8. Maximally symmetric solutions and RW metric

It can be shown that for an n-dim spacetime M the maximal possible number of symmetries (mathematically isometries $M \to M$) is given by

$$(8.1) \qquad \frac{n(n+1)}{2}$$

(See Carroll pages 134–144 for details). A maximally symmetric spacetime is one for which the number of symmetries is given by (8.1).

For $n=4$, there are three maximally symmetric spacetimes: Minkowski; de Sitter (dS) and anti-de Sitter (AdS)

The Minkowski is easy to check: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ is invariant under:

4 translations $\quad x^\mu \to x^\mu + a^\mu$

6 Lorentz-transf. $\quad x^\mu \to \Lambda^\mu{}_\nu x^\nu$

$\Rightarrow$ $4+6=10$ symmetries, i.e. 10 generators of the symmetry transformations

The two other maximally symmetric spacetimes, dS and AdS, have the same symmetries but they are not flat.

The Riemann tensor for a maximally symmetric spacetime must be of the form (see Carroll for the derivation)

$$(8.2) \qquad R_{\rho\sigma\mu\nu} = \underset{\uparrow \atop \text{constant}}{\kappa}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)$$

This yields $R_{\sigma\nu} = R^{\mu}{}_{\sigma\mu\nu} = \mathcal{K}\left(g^{\mu}{}_{\mu} g_{\sigma\nu} - g^{\mu}{}_{\nu} g_{\sigma\mu}\right)$

$$= \mathcal{K}(n g_{\sigma\nu} - g_{\sigma\nu}) \qquad , \; n = \dim M = g^{\mu}{}_{\mu}$$

$$= \mathcal{K}(n-1)\, g_{\sigma\nu}$$

(8.3) $$R = R^{\nu}{}_{\nu} = \mathcal{K}(n-1)n \quad \Rightarrow \quad \mathcal{K} = \frac{R}{(n-1)n}$$

Hence $R = \text{const.}$ for the max. symm. sol's. ∇

The 4-d cases are:

$\mathcal{K} = 0 \iff$ Minkowski

$\mathcal{K} > 0 \iff$ dS

$\mathcal{K} < 0 \iff$ AdS

8.1 de Sitter space $\quad (R = \text{const} > 0)$

The de Sitter spacetime can be embedded in a 5-d Minkowski spacetime

$$ds^2 = -du^2 + dx^2 + dy^2 + dz^2 + dw^2$$

as the hyperboloid

(8.4) $$-u^2 + x^2 + y^2 + z^2 + w^2 = \alpha^2 = \text{const.}$$

The metric evaluated on the 4-d surface of the hyperboloid (= embedded metric) gives the dS metric. To find its expression, introduce crd's (t, χ, θ, ϕ) on the hyperboloid (8.4) as:

(8.5)
$$u = \alpha \sinh(t/\alpha) \qquad x = \alpha \cosh(t/\alpha) \sin\chi \cos\theta$$
$$w = \alpha \cosh(t/\alpha) \cos\chi \qquad y = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi$$
$$z = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi$$

check:

$$-u^2+x^2+y^2+z^2+w^2 = \alpha^2\big(-\sinh^2(t/\alpha)+\cosh^2(t/\alpha)\sin^2\chi\cos^2\theta +$$
$$\cosh^2(t/\alpha)\sin^2\chi\sin^2\theta\cos^2\phi + \cosh^2(t/\alpha)\sin^2\chi\sin^2\theta\sin^2\phi$$
$$+\cosh^2(t/\alpha)\cos^2\chi\big)$$
$$= \alpha^2\big(-\sinh^2(t/\alpha)+\cosh^2(t/\alpha)\sin^2\chi\cos^2\theta +$$
$$\cosh^2(t/\alpha)\sin^2\chi\sin^2\theta + \cosh^2(t/\alpha)\cos^2\chi\big)$$

$$= \alpha^2\big(-\sinh^2(t/\alpha)+\cosh^2(t/\alpha)\sin^2\chi + \cosh^2(t/\alpha)\cos^2\chi\big)$$
$$= \alpha^2 \quad \text{ok, (8.5) satisfies (8.4)}$$

