# 7. Gravitational waves (GW)

Gravitational waves are wavelike perturbations of the metric, ripples in the spacetime. The first direct detection of gravitational waves was announced on Feb 11, 2016 by the LIGO interferometer which measured the gravitational wave signal produced by coalescence of $M \sim 30 M_\odot$ black holes. The GW produced by this violent process modify distance scales by $\delta l / l \sim 10^{-21}$ as they propagate through the earth $\Rightarrow$ GW are very weak! Yet the effect is measurable by the carefully constructed interferometer apparatus of LIGO.

Before LIGO the GW were indirectly detected already in the 1970's by radio observations of binary pulsars. The binary system emit GW which reduce its energy causing the orbit time to decline. This was detected by Hulse & Taylor in 1974 and they were awarded the Nobe Prize in 1993.

Recall that 10 of the 20 dofs of the Riemann tensor are encoded in the Ricci tensor $R_{\mu\nu}$ and the other 10 in the Weyl tensor. The gravitational waves are included in the Weyl part. The Ricci is directly determined by the local matter distribution through the Einstein eq. $R_{\mu\nu} = 8\pi G (T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} T)$. The Weyl part i.e. gravitational waves carry information about non-local properties. The GW propagate with the speed of light: if you change the matter distribution the spacetime does not immediately change everywhere but the information is carried by GW's. The GW propagate even in the empty space $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$. We shall concentrate on this case first, i.e. consider small perturbations around the Minkowski space.

## 7.1 Linear perturbations around the Minkowski space

Consider small perturbations around the Minkowski space

(7.1) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad |h_{\mu\nu}| \ll 1$ but unlike in Chapter 4, we do not require static spacetime here: $\partial_0 h_{\mu\nu} \neq 0$.

$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$

Work to linear order in perturbations, i.e. drop all $\mathcal{O}(\delta^2)$ terms. All equalities in the following hold to linear precision.

The inverse metric is:

(7.2) $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ where $h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}$

Check:
$$
\begin{aligned}
g^{\mu\nu} g_{\nu\sigma} &= (\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) \\
&= \delta^{\nu}{}_{\sigma} + \eta^{\mu\nu} h_{\nu\sigma} - h^{\mu\nu}\eta_{\nu\sigma} + \cancelto{0}{\mathcal{O}(h^2)} \\
&= \delta^{\nu}{}_{\sigma} + \eta^{\mu\nu} h_{\sigma\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}\eta_{\nu\sigma} h_{\alpha\beta} \\
&= \delta^{\nu}{}_{\sigma} + \eta^{\mu\nu} h_{\sigma\nu} - \eta^{\mu\alpha} h_{\alpha\sigma} \\
&= \delta^{\nu}{}_{\sigma} \quad \text{OK}
\end{aligned}
$$

Indices of perturbations raised/lowered by the background metric $\eta_{\mu\nu}$:

$V^{\mu}$ = 1st. order perturbation

$$V_{\mu} = g_{\mu\nu} V^{\nu} = \eta_{\mu\nu} V^{\nu} + \underbrace{h_{\mu\nu} V^{\nu}}_{\mathcal{O}(\delta^2) \text{ term which we drop in the linear perturbation theory.}}$$

The splitting (7.1) of the metric into background and perturbation is not coordinate invariant, consequently $\eta_{\mu\nu}$ and $h_{\mu\nu}$ are not tensors but their sum $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is. Choosing different nearly cartesian coordinates $\tilde{x}^{\mu}$ for our nearly Minkowski space results a different definition of the perturbation $\tilde{h}_{\mu\nu}$.

Let us see how this works in practise by considering a small coordinate transformation $\tilde{x}^\mu(x^\nu)$ that can be expanded as:

(7.3) $$\tilde{x}^\mu = x^\mu + \underbrace{\xi^\mu(x^\nu)}_{\text{1st. order small perturbation}} + \cancel{\mathcal{O}(\xi^2)}^{\,0}$$

The inverse transformation to linear order is:

(7.4) $$x^\mu = \tilde{x}^\mu - \xi^\mu(x^\nu) = \tilde{x}^\mu - \xi^\mu(\tilde{x}^\nu)$$

The corresponding Jacobians are thus given by:

(7.5) $$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \delta^\mu{}_\nu + \partial_\nu \xi^\mu \quad , \quad \frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \delta^\mu{}_\nu - \partial_\nu \xi^\mu$$

