General Relativity (FYSS7320), 2022 **Excercise 2.**

Return by 12.00, Wednesday 26.1.2021 (electronically to pyry.m.rahkila@jyu.fi).

1. **Manipulating tensors and vectors:** Consider a tensor A and vector v with components

$$A^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \ , \qquad v^\mu = (-1, 2, 0, -2) \ .$$

Compute the components of $A^\mu{}_\nu$, $A_\mu{}^\nu$, $A^{(\mu\nu)}$, $A_{[\mu\nu]}$, $A^\mu{}_\mu$ and $v^\mu v_\mu$, $v_\mu A^{\mu\nu}$.

2. **Four-force:** The four-force acting on a particle of mass m is defined as the four-vector

$$f^\mu = \frac{dp^\mu}{d\tau} = ma^\mu \ .$$

We can now define components of the three-force (three-vector, not Lorentz invariant) as $F^i = f^i/\gamma$ where $\gamma = 1/\sqrt{1-v^2}$ and v^i is three-velocity of the particle. Show that

$$f^0 = \gamma \sum_{i=1}^{3} F^i v^i = \gamma \boldsymbol{F} \cdot \boldsymbol{v}.$$

Derive a connection between the three acceleration $\boldsymbol{a} \equiv d\boldsymbol{v}/dt$ and the three-force: $m\gamma\boldsymbol{a} = \boldsymbol{F} - (\boldsymbol{F}\cdot\boldsymbol{v})\boldsymbol{v}$. Study the behaviour of the acceleration as a function of (direction and magnitude) of $\boldsymbol{v}$ with respect to $\boldsymbol{F}$.

3. **Aberration:** Inertial frame K' is moving with velocity $v\hat{\mathbf{x}}$ with respect to frame K ($\hat{\mathbf{x}}$ is the unit vector along the x-axis). Show that the angles θ' and θ between the direction of light ray and x-axis in the two frames are related by

$$\tan\theta' = \frac{\tan\theta}{\gamma(1 - v/\cos\theta)} \ .$$

4. **Lorentz transformations of the electromagnetic field:** Components of the electromagnetic field strength tensor are given in terms of the electric E^i and magnetic B^i fields as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

   a) Using the tensor transformation law $F_{\mu'\nu'} = \Lambda^\rho{}_{\mu'}\Lambda^\sigma{}_{\nu'}F_{\rho\sigma}$ derive the transformation laws for E^i and B^i under a boost in the x-direction.

   b) Find the electric and magnetic feld of a charge moving at a constant velocity v in the x direction, by doing a Lorentz transformation on the feld of a nonmoving charge.