The induced metric on (8.4) is obtained by substitution (8.5) into $ds^2 = -du^2+dx^2+dy^2+dz^2+dw^2$ which yields (exercise)

$$(8.6) \qquad ds^2 = -dt^2 + \underbrace{\alpha^2\cosh^2(t/\alpha)}_{\text{scale factor}}\Big(\underbrace{d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\, d\phi^2)}_{=d\Omega_3^2}\Big)$$

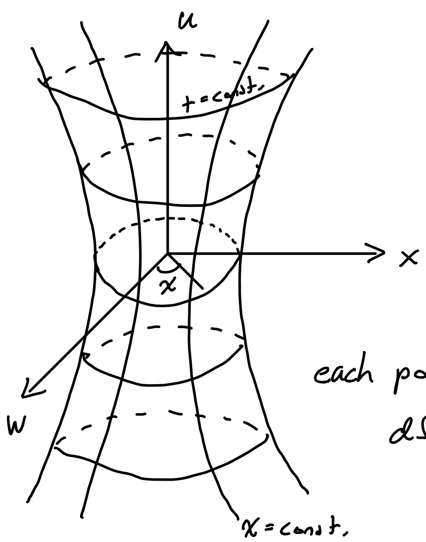
$t \in]-\infty,\infty[$
$\chi \in [0,\pi]$
$\theta \in [0,\pi]$
$\phi \in [0,2\pi]$

scale factor
$\to \infty$ as $t \to \pm\infty$
$= 1$ a $t=0$

$= d\Omega_3^2$ metric of a 3-sphere (i.e. 3-d surface)

The spatial part of (8.6) describes a 3-sphere that shrinks until $t=0$ and then grows.

The crd's χ, θ must both be constrained to $[0,\pi]$ since $g_{\theta\theta}, g_{\phi\phi} \to 0$ as $\chi \to \pi$ and $g_{\phi\phi} \to 0$ as $\theta \to \pi$. The metric becomes non-invertible ($\det g = 0$) and therefore we cannot extend the crd range beyond $[0,\pi]$.



each point reprents a 2-sphere

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

As always, we can use different crd's instead of (t, χ, θ, ϕ). Start from

$$ds^2 = -du^2 + dx^2 + dy^2 + dz^2 + dw^2$$

$$-u^2 + x^2 + y^2 + z^2 + w^2 = (w-u)(w+u) + x^2 + y^2 + z^2 = \alpha^2$$

and define

$$\tilde{t} \equiv w + u$$

Then $\quad w - u = \dfrac{\alpha^2 - r^2}{\tilde{t}} \quad , \; r^2 \equiv x^2 + y^2 + z^2$

and we can use crd's $(\tilde{t}, x, y, z)$ for the dS. However, these cover only half of the manifold as

$$\tilde{t} = 0 \iff w = -u \Rightarrow \underbrace{(w-u)}_{=0}(w+u) + x^2 + y^2 + z^2 = \alpha^2$$
$$x^2 + y^2 + z^2 = \alpha^2$$
$$r = \pm\alpha$$

So $\tilde{t} = 0 \iff w = -u, \; r = \alpha$ or $w = -u, \; r = -\alpha$ i.e. the mapping is not single valued $\Rightarrow (\tilde{t}, x, y, z)$ charts cannot be extended beyond $w = -u$ lines. We need two separate patches to cover the manifold: $0 < \tilde{t} < \infty$

and $-\infty < \tilde{f} < 0$ and neither of these includes the boundary. (175)