Under (7.3) the metric transforms in the usual way:

$$\begin{aligned}
g_{\mu\nu}(\tilde{x}) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) \\
&= (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta) g_{\alpha\beta}(x) \\
&= (\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\mu \partial_\nu \xi^\beta - \delta^\beta_\nu \partial_\mu \xi^\alpha)(\eta_{\alpha\beta} + h_{\alpha\beta}(x)) \qquad \left(\text{drop } \xi h = \mathcal{O}(\delta^2)\right) \\
&= \eta_{\mu\nu} + h_{\mu\nu}(x) - \eta_{\mu\beta}\partial_\nu \xi^\beta - \eta_{\nu\alpha}\partial_\mu \xi^\alpha \\
&= \eta_{\mu\nu} + h_{\mu\nu}(x) - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \qquad \eta_{\mu\nu} = \text{const.}, \text{ indices of the 1st. order perturbation } \xi^\mu \text{ lowered by } \eta_{\mu\nu}.
\end{aligned}$$

Now $$\begin{aligned} g_{\mu\nu}(\tilde{x}) &\equiv \eta_{\mu\nu} + \tilde{h}_{\mu\nu}(\tilde{x}) \\ &= \eta_{\mu\nu} + \tilde{h}_{\mu\nu}(x) \end{aligned}$$ splitting into background + perturbations in the new crd system $\tilde{x}^\mu$.

so that we get:

(7.6) $$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu$$

A perturbative crd transformation of type (7.3) is called gauge transformation. A crd system $x^\mu$ defines the gauge where perturbations, such as $h_{\mu\nu}$, are defined. In general, perturbations change under gauge transformations as we see in eq. (7.6). The gauge transformation however does not change the physical setup. This may sound somewhat paradoxical. The resolution is that perturbations defined in a given gauge are not directly physical quantities, there are more perturbative degrees of freedom than there are dynamical equations. The extra degrees of freedom are spurious gauge modes that originate from the arbitrariness in splitting quantities into background + perturbations. This is not a problem for us, we are free to choose any gauge in which we study a given physical problem. The gauge modes will always drop out from the final result and appear only in the intermediate steps which look different in different gauges.

## 7.2 Linearised field equations

Let us the compute the linearised Einstein eqs. for the metric (7.1). The connection coefficients are given by:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda}\left(\partial_\nu g_{\sigma\lambda} + \partial_\sigma g_{\lambda\nu} - \partial_\lambda g_{\sigma\nu}\right) \quad , \quad \eta_{\mu\nu} = \text{const}$$

$$= \frac{1}{2} \eta^{\mu\lambda}\left(\partial_\nu h_{\sigma\lambda} + \partial_\sigma h_{\lambda\nu} - \partial_\lambda h_{\sigma\nu}\right)$$

$$= \frac{1}{2}\left(\partial_\nu h_\sigma{}^{\mu} + \partial_\sigma h^{\mu}{}_{\nu} - \partial^{\mu} h_{\sigma\nu}\right) \quad , \quad \text{note that} \quad h^{\mu}{}_{\nu} = \eta^{\mu\alpha} h_{\alpha\nu} \neq \delta^{\mu}{}_{\nu}$$

The linearised Ricci tensor becomes:

$$R_{\mu\nu} = \partial_\sigma \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\sigma\mu} + \underbrace{\cancelto{0}{\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}}}_{=\mathcal{O}(\delta^2)}$$

$$= \frac{1}{2}\left(\partial_\sigma\partial_\mu h_\nu{}^{\sigma} + \cancel{\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu}} - \partial_\sigma\partial^\sigma h_{\nu\mu}\right) - \frac{1}{2}\left(\cancel{\partial_\nu\partial_\sigma h_\mu{}^{\sigma}} + \partial_\nu\partial_\mu h^{\sigma}{}_{\sigma} - \partial_\nu\partial^\sigma h_{\mu\sigma}\right)$$

$$= \frac{1}{2}\left(\partial^\sigma\partial_\mu h_{\nu\sigma} - \partial^\sigma\partial_\sigma h_{\mu\nu} - \partial_\nu\partial_\mu h^{\sigma}{}_{\sigma} + \partial_\nu\partial^\sigma h_{\mu\sigma}\right)$$

