General Relativity (FYSS7320), 2022 **Excercise 3.**

Return by 12.00, Wednesday 2.2.2022 (electronically to pyry.m.rahkila@jyu.fi).

1. **Commutator of vector fields:** The composition of two vector fields $u \circ v$ and their commutator $[u, v]$ are defined by their action on an arbitrary function f as

$$(u \circ v)(f) \equiv u(v(f)), \qquad [u, v](f) = u(v(f)) - v(u(f)).$$

Show that the commutator $[u, v]$ transforms like a vector under a generic coordinate transformation whereas the composition $u \circ v$ does not.

2. **Horizon size:** A homogeneous and isotropic universe is described by the metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2).$$

For a radiation dominated universe $a(t) \propto t^{1/2}$. Compute the distance travelled by light rays from $t = 0$ to the formation of Cosmic Microwave Background (CMB) at $t_{\mathrm{CMB}} = 3 \times 10^5$ years. This determines the *particle horizon*, the size of causally connected patches.

3. **Locally inertial coordinates:** Geometry of a sphere is represented by the metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

where (θ, ϕ) are the usual spherical coordinates. Consider a coordinate transformation

$$x = \theta\cos\phi\ , \qquad y = \theta\sin\phi,$$

and show that (x, y) are locally inertial coordinates at the north pole $\theta = 0$. (In the neighbourhood of the north pole, x and y are small, so you can write the metric components as an expansion in powers of x and y and include only the lowest powers, if this helps.)

4. **Covariant derivatives:** In curved spacetime we can define the covariant derivative of a vector as

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^\nu_{\mu\sigma} v^\sigma\ .$$

None of the terms on the right hand side is a tensor separately but their sume is if we choose the *connection coefficients* $\Gamma^\nu_{\mu\sigma}$ appropriately. Requiring that $\nabla_\mu v^\nu$ transforms as a tensor derive the implied transformation rule for the connection coefficients

$$\Gamma'^\gamma_{\alpha\beta} = \Gamma^\mu_{\nu\rho} X^\gamma{}_\mu (X^{-1})^\nu{}_\alpha (X^{-1})^\rho{}_\beta - (X^{-1})^\mu{}_\alpha (X^{-1})^\nu{}_\beta \partial_\nu X^\gamma{}_\mu,$$

where

$$X^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}.$$