Let us concentrate on the patch:

$$0 < \tilde{f} < \infty$$
$$-\infty < x, y, z < \infty$$

From
$$w + u = \tilde{f}, \quad w - u = \frac{\alpha^2 - r^2}{\tilde{f}} \quad \Rightarrow \quad w = \frac{1}{2}\left(\tilde{f} + \frac{\alpha^2 - r^2}{\tilde{f}}\right), \quad u = \frac{1}{2}\left(\tilde{f} - \frac{\alpha^2 - r^2}{\tilde{f}}\right)$$

$$dw = \frac{1}{2}\left(1 - \frac{\alpha^2 - r^2}{\tilde{f}^2}\right)d\tilde{f} - \frac{1}{\tilde{f}}(x\,dx + y\,dy + z\,dz)$$
$$du = \frac{1}{2}\left(1 + \frac{\alpha^2 - r^2}{\tilde{f}^2}\right)d\tilde{f} + \frac{1}{\tilde{f}}(x\,dx + y\,dy + z\,dz)$$
$$dw + du = d\tilde{f}$$
$$dw - du = -\frac{\alpha^2 - r^2}{\tilde{f}^2}d\tilde{f} - \frac{2}{\tilde{f}}(x\,dx + y\,dy + z\,dz)$$

Substituting these into the line element we get:

$$
\begin{aligned}
ds^2 &= -du^2 + dx^2 + dy^2 + dz^2 + dw^2 \\
&= (dw - du)(dw + du) + dx^2 + dy^2 + dz^2 \\
&= -\frac{\alpha^2 - r^2}{\tilde{f}^2}d\tilde{f}^2 - \frac{2}{\tilde{f}}(x\,dx + y\,dy + z\,dz)\,d\tilde{f} + dx^2 + dy^2 + dz^2 \\
&= \underbrace{-\frac{\alpha^2}{\tilde{f}^2}d\tilde{f}^2}_{\equiv d\hat{f}^2} + \underbrace{\left(-\frac{x}{\tilde{f}}d\tilde{f} + dx\right)^2}_{\equiv \frac{\tilde{f}}{\alpha}d\hat{x}} + \underbrace{\left(-\frac{y}{\tilde{f}}d\tilde{f} + dy\right)^2}_{\equiv \frac{\tilde{f}}{\alpha}d\hat{y}} + \underbrace{\left(-\frac{z}{\tilde{f}}d\tilde{f} + dz\right)^2}_{\equiv \frac{\tilde{f}}{\alpha}d\hat{z}}
\end{aligned}
$$

$$\Rightarrow \quad \hat{f} = \alpha \ln\frac{\tilde{f}}{\alpha}, \quad \hat{x} = \frac{\alpha x}{\tilde{f}}, \quad \hat{y} = \frac{\alpha y}{\tilde{f}}, \quad \hat{z} = \frac{\alpha z}{\tilde{f}}$$

In the new variables $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ the metric then reads:

(8.7) $$ds^2 = -d\hat{t}^2 + e^{2\hat{t}/\alpha}(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)$$

This describes an exponentially expanding spatially flat (= t=const. surfaces have Euclidean geometry) spacetime which corresponds to $t>0$ half of the full dS manifold. In cosmology inflation in the very early universe and dark energy domination in the late universe are approximatively described by dS metric of the form (8.7).

The causal structure of a spacetime is often described in terms of conformal diagrams (Penrose diagrams) where crd's are chosen s.t. an infinite spacetime is mapped into a finite crd patch. To construct the conformal diagram of the dS space, start from (8.6)

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha)\left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

$t \in]-\infty, \infty[$

$\chi \in [0, \pi]$

$\theta \in [0, \pi]$

$\phi \in [0, 2\pi]$

Define a new time crd t' as:

(8.8) $$\cosh \frac{t}{\alpha} = \frac{1}{\cos t'}$$

The full interval $-\infty < t < \infty$ can now be mapped to

$$t' \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad t' = \frac{\pi}{2} \Rightarrow t \to \infty$$
$$t' = -\frac{\pi}{2} \Rightarrow t \to -\infty$$

The metric in (t', χ, θ, ϕ) becomes

$$ds^2 = -\underbrace{\left(\frac{\alpha}{\sinh\frac{t}{\alpha}} \frac{\sin t'}{\cos^2 t'}\right)^2}dt'^2 + \frac{\alpha^2}{\cos^2 t'}\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