Denote:

$$h \equiv h^{\mu}{}_{\mu}$$

$$\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial^\mu\partial_\mu = -\frac{\partial^2}{\partial t^2} + \sum_{i=1}^{3}\frac{\partial^2}{\partial x^{i2}} \equiv -\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$(7.7) \quad R_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial^\sigma h_{\nu\sigma} + \partial_\nu\partial^\sigma h_{\mu\sigma} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right)$$

The Ricci scalar is given by:

$$R = g^{\mu\nu}R_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu}$$

$$= \frac{1}{2}\left(2\partial_\mu\partial^\sigma h^{\mu}{}_{\sigma} - \Box h - \Box h\right)$$

$$(7.8) \quad R = \partial^\mu\partial^\sigma h_{\mu\sigma} - \Box h$$

The Einstein tensor reads:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$
$$= \frac{1}{2}\left(\partial_\mu \partial^\sigma h_{\nu\sigma} + \partial_\nu \partial^\sigma h_{\mu\sigma} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right) - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\sigma \partial^\lambda h_{\sigma\lambda} - \Box h\right)$$
$$= \frac{1}{2}\left(\partial_\mu \partial^\sigma h_{\nu\sigma} + \partial_\nu \partial^\sigma h_{\mu\sigma} - \Box h_{\mu\nu} + \eta_{\mu\nu} \Box h - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial^\sigma \partial^\lambda h_{\sigma\lambda}\right)$$

Define the trace reversed perturbation $\bar{h}_{\mu\nu}$ by:

(7.9) $$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad \bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - \frac{4}{2}h = -h$$

Rewriting $G_{\mu\nu}$ in terms of $\bar{h}_{\mu\nu}$ we get:

$$G_{\mu\nu} = \frac{1}{2}\Big(\partial_\mu \partial^\sigma \bar{h}_{\nu\sigma} + \cancel{\frac{1}{2}\partial_\mu \partial_\nu h} + \partial_\nu \partial^\sigma \bar{h}_{\mu\sigma} + \cancel{\frac{1}{2}\partial_\mu \partial_\nu h} - \Box \bar{h}_{\mu\nu} - \cancel{\frac{1}{2}\eta_{\mu\nu}\Box h} + \cancel{\eta_{\mu\nu}\Box h}$$
$$- \cancel{\partial_\mu \partial_\nu h} - \eta_{\mu\nu}\partial^\sigma \partial^\lambda \bar{h}_{\sigma\lambda} - \cancel{\frac{\eta_{\mu\nu}}{2}\Box h}\Big)$$

(7.10) $$G_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial^\sigma \bar{h}_{\nu\sigma} + \partial_\nu \partial^\sigma \bar{h}_{\mu\sigma} - \eta_{\mu\nu}\partial^\sigma \partial^\lambda \bar{h}_{\sigma\lambda} - \Box \bar{h}_{\mu\nu}\right)$$

The energy momentum tensor $T_{\mu\nu}$ is 1st. order perturbation since $T_{\mu\nu} = 0$ for $g_{\mu\nu} = \eta_{\mu\nu}$.

Therefore, the linearised Einstein equation takes the form:

(7.11) $$\partial_\mu \partial^\sigma \bar{h}_{\nu\sigma} + \partial_\nu \partial^\sigma \bar{h}_{\mu\sigma} - \eta_{\mu\nu}\partial^\sigma \partial^\lambda \bar{h}_{\sigma\lambda} - \Box \bar{h}_{\mu\nu} = 16\pi G\, T_{\mu\nu}$$

## Choosing the gauge

Eq. (7.11) can be further simplified by choosing a particular gauge. Under the gauge transformation $\tilde{x}^\mu = x^\mu + \xi^\mu$ we obtain from (7.6) the transformation properties:

(7.12) $$\tilde{h} = h - 2\partial^\mu \xi_\mu$$

$$\tilde{\bar{h}}_{\mu\nu} = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu - \frac{1}{2}\eta_{\mu\nu}\left(h - 2\partial^\sigma \xi_\sigma\right)$$

$$(7.13) \qquad \tilde{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\sigma \xi_\sigma$$

In eq. (7.11) the three first terms on the LHS are proportional to $\partial^\sigma \bar{h}_{\nu\sigma}$ which can be set to zero by a suitable choice of the gauge parameter $\xi^\mu$.

