal and t a spatial coordinate. Consider the motion (not necessarily geodesic) of a massive object in the regime $r < 2GM$ and show that r must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geqslant \sqrt{\frac{2GM}{r} - 1} \ .$$

   Calculate the maximum proper time for a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses $M_\odot$.

   a) What is the time for a typical stellar size black hole $M = 10M_\odot$?

   b) What is the time for the supermassive black hole at the center of the M87 galaxy $M \sim 2 \times 10^9 M_\odot$?

3. **Lifetime of black holes.** According to curved space quantum field theory, a black hole of mass M radiates (as observed by an asymptotic observer at $r \to \infty$) like a blackbody with an area A and temperature T given by

$$A = 4\pi(2GM)^2 \qquad T = \frac{\hbar}{8\pi k_B GM} \ ,$$

   where k_B is the Boltzmann constant. This radiation is called the Hawking radiation. Total energy is conserved in the process so that the mass decreases at the rate energy is radiated away.

a) Find the lifetime of a black hole with an initial mass M.

b) What is the temperature (initial) and lifetime of a stellar black hole $M = 10M_\odot$?

c) Small primordial black holes can have been produced in the very early universe. The age of the universe is $t_0 = 13.8 \times 10^9$ years. What is the minimum initial mass of primordial black holes that could have survived until today?