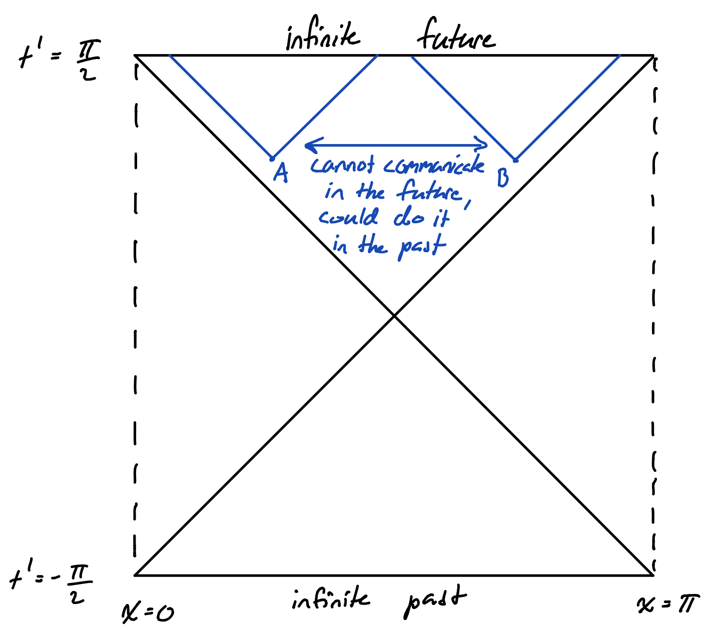
$$= \frac{\alpha^2}{\cos^4 t'} \frac{\sin^2 t'}{\cosh^2\frac{t}{\alpha} - 1} = \frac{\alpha^2}{\cos^4 t'} \frac{1 - \cos^2 t'}{\frac{1}{\cos^2 t'} - 1} = \frac{\alpha^2}{\cos^2 t'}$$

(8.9) $$ds^2 = \frac{\alpha^2}{\cos^2 t'}\left(-dt'^2 + d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

$t' \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
$\chi \in [0, \pi]$
$\theta \in [0, \pi]$
$\phi \in [0, 2\pi]$

Light cones correspond to $t' = \pm\chi$ (can rotate θ, ϕ s.t. motion along χ)

and we can plot the entire spacetime as:



Each point corresponds to a two sphere (θ, ϕ) except the edges $\chi = 0$, $\chi = \pi$ which are points $(\sin^2\chi = 0)$

What is the $T_{\mu\nu}$ that gives rise to dS solution?

$$R_{\mu\nu} = \mathcal{H}(n-1) g_{\mu\nu}$$
$$R = \mathcal{H}(n-1)n > 0 \qquad , n = 4$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = (3\mathcal{H} - \frac{1}{2} 4 \cdot 3\mathcal{H}) g_{\mu\nu} = -3\mathcal{H} g_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow T_{\mu\nu} = -\frac{3\mathcal{H}}{8\pi G} g_{\mu\nu}$$

Assuming the ideal fluid form $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$ we see that

$$\rho + p = 0 \quad \text{and} \quad p = -\frac{3\mathcal{H}}{8\pi G}$$

$$\Rightarrow \quad p = -\rho = -\frac{3\mathcal{H}}{8\pi G} \qquad \text{and} \quad T_{\mu\nu} = -\frac{3\mathcal{H}}{8\pi G} g_{\mu\nu}$$

This kind of matter corresponds to vacuum energy with a positive energy density.