$$(7.14) \qquad \partial^\sigma \tilde{\bar{h}}_{\nu\sigma} = \partial^\sigma\left(\bar{h}_{\nu\sigma} - \partial_\nu \xi_\sigma - \partial_\sigma \xi_\nu + \eta_{\nu\sigma}\partial^\mu \xi_\mu\right) = 0$$

$$\partial^\sigma \bar{h}_{\nu\sigma} - \partial_\nu \cancel{\partial^\sigma \xi_\sigma} - \Box \xi_\nu + \partial_\nu \cancel{\partial^\mu \xi_\mu} = 0$$

$$\Box \xi_\nu = \partial^\sigma \bar{h}_{\nu\sigma}$$

In the gauge specified by the condition (7.14), the linearised Einstein eqs. (7.11) take the form:

$$(7.15) \qquad \boxed{\Box \tilde{\bar{h}}_{\mu\nu} = -16\pi G \tilde{T}_{\mu\nu} \quad , \quad \partial_\sigma \tilde{\bar{h}}_\mu{}^\sigma = 0}$$

__Note added:__

In this gauge $\tilde{G}_{\mu\nu} = -\frac{1}{2}\Box \tilde{\bar{h}}_{\mu\nu}$

and the constraint eq.

$$\nabla^\mu \tilde{G}_{\mu\nu} = 0 \iff \partial^\mu \partial^\sigma \partial_\sigma \tilde{\bar{h}}_{\mu\nu} = 0$$

$$\Box \partial^\mu \tilde{\bar{h}}_{\mu\nu} = 0$$

$$\Rightarrow \partial^\mu \tilde{\bar{h}}_{\mu\nu} = 0 + f_\nu(x) \text{ , where } \Box f_\nu = 0$$

The gauge condition $\partial^\mu \tilde{\bar{h}}_{\mu\nu} = 0$ now imposes the condition $\nabla^\mu G_{\mu\nu} = 0$ but does not yet fully fix the gauge, see the next page.

For an empty spacetime $\tilde{T}_{\mu\nu} = 0$ this yields:

$$\Box \tilde{\bar{h}}_{\mu\nu} = -\frac{\partial^2}{\partial t^2}\tilde{\bar{h}}_{\mu\nu} + \nabla^2 \tilde{\bar{h}}_{\mu\nu} = 0$$

which is just the wave equation. The perturbations $\tilde{\bar{h}}_{\mu\nu}$ describe gravitational waves which propagate at the speed of light $c=1$. Eq. (7.15) tells how the GW are sourced by matter.

From now on we assume that all perturbations are given in the gauge

$$(7.16) \qquad \partial_\sigma \bar{h}_\mu{}^\sigma = 0$$

and drop the tilde $\tilde{\bar{h}} \equiv \bar{h}$.

The condition (7.16) actually does not fully determine the gauge but there is some gauge freedom still left.

Any gauge transformation:

(7.17) $\hat{x}^\mu = x^\mu + \xi^\mu$ where $\Box \xi^\mu = 0$

preserve the condition (7.16)

$$\frac{\partial}{\partial \hat{x}^\sigma} \hat{\bar{h}}_\mu{}^\sigma = \frac{\partial x^\alpha}{\partial \hat{x}^\sigma} \frac{\partial}{\partial x^\alpha} \left( \bar{h}_\mu{}^\sigma - \partial_\mu \xi^\sigma - \partial^\sigma \xi_\mu + \eta_\mu{}^\sigma \partial^\beta \xi_\beta \right)$$

$$= \left( \delta^\alpha_\sigma - \xi^\alpha{}_\sigma \right) \partial_\alpha \left( \bar{h}_\mu{}^\sigma - \partial_\mu \xi^\sigma - \partial^\sigma \xi_\mu + \eta_\mu{}^\sigma \partial^\beta \xi_\beta \right)$$

$$= \partial_\sigma \bar{h}_\mu{}^\sigma - \cancel{\partial_\sigma \partial_\mu \xi^\sigma} - \Box \xi_\mu + \cancel{\partial_\mu \partial^\beta \xi_\beta}$$

$$= \partial_\sigma \bar{h}_\mu{}^\sigma - \Box \xi_\mu = 0$$

How many physical degrees of freedom we have? (when $T_{\mu\nu} = 0$)

$\bar{h}_{\mu\nu}$ symm. 4x4 matrix $\Rightarrow$ 10 ind. components

$\xi^\mu$ 4 components associated to gauge transformations

$\nabla^\mu G_{\mu\nu} = 0$ 4 non-dynamical constraint equations

$\Rightarrow$ 10 - 4 - 4 = 2 physical dof's

To fully fix the gauge we need to impose 8 conditions. The gauge condition $\partial_\sigma \bar{h}_\mu{}^\sigma = 0$ fixes 4 of the 8 gauge dof's. The remaining 4 are fixed in the following by going to the so called transverse traceless gauge.

