General Relativity (FYSS7320), 2022 **Excercise 1.**

Return by 12.00, Wednesday 19.1.2022 (electronically to pyry.m.rahkila@jyu.fi).

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1. **Boosts:** Show that the transformation

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\psi & -\sinh\psi & & \\ -\sinh\psi & \cosh\psi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

satisfies the condition $\eta = \Lambda^T \eta \Lambda$.

2. **Twin "paradox":** Alice stays on Earth. Betty leaves Alice, travels to Alpha Centaury (distance 4 lightyears) at the speed $v = 0.8c$, turns around, and returns at the same speed. How much have Alice and Betty aged when they meet again?

3. **General boosts:** A general Lorentz boost ($K \to K'$), .i.e. a Lorentz transformation without spatial rotations, is represented by (here $c = 1$ and latin indices run over $i = 1, 2, 3$)

$$\begin{aligned} \Lambda^{0'}{}_{0} &= \frac{1}{\sqrt{1-v^2}} \equiv \gamma \\ \Lambda^{j'}{}_{0} = \Lambda^{0'}{}_{j} &= -v\gamma n^j \\ \Lambda^{j'}{}_{k} = \Lambda^{k'}{}_{j} &= (\gamma - 1) n^j n^k + \delta^j_k \ . \end{aligned}$$

Components of the inverse transformations are given by $\Lambda^{\mu}{}_{\nu'} = \Lambda^{\nu'}{}_{\mu}$ with the replacement $(v \to -v)$. Here v, n^i are parameters satisfying $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$.

   a) Show that $\Lambda^T \eta \Lambda = \eta$.

   b) Show that K' moves at velocity $v\mathbf{n}$ (here $\mathbf{n}$ is just a usual Euclidean 3-vector) with respect to K, and that K moves at velocity $-v\mathbf{n}$ with respect to K'.

   c) Show that if $\mathbf{n}$ is parallel to a coordinate axis, we get the "ordinary" Lorentz boost.

4. **Tangent vectors and gradients:** In Euclidean 3-space, let p be the point with coordinates (x,y,z)=(1,0,-1). Consider the following curves that pass through p:

$$\begin{aligned} x^i(\lambda) &= (\lambda, (\lambda-1)^2, -\lambda) \\ x^i(\mu) &= (\cos\mu, \sin\mu, \mu - 1) \\ x^i(\sigma) &= (\sigma^2, \sigma^3 + \sigma^2, \sigma) \ . \end{aligned}$$

   a) Calculate the components of the tangent vectors to these curves at p in the coordinate basis $\{e_x, e_y, e_z\}$.

   b) Let $f = x^2 + y^2 - yz$. Calculate $df/d\lambda, df/d\mu$ and $df/d\sigma$.