General Relativity (FYSS7320), 2022 **Excercise 4.**

Return by 12.00, Wednesday 9.2.2022 (electronically to pyry.m.rahkila@jyu.fi).

1. **Metric compatibility:** Show that the metric compatibility $\nabla_\sigma g_{\mu\nu} = 0$ implies

$$\begin{aligned} \nabla_\sigma g^{\mu\nu} &= 0 \ , \\ g_{\mu\nu}\nabla_\sigma u^\nu &= \nabla_\sigma u_\mu \ . \end{aligned}$$

2. **Divergences:** Denote the determinant of the metric by $g \equiv \det(g_{\mu\nu})$. Show that the following identities, where $\Gamma^\alpha_{\beta\gamma}$ are Christoffel symbols, hold:

$$\begin{aligned} \partial_\mu g &= g^{\alpha\beta} g \partial_\mu g_{\alpha\beta} \ , \\ \Gamma^\alpha_{\alpha\beta} &= \partial_\beta(\ln\sqrt{-g}) \ , \\ \nabla_\mu u^\mu &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}u^\mu) \ . \end{aligned}$$

   Hint: One possibility is to use the relation $\ln(\det M) = \mathrm{Tr}(\ln M)$, that holds for any $n\times n$ matrix M, and differentiate this.

3. **Geodesics and connection:** Consider a static spherically symmetric spacetime for which the line element can be written in the form

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ ,$$

   where $\alpha(r)$ and $\beta(r)$ are some functions. Derive the geodesic equations by using the Euler-Lagrange equations for the Lagrangian $L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ (where $\dot{x}^\mu \equiv dx^\mu/d\tau$). Find the non-zero connection coefficients.

4. **Parallel transport:** Consider parallel transports on a two sphere of constant radius R

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

   a) Show that the longitudes (ϕ =const.) are geodesics but the only latitude (θ = const.) that is a geodesic is the equator $\theta = \pi/2$.

   b) Take a vector $v^\mu = (1, 0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector as a function of θ?