When there is a source present $T_{\mu\nu} \neq 0$ and there are more dof's in total.

## 7.3 Gravitational waves in vacuum

In vacuum $T_{\mu\nu}=0$, eq. (7.15) reads:

(7.18) $$\Box \bar{h}_{\mu\nu} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = 0 \quad , \quad \partial^\nu \bar{h}_{\mu\nu} = 0$$

The solutions of this wave equation are plane waves:

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left(\bar{A}_{\mu\nu} e^{ik_\sigma x^\sigma}\right) \quad \text{where} \quad k^\mu k_\mu = 0 \ \text{lightlike wave vector}$$
$$\bar{A}_{\mu\nu} = \bar{A}_{\nu\mu} = \text{const. } 4\times 4 \text{ matrix}$$

Check:
$$\begin{aligned}\Box \bar{h}_{\mu\nu} &= \eta^{\alpha\beta}\partial_\alpha\partial_\beta \,\mathrm{Re}\,\bar{A}_{\mu\nu} e^{ik_\sigma x^\sigma} \\ &= \mathrm{Re}\,\bar{A}_{\mu\nu}\,\eta^{\alpha\beta}\partial_\alpha\, ik_\beta e^{ik_\sigma x^\sigma} \\ &= \mathrm{Re}\,\bar{A}_{\mu\nu}\,\eta^{\alpha\beta}\, ik_\beta\, ik_\alpha\, e^{ik_\sigma x^\sigma} \\ &= -k_\alpha k^\alpha \bar{h}_{\mu\nu} = 0\end{aligned}$$

The gauge condition $\partial^\nu \bar{h}_{\mu\nu}=0$ implies

$$\partial_\nu \bar{h}_\mu{}^\nu = \mathrm{Re}\left(\bar{A}_\mu{}^\nu ik_\nu e^{ik_\sigma x^\sigma}\right) = 0 \ \Rightarrow\ \bar{A}_\mu{}^\nu k_\nu = \bar{A}_{\mu\nu}k^\nu = 0$$

Therefore, we find that the solution of (7.18) is given by:

(7.19) $$\bar{h}_{\mu\nu} = \mathrm{Re}\left(\bar{A}_{\mu\nu} e^{ik_\sigma x^\sigma}\right) \quad \text{where} \quad k_\mu k^\mu = 0,\ \bar{A}_{\mu\nu}k^\nu = 0$$

Consider a single plane wave propagating into the direction of z-axis (or any other direction, our background is invariant under rotations):

(7.20) $$k^\mu = (k,0,0,k) \qquad k^\mu k_\mu = -k^2 + k^2 = 0 \quad \text{ok.}$$

Gauge cond: $\bar{A}_{\mu\nu}k^\nu = \bar{A}_{\mu 0}k + \bar{A}_{\mu 3}k = 0 \ \Rightarrow\ \bar{A}_{\mu 0} = -\bar{A}_{\mu 3}$

The coefficient matrix thus takes the form:

$$\bar{A}_{\mu\nu} = \begin{bmatrix} \bar{A}_{00} & \bar{A}_{01} & \bar{A}_{02} & -\bar{A}_{00} \\ \bar{A}_{01} & \bar{A}_{11} & \bar{A}_{12} & -\bar{A}_{01} \\ \bar{A}_{02} & \bar{A}_{12} & \bar{A}_{22} & -\bar{A}_{02} \\ -\bar{A}_{00} & -\bar{A}_{01} & -\bar{A}_{02} & \bar{A}_{00} \end{bmatrix} \tag{7.21}$$

6 independent components
$\bar{A}_{00}, \bar{A}_{01}, \bar{A}_{02}, \bar{A}_{11}, \bar{A}_{12}, \bar{A}_{22}$

## Transverse traceless (TT) gauge

Recall that (7.16) fixes only 4 of the 8 gauge modes* in $\bar{h}_{\mu\nu}$. To fix the remaining 4 we consider gauge transformations with $\Box^2 \xi^\mu = 0$ which preserve (7.16).