8.2 Anti-de Sitter spacetime (AdS)

The AdS space corresponds to $\mathcal{k} < 0$ in (8.2). Embedded into a 5-d space with the metric

$$(8.10) \qquad ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$$

note !

this corresponds to the hyperboloid:

$$(8.11) \qquad -u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2$$

Then define crd's (t', ρ, θ, ϕ) on the hyperboloid as:

$$(8.12) \qquad \begin{aligned} u &= \alpha \sin t' \cosh \rho \\ v &= \alpha \cos t' \cosh \rho \\ x &= \alpha \sinh \rho \cos \theta \\ y &= \alpha \sinh \rho \sin \theta \cos \phi \\ z &= \alpha \sinh \rho \sin \theta \sin \phi \end{aligned}$$

which satisfy (8.11) (check). Substituting (8.12) into (8.10) one obtains (exercise) the AdS metric:

$$(8.13) \qquad ds^2 = \alpha^2 \left(-\cosh^2 \rho \, dt'^2 + d\rho^2 + \sinh^2 \rho \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right)$$

$$-\infty < t' < \infty$$
$$0 < \rho < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$

(t' and $t' + 2\pi$ are NOT the same points although (8.12) appears periodic, periodicity in the embedding space is not necessarily a real property of the embedded surface)

see the comments in dS case below eq. (8.6)

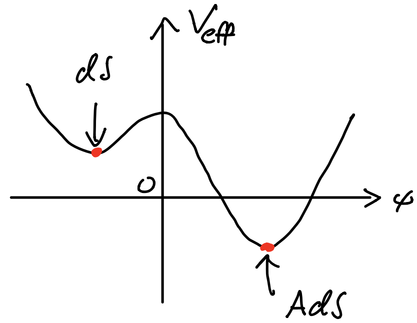
$\sinh \rho \big|_{\rho = 0} = 0$
$\Rightarrow$ cannot extend to $\rho < 0$ in these crd's

Using (8.2) we again get

$$G_{\mu\nu} = -3\mathcal{H} g_{\mu\nu} \quad \Rightarrow \quad T_{\mu\nu} = -\frac{3\mathcal{H}}{8\pi G} g_{\mu\nu}$$

$$p = -\rho = -\frac{3\mathcal{H}}{8\pi G}$$

But now $\mathcal{H} < 0$ implying that $\rho < 0$. The AdS space therefore corresponds to vacuum energy with negative energy density.



Let us then work out the conformal digram of the AdS space. We would like to map $-\infty < t' < \infty$ and $-\infty < \rho < \infty$, and have light cones in $45°$ angles. It turns out that we cannot achive all three simultaneously, thus we will leave t' uncompactified. To compactify ρ, we define a new crd ρ' as:

$$\cosh \rho = \frac{1}{\cos \chi} \qquad \text{s.t.} \qquad 0 < \rho < \infty \;\rightarrow\; \chi \in [0, \tfrac{\pi}{2}]$$

The metric becomes

$$ds^2 = \alpha^2 \left(-\frac{1}{\cos^2\chi} dt'^2 + \underbrace{\frac{1}{\sinh^2\rho} \frac{\sin^2\chi}{\cos^4\chi}}_{= \frac{1}{\cos^2\chi}} d\chi^2 + \underbrace{\left(\frac{1}{\cos^2\chi} - 1\right)}_{= \frac{\sin^2\chi}{\cos^2\chi}} \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right)$$

$$(8.14) \qquad ds^2 = \frac{\alpha^2}{\cos^2\chi}\left(-dt'^2 + d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

$t' \in\ ]-\infty, \infty[$

$\chi \in [0, \frac{\pi}{2}]$ lightcones $t' = \pm\chi$

$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi]$

Compare this to the dS metric (8.9). The differences are:

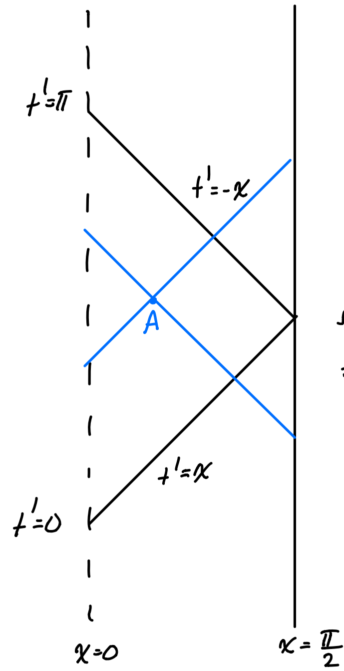
$$ds^2 = \frac{\alpha^2}{\cos^2\chi}\left(-dt'^2 + d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

