General Relativity (FYSS7320), 2022 **Excercise 5.**

Return by 12.00, Wednesday 16.2.2022 (electronically to pyry.m.rahkila@jyu.fi).

1. **Connection coefficients:** The spacetime outside a spherically symmetric object of mass M is described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ ,$$

where G is Newton's constant. Compute the components of Christoffel connection for this metric.

2. **Clocks and satellites:** The metric outside the surface of the Earth can be approximated by

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Phi(r))dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ ,$$

where

$$\Phi(r) = -\frac{GM}{r}$$

can be thought as the Newtonian gravitational potential and M is Earth's mass. Here $\Phi \ll 1$ and it suffices to work to first order in Φ. (This is just the Schwarzschild metric linearised in $2GM/r$.)

a) Consider two clocks, one on the surface of the Earth at distance R_1 from the Earth's center and another on a tall buliding at distance R_2. Calculate the (proper) time elapsed on each clock as a function of the coordinate time t. Which clock ticks faster?

b) How much proper time elapses in a clock located on a satellite while the satellite at radius R_1 (skimming along the surface of the Earth, neglect air resistance) completes one orbit? How does this number compare to the proper time elapsed on a clock stationary on the surface of the Earth? Put in the numbers and express the results in seconds.

3. **Observers in a homogenous and isotropic universe:** Consider a spatially flat, expanding universe for which the metric in comoving coordinates reads

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ , \qquad a(t) = \left(\frac{t}{t_0}\right)^{2/3} \ .$$

Consider two observers A and B, and an object C all located at $x^\mu = (t_0/(2\sqrt{2}), 0, 0, 0)$. Their four-velocities expressed in the comoving coordinate system are given by

$$\begin{aligned} u_A^\mu &= (1, 0, 0, 0) \\ u_B^\mu &= (u_B^0, 1/2, 0, 0) \\ u_C^\mu &= (u_C^0, 3/2, 0, 0) \ . \end{aligned}$$

What are the values of u_B^0 and u_C^0 (use that $u_\mu u^\mu = -1$)? What is the coordinate velocity dx/dt of C? What is the velocity of the object C as measured by A? What is the velocity of the object C as measured by B? (Hint: Recall that in the local Lorentz frame all physics at a point reduces to special relativity.)

4. **Riemann tensor:** Let A,B,C and D be four points on a manifold with coordinates $x^\mu, x^\mu + \xi^\mu, x^\mu + \xi^\mu + \eta^\mu$ and $x^\mu + \eta^\mu$ respectively. Here ξ^μ and η^μ are small so that ABCDA forms a small parallelogram. Parallel transport a vector v^μ from A to C along the sides of the parallelogram via the different routes, ABC and ADC. Show that to leading order in ξ^μ and η^μ the difference between the two resulting vectors is

$$\delta v^\mu = -R^\mu{}_{\nu\sigma\rho} v^\nu \xi^\sigma \eta^\rho \ ,$$

where $R^\mu{}_{\nu\sigma\rho}$ has the form in terms of $\Gamma^\mu_{\nu\sigma}$ given in the lectures.