Perform a gauge transformation:

$$\hat{x}^\mu = x^\mu + \xi^\mu \quad , \quad \xi^\mu = -\mathrm{Re}\left(i \overset{\text{constant}}{\epsilon^\mu} e^{ik_\nu x^\nu}\right) \Rightarrow \Box \xi^\mu = 0 \tag{7.22}$$

Under this $\bar{h}_{\mu\nu}$ transforms as: (c.f. (7.13))

$$\hat{\bar{h}}_{\mu\nu} = \mathrm{Re}\left[\left(\bar{A}_{\mu\nu} - k_\mu \epsilon_\nu - k_\nu \epsilon_\mu + \eta_{\mu\nu} k^\sigma \epsilon_\sigma\right) e^{ik_\lambda x^\lambda}\right]$$
$$= \mathrm{Re}\left(\hat{\bar{A}}_{\mu\nu} e^{ik_\sigma x^\sigma}\right)$$

so that:

$$\hat{\bar{A}}_{\mu\nu} = \bar{A}_{\mu\nu} - k_\mu \epsilon_\nu - k_\nu \epsilon_\mu + \eta_{\mu\nu} k^\sigma \epsilon_\sigma \tag{7.23}$$

Apply this to (7.21): $\quad (k_\mu = (-k, 0, 0, k))$

$$\left\{ \begin{aligned} \hat{\bar{A}}_{00} &= \bar{A}_{00} + 2k\epsilon_0 - k(\epsilon_0 + \epsilon_3) = \bar{A}_{00} + k(\epsilon_0 - \epsilon_3) \\ \hat{\bar{A}}_{01} &= \bar{A}_{01} + k\epsilon_1 \\ \hat{\bar{A}}_{02} &= \bar{A}_{02} + k\epsilon_2 \\ \hat{\bar{A}}_{11} &= \bar{A}_{11} + k(\epsilon_0 + \epsilon_3) \\ \hat{\bar{A}}_{12} &= \bar{A}_{12} \\ \hat{\bar{A}}_{22} &= \bar{A}_{22} + k(\epsilon_0 + \epsilon_3) \end{aligned} \right. \tag{7.24}$$

*Because in vacuum the 4 constraints $\nabla^\mu G_{\mu\nu} = 0$ all act on $h_{\mu\nu}$ only.

We now fix the remaining 4 gauge parameters $\epsilon^\mu$ by setting:

(7.25) $\hat{A}^\mu{}_\mu = 0 \iff \hat{A}^1{}_1 + \hat{A}^2{}_2 = \hat{A}_{11} + \hat{A}_{22} = 0$

$\left( = \eta^{00}\hat{A}_{00} + \eta^{11}\hat{A}_{11} + \eta^{22}\hat{A}_{22} + \eta^{33}\hat{A}_{33} \right)$

$\iff \epsilon_0 + \epsilon_3 = -\dfrac{\bar{A}_{11} + \bar{A}_{22}}{2k}$ 1 cond.

(7.26) $\hat{A}_{00} = 0,\ \hat{A}_{01} = 0,\ \bar{A}_{02} = 0$ 3 cond.

$$\iff \begin{cases} \epsilon_0 - \epsilon_3 = -\dfrac{\bar{A}_{00}}{k} \\ \epsilon_1 = -\dfrac{\bar{A}_{01}}{k} \\ \epsilon_2 = -\dfrac{\bar{A}_{02}}{k} \end{cases}$$

The conditions (7.25) & (7.26) specify the transverse and traceless gauge where:

(7.27)
$$\hat{A}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat{A}_{11} & \hat{A}_{12} & 0 \\ 0 & \hat{A}_{21} & -\hat{A}_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Denoting $\hat{A}_{11} \equiv \alpha$, $\hat{A}_{12} \equiv \beta$ and defining the polarization matrices:

(7.28)
$$e^+_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e^\times_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the perturbation in the TT gauge reads:

$$\hat{\bar{h}}_{\mu\nu} = \mathrm{Re}\left[ \left( \alpha e^+_{\mu\nu} e + \beta e^\times_{\mu\nu} \right) e^{ik_\sigma x^\sigma} \right] \qquad \alpha, \beta \in \mathbb{C} \text{ constants}$$

