General Relativity (FYSS7320), 2022 **Excercise 6.**

Return by 12.00, Wednesday 23.2.2022 (electronically to pyry.m.rahkila@jyu.fi).

1. **Curvature:** Consider a homogeneous and isotropic universe with the Robertson-Walker metric
$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \ .$$
In the lectures we found that the non-zero connection coefficients are given by
$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} \ , \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j \ ,$$
where $\dot{a} \equiv da/dt$. Compute the Riemann tensor, Ricci tensor, Ricci scalar and Einstein tensor for this metric. Using Mathematica or its equivalent is warmly recommended. Suppose that $a(t) \propto (t/t_0)^q$ where t_0 and $q \in [0,1]$ are constants. Does the curvature vanish for some values of q?

2. **Motion of dust observers:** Consider ideal fluid, with
$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \ .$$
Show that if the matter is dust ($p = 0$), observers comoving with the fluid follow geodesics, i.e. the fluid four-velocity satisfies $u^\mu \nabla_\mu u^\nu = 0$. Are there any cases with non-zero pressure where they would follow geodesics? (Hint: start from the covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$.)

3. **Euler equation:** A projection tensor to surfaces orthogonal to the four-velocity u^μ is defined by
$$P_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \ , \qquad P_{\mu\nu}u^\nu = 0 \ .$$
Show that for ideal fluid in Minkowski space and in the non-relativistic limit $p \ll \rho$, $v^i \ll 1$, the spatial components of $P^{\sigma\nu}\nabla_\mu T^\mu{}_\nu = 0$ reduce to the Euler equation
$$\rho(\partial_t v^i + v^j \partial_j v^i) = -\partial^i p \ ,$$
which you may have encountered in fluid dynamics or statistical physics.

4. **Unimodular gravity:** Let us postulate that (instead of the Einstein-Hilbert action) the geometry and matter are described by the action
$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{1}{16\pi G_N} \left(R + \chi\Big(1 - \frac{\xi}{\sqrt{-g}}\Big) \right) + \mathcal{L}_m \right]$$
where χ is some new scalar field and ξ is a constant. Show that in this model the metric has one extra constraint and that instead of the usual Einstein equation we now get the following trace-free equation:
$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 8\pi G \left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T \right) .$$
On the way you probably observed that this theory is equivalent to Einstein equations with one additional term, what is it?