**General Relativity (FYSS7320), 2022** **Excercise 10.**

Return by 12.00, Wednesday 23.3.2022 (electronically to pyry.m.rahkila@jyu.fi).

1. **Gravitational wave detector.** Consider a space-based laser interferometer (such as the planned LISA experiment) consisting of three satellites, located at $x^i = (0,0,0)$, $x^i = (L,0,0)$ and $x^i = (0,L,0)$. From the first satellite, two laser beams with wavelength $\lambda$ and originally at the same phase, are sent along the $x$- and $y$-axes. They are reflected by mirrors at the two other satellites and return back to the first satellite, where the phase difference between the two light beams is measured by interferometry. Assume the background metric is Minkowski. Consider a gravitational wave with + polarisation, amplitude $\alpha$ and frequency $f = \omega/(2\pi)$, travelling in the $x^3$ direction and passing through the interferometer. All quantities are given in the TT gauge and the three satellites stay at constant spatial co-moving coordinates.

   a) What is the observed phase shift $\Delta\phi$? Note that the answer depends on the phase of the gravitational wave at the time $t_e$ when laser beams where sent out.

   b) Take $L = 5 \times 10^6$km, $\lambda = 500 \times 10^{-9}$ m and assume that $\Delta\phi = 10^{-9}$ is the smallest phase shift you can observe. What is the weakest amplitude $\alpha$ that can be detected? For what frequency range do you achieve this sensitivity?

2. **Projection to TT gauge.** Far away from the gravitational wave source, the trace reversed perturbation $\bar{h}_{\mu\nu}$ in a generic Lorenz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ takes the plane wave form $\bar{h}_{\mu\nu} = \mathrm{Re}(A_{\mu\nu} e^{-i n_\sigma x^\sigma})$. Define $h_{ij}^{\mathrm{TT}}$ as

$$h_{ij}^{\mathrm{TT}} \equiv (P_i{}^k P_j{}^\ell - \frac{1}{2} P_{ij} P^{k\ell}) \bar{h}_{k\ell} \ ,$$

   where the projection operator $P_{ij} = \delta_{ij} - n^i n^j$ and $n^i$ is the spatial part of the wave vector $n^\sigma$, and $n^i n_i = 1$. Show that $h_{ij}^{\mathrm{TT}}$ satisfies the TT gauge conditions

$$\partial^i h_{ij}^{\mathrm{TT}} = 0 \ , \qquad (h^{\mathrm{TT}})^i{}_i = 0 \ .$$

3. **Perturbed Ricci tensor.** Expand the metric to second order around flat spacetime as $g_{\mu\nu} = \eta_{\mu\nu} + \delta h_{\mu\nu} + \delta^2 h_{\mu\nu}$ and fix the first order part of the gauge by imposing the TT conditions

$$\delta h^\mu{}_\mu = 0 \ , \qquad \delta h_{0\mu} = 0 \ , \qquad \partial^\mu \delta h_{\mu\nu} = 0 \ .$$

   Show that with this gauge choice the second order perturbation of the Ricci tensor is given by eq. (7.44) in the lecture notes.