In the TT gauge $\hat{A}^\mu{}_\mu = 0 \Rightarrow \hat{\bar{h}}^\mu{}_\mu = 0$ so that

$$\hat{\bar{h}}_{\mu\nu} = \hat{h}_{\mu\nu}$$

In this section, we will use the TT gauge and drop the hat. Therefore, the metric perturbation reads:

$$\boxed{h_{\mu\nu} = \mathrm{Re}\left[\left(\alpha e^{+}_{\mu\nu} + \beta e^{\times}_{\mu\nu}\right) e^{ik_\sigma x^\sigma}\right]} \tag{7.29}$$

$\alpha, \beta \in \mathbb{C}$ constants

In this form the gauge is fully fixed and $h_{\mu\nu}$ has 2 degrees of freedom left.

## Gravitational waves and test particles

Our next task is to consider the effect of GW's of the form (7.29) on a cloud of test particles, i.e. how do distances btw particles change when the GW passes through? We assume $T_{\mu\nu} = 0$ all the time here.

There are two parts in the problem: 1) the motion determined by geodesics and 2) distances determined by $ds^2$.

### Geodesics:

To find the geodesic eqs we need the Christoffels:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha h_\beta{}^\mu + \partial_\beta h_\alpha{}^\mu - \partial^\mu h_{\alpha\beta}\right)$$

In the TT gauge where $h_{\mu 0} = 0$ so that:

$$\Gamma^{\mu}_{00} = \frac{1}{2}\left(\partial_0 \cancel{h_0{}^\mu} + \partial_0 \cancel{h_0{}^\mu} - \partial^\mu h_{00}\right) = 0 \; ,$$

$$\Gamma^{\mu}_{0i} = \frac{1}{2}\left(\partial_0 h_i{}^\mu + \partial_i \cancel{h_0{}^\mu} - \partial^\mu \cancel{h_{0i}}\right) = \frac{1}{2}\partial_0 h_i{}^\mu$$

The geodesic eq. then simplify to:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{x}^\mu + 2\Gamma^\mu_{0i}\dot{x}^0\dot{x}^i + \Gamma^\mu_{ij}\dot{x}^i\dot{x}^j = 0 \qquad (7.30)$$

A solution of (7.30) is given by $x^i = \text{const.}$

check:

$\mu = i: \quad \ddot{x}^i + 2\Gamma^i_{0j}\dot{x}^0\dot{x}^j + \Gamma^i_{\ell m}\dot{x}^\ell\dot{x}^m = 0$

$\mu = 0: \quad \ddot{x}^0 + 2\Gamma^0_{0j}\dot{x}^0\dot{x}^j + \Gamma^0_{\ell m}\dot{x}^\ell\dot{x}^m = \ddot{x}^0$

but from $u^\mu u_\mu = (\eta_{00} + \underbrace{h_{00}}_{=0})\dot{x}^0\dot{x}^0 = -1$

$(\dot{x}^0)^2 = 1$

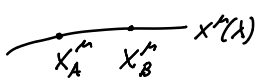
$\Rightarrow \ddot{x}^0 = 0$

OK

Therefore, we find that the GW does change the crd locations $x^i$ of test particles in the TT gauge, they stay put at $x^i = \text{const}$ (no ext. forces, assume the test particle cloud at rest initially).
The distances btw particles however do change because $h_{\mu\nu}$ affects $ds^2$.

Distances:

Consider two test particles infinitesimally close to each other




$X_A^\mu = (X_A^0, X_A^j)$, $X_A^j = \text{const.}$ ← sol. of the geodesic eqs.
$X_B^\mu = (X_B^0, X_B^j)$, $X_B^j = \text{const.}$ ↙

Infinitesimal physical distance at fixed time $t$:

$$ds^2_{AB} = g_{\mu\nu}\,dX^\mu_{AB}\,dX^\nu_{AB} \qquad dt = 0$$

$$= g_{ij}\,dX^i_{AB}\,dX^j_{AB}, \qquad dX^j_{AB} \equiv X^j_B - X^j_A$$

$$ds_{AB} = (g_{ij}\, dx^i_{AB}\, dx^j_{AB})^{1/2}$$

Here $g_{ij} = \delta_{ij} + h_{ij}$ and working to linear order in $h_{ij}$, this can be recast as:

$$ds_{AB} = ((\delta_{ij} + h_{ij})\, dx^i_{AB} dx^j_{AB})^{1/2} = \left(\delta_{ij}\left(dx^i_{AB} + \frac{1}{2} h^i{}_k\, dx^k_{AB}\right)\left(dx^j_{AB} + \frac{1}{2} h^j{}_k\, dx^k_{AB}\right)\right)^{1/2}$$