$\chi \in [\ , \frac{\pi}{2}]$ in AdS ($[0, \pi]$ in dS)

The conformal factor depends on χ in AdS (on t' in dS)

In these crd's the AdS space is static but shrinks as function of the radial crd χ. Constant time slices $t' = \text{const.}$ are not Euclidean and not even 3-spheres as χ does not extend up to π but $\chi \in [0, \frac{\pi}{2}]$

The conformal diagram for AdS is given by



Each point represent a two-sphere (θ,ϕ) except for the dashed line $\chi=0$ which is a point ($\sin^2\chi=0$).

spatial infinity $\rho\to\infty \Leftrightarrow \chi=\frac{\pi}{2}$
timelike surface ($ds^2=-\alpha^2 dt'^2$)

In AdS any observer sees the spatial infinity (reache in finite time t') and it is possible for any two observers to communicate in the future and have communicated in the past. The fact that the spatial infinity is timelike also means that initial value problem are not well posed with initial data given on spacelike surfaces, information can leak in from the spatial infinity to any process.

8.3 Robertson-Walker metric

The observed universe appears to be homogeneous and isotropic on distance scales $d \gtrsim 100$ Mpc (galaxies $d \sim 0.1$ Mpc) but it is not stationary. From observations we know that the universe grows in time and the matter properties have also evolved in time implying that $R \neq$ const. The universe is therefore not described by a maximally symmetric spacetime, there is no timelike symmetry.

What is the metric describing a homogeneous & isotropic (in spatial crd's x^i) but time evolving spacetime? In the comoving coordinates (t, x^i) where homogeneity & isotropy are manifest the metric must take the form:

$$(8.16) \quad ds^2 = -dt^2 + \underset{\substack{\uparrow \\ \text{time-dependent} \\ \text{scale factor}}}{a^2(t)} \underbrace{\gamma_{ij}\, dx^i dx^j}_{\substack{\text{maximally symmetric (Euclidean)} \\ \text{3-space, invariant under spatial} \\ \text{translations and rotations}}}$$

and since homogeneity & isotropy are real symmetries of the spacetime this crd system should cover the entire manifold.
The metric on the $t =$ const. surfaces

$$(8.17) \quad ds^2\Big|_{dt=0} = a^2(t)\gamma_{ij}dx^i dx^j \equiv g_{ij}(t)\, dx^i dx^j$$

is maximally symmetric so that the Riemann tensor of these 3-d surfaces must take the form:

$$(8.18) \quad {}^{(3)}R_{ijkl} = \underset{\substack{\uparrow \\ \text{const.}}}{k}\,(g_{ik}g_{jl} - g_{il}g_{jk}) = k\,a^4(t)(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$$

From this we get:

$$(8.19) \qquad {}^{(3)}R_{jl} = k\,(g^{i}{}_{i}\,g_{jl} - g_{il}\,g_{j}{}^{i})$$
$$= k\,(3\,g_{jl} - g_{jl})$$
$$= 2k\,a^2(t)\,\gamma_{jl}$$

$$(8.20) \qquad {}^{(3)}R = {}^{(3)}R^{i}{}_{i} = 6k$$

The maximally symmetric 3-space is invariant under rotations (= spherically symmetric) so that in spherical crd's: (see Chapter 5)

$$(8.21) \qquad ds^2\Big|_{t=\text{const.}} = a^2(t)\,(e^{2\beta(r)}dr^2 + r^2 d\Omega^2) \;,\quad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$\Rightarrow \quad {}^{(3)}R_{11} = a^2(t)\,\frac{2}{r}\,\partial_r\beta$$
$${}^{(3)}R_{22} = a^2(t)\,e^{-2\beta}(r\partial_r\beta - 1) + 1$$
$${}^{(3)}R_{33} = a^2(t)\,(e^{-2\beta}(r\partial_r\beta - 1) + 1)\sin^2\theta$$
$${}^{(3)}R = e^{-2\beta}\,\frac{2}{r}\,\partial_r\beta +$$

Comparing these to (8.19) we get:

$$\frac{2}{r}\partial_r\beta = 2k\,\gamma_{11} = 2k\,e^{2\beta} \qquad ,\ \gamma_{11} = e^{2\beta}$$
$$e^{-2\beta}(r\partial_r\beta - 1) + 1 = 2k\,\gamma_{22} = 2k\,r^2 \qquad ,\ \gamma_{22} = r^2$$
$$\gamma_{22}\sin^2\theta = \gamma_{33}$$

$$\Rightarrow \begin{cases} e^{2\beta} = \dfrac{1}{kr}\dfrac{d\beta}{dr} \\[2ex] e^{-2\beta}\left(1 - r\dfrac{d\beta}{dr}\right) = 1 - 2kr^2 \end{cases} \quad \Rightarrow \quad e^{-2\beta}\left(1 - r\,e^{2\beta}kr\right) = 1 - 2kr^2$$

$$e^{-2\beta} = 1 - 2kr^2 + kr^2$$
$$= 1 - kr^2$$

$$\beta = -\frac{1}{2}\ln(1-kr^2)$$

Substituting this into (8.21) yields:

$$(8.22) \qquad ds^2\Big|_{t=\text{const.}} = a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

and the full metric (8.16) becomes:

$$(8.23) \qquad ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \qquad k = \text{const.}$$

This is the Robertson-Walker metric for a homogeneous & isotropic spacetime written in comoving coordinates.

Note that the metric is invariant under scalings:

$$\begin{aligned} r &\to \lambda r \\ a &\to a\lambda^{-1} \\ k &\to k\lambda^{-2} \end{aligned} \qquad \text{where } \lambda = \text{const.}$$

There are two common ways to rescale:

1) $k=0,\ k>0,\ k<0 \quad \longrightarrow \quad k = 0, \pm 1$

2) Set $a(t_0) = 1$ at some reference time t_0 (often chosen to be today)

The three different values of k correspond to different geometries of the $t=\text{const}$ surface:

$$k=1 \iff {}^{(3)}R>0 \quad \text{"open"}$$
$$k=0 \iff {}^{(3)}R=0 \quad \text{"flat"}$$
$$k=-1 \iff {}^{(3)}R<0 \quad \text{"closed"}$$

The physical meaning of these can be illuminated by switching to a new radial coordinate χ defined by:

$$(8.24) \quad d\chi = \frac{dr}{(1-kr^2)^{1/2}}$$

so that (8.23) becomes:

$$(8.25) \quad ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + S_k(\chi)^2 d\Omega^2\right)$$

where $S_k(\chi) \equiv r(\chi)$ determined by (8.24):

$$\chi = \int^r \frac{dr}{\sqrt{1-kr^2}} \equiv S_k^{-1}(r) \quad \Rightarrow \quad r = S_k(\chi)$$

$$(8.26) \quad S_k(\chi) = \begin{cases} \sin\chi & , k=1 \\ \chi & , k=0 \\ \sinh\chi & , k=-1 \end{cases}$$

For $k=0$:

$$ds^2 = -dt^2 + a^2(t)\underbrace{\left(d\chi^2 + \chi^2 d\Omega^2\right)}_{\text{Euclidean 3-space}}$$

$t=\text{const}$ slices have flat Euclidean geometry, topology can be $\mathbb{R}^3$ (infinite) or e.g. 3-torus (finite)

For $k>0$:

$$ds^2 = -dt^2 + a^2(t)\underbrace{\left(d\chi^2 + \sin^2\chi\, d\Omega^2\right)}_{\text{3-sphere } S^3}$$

$t=$ const. slices have the geometry and topology of 3-sphere, the manifold is finite ($0 < r < 1/k$).

For $k<0$:

$$ds^2 = -dt^2 + a^2(t)\underbrace{\left(d\chi^2 + \sinh^2\chi\, d\Omega^2\right)}$$

3-d space w. const. negative curvature H^3 (3-hyperboloid)
the manifold is infinite (assuming the simplest topology)

8.31 Friedmann equations

The non-zero components of the Ricci tensor and the Ricci scalar computed for the RW metric (8.23) are given by (exercise)

(8.27)
$$R_{00} = -\frac{3\ddot{a}}{a} \qquad R_{22} = r^2(1-kr^2)R_{11}$$
$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \qquad R_{33} = R_{22}\sin^2\theta$$
$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \qquad \dot{} \equiv \frac{d}{dt}$$

Assume the matter consists of ideal fluid(s) with:

(8.28)
$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\, g_{\mu\nu} \quad , \quad u^\mu = (1, \bar{0}) \text{ in comoving crd's}$$

all fluids must have same u^μ due to homogeneity + isotropy

(if the fluid moves wrt comoving crd's the universe will not be homogeneous + isotropic)

Substituting (8.27) and (8.28) into the Einstein eqs.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

results two independent eqs. known as the Friedmann eqs.

(8.29) $$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

(8.30) $$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

These govern the evolution of the scale factor $a(t)$. The time-dep. quantity $\frac{\dot a}{a}$ is called the Hubble rate:

(8.31) $$H(t) \equiv \frac{\dot a}{a}$$

Defining the critical density as:

(8.32) $$\rho_c \equiv \frac{3H^2}{8\pi G}$$

$$1 = \frac{\rho}{\rho_c} - \frac{k}{a^2 H^2} \qquad k = a^2 H^2 \left(\frac{\rho}{\rho_c} - 1\right)$$

We can recast (8.29) into the form:

(8.33) $$k = a^2 H^2 \left(\frac{\rho}{\rho_c} - 1\right) \quad \Rightarrow \quad \begin{aligned} \rho = \rho_c &\Leftrightarrow k = 0 \\ \rho > \rho_c &\Leftrightarrow k > 0 \\ \rho < \rho_c &\Leftrightarrow k < 0 \end{aligned}$$

Cosmological observations tell that $\rho = \rho_c$, up to observational errors, and hence our universe is described by the case $k = 0$.

The energy momentum tensor (8.28) satisfies the continuity equation

$$\nabla^\mu T_{\mu\nu} = 0$$

Contracting this with u^ν yields:

$$u^\nu \left(\partial^\mu(\rho+p) u_\mu u_\nu + (\rho+p)\left((\nabla^\mu u_\mu) u_\nu + u_\mu \nabla^\mu u_\nu + g_{\mu\nu}\partial^\mu p\right) \right) = 0$$

$$-(\dot\rho + \dot p) + (\rho+p)(-\underbrace{\nabla^\mu u_\mu}_{=3H} + u_\mu \underbrace{(u^\nu \nabla^\mu u_\nu)}_{=\frac{1}{2}\nabla^\mu(\underbrace{u^\nu u_\nu}_{=-1}) = 0}) + \dot p = 0$$

$$\Rightarrow \quad \dot\rho + 3H(\rho + p) = 0 \qquad (8.34)$$

This follows also from (8.31) and (8.32) but the form (8.34) is often useful. Assuming the equation of state is constant

$$p = w\rho \quad , \quad w = \text{const.}$$

we can integrate (8.34) to get

$$(8.35) \qquad \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \quad ; \ \rho_0, a_0 = \text{constants}$$

Substituting this into the Friedmann (8.31) one can then solve for $a(t)$. The common cases in cosmology are:

radiation $\quad p = \frac{1}{3}\rho \quad , \quad \rho \propto a^{-4} \quad , \quad a \propto t^{1/2}$

matter $\quad p = 0 \quad , \quad \rho \propto a^{-3} \quad , \quad a \propto t^{2/3}$

vacuum energy $\quad p = -\rho = \text{const.} \quad , \quad \rho = \text{const,} \quad a \propto e^{Ht} \quad , \quad H = \text{const.}$

A more detailed discussion of the Friedmann cosmology and RW geometry is postponed to the Cosmology course.