Now dropping the differentials (implicitly understood that we are referring to infinitesimal distances, we can express the physical distance btw. any test particles A and B at a time $t$ as:

(7.30) $$s_{AB}(t) = (\delta_{ij}\, \tilde{\zeta}^i_{AB}(t)\, \tilde{\zeta}^j_{AB})^{1/2}$$

where we have defined

(7.31) $$\tilde{\zeta}^i_{AB}(t) = (x^i_B - x^i_A) + \frac{1}{2} h^i{}_k(t)(x^k_B - x^k_A) \equiv x^i_{AB} + \frac{1}{2} h^i{}_k(t)\, x^k_{AB}$$

(physical dist)$^{1/2}$ coordinate distance time dependent modulation

Because of the time dependent modulation due to $h_{ij}(t)$, the physical distances change in time although the crd distances $x^i$ remain constant. That means the GW causes the test particles to accelerate.

Inserting into eq. (7.31) the solution (7.29) for GW propagating into the $x^3$-direction, we get

$$\tilde{\zeta}^i_{AB} = \left(x^1_{AB} + \frac{1}{2}(h_{11}\, x^1_{AB} + h_{12}\, x^2_{AB}),\ x^2_{AB} + \frac{1}{2}(h_{21}\, x^1_{AB} + h_{22}\, x^2_{AB}),\ x^3_{AB}\right)$$

distances transverse to the propagation direction change

distances along the GW direction not affected

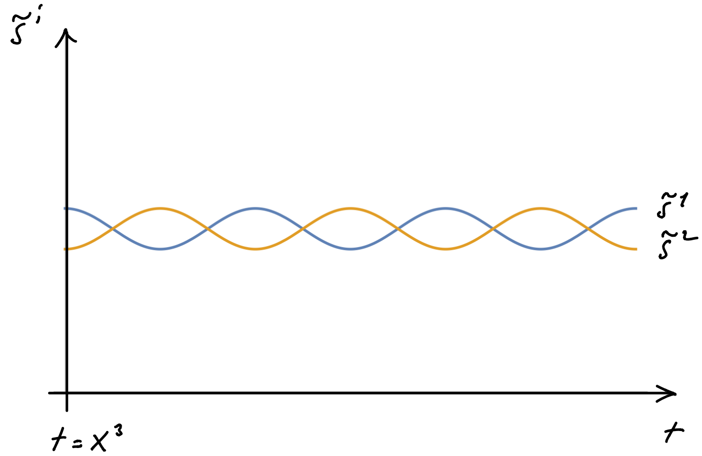
Consider then a set of test particles. Choose one of them as the reference point and investigate how distances relative to it change as a GW passes through.

Consider first the case $\beta = 0$ in (7.29), and choose $\alpha > 0$ : + polarisation

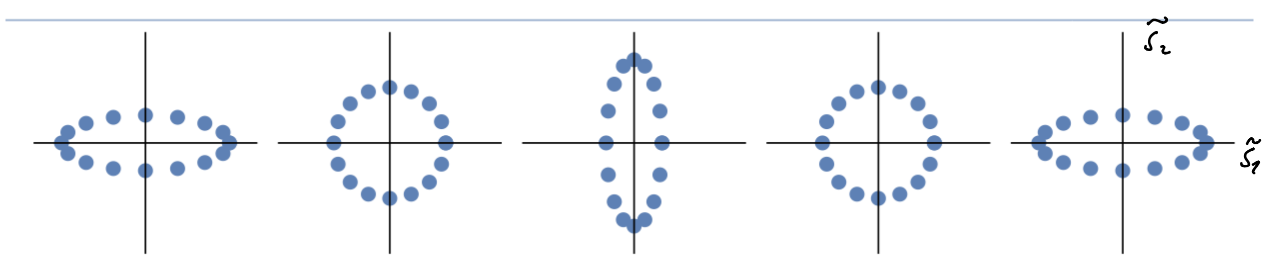
$$h_{\mu\nu} = Re\left( \alpha e^{+}_{\mu\nu} e^{ik_\sigma x^\sigma} \right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos(k(-x^0 + x^3))$$

$$\tilde{\zeta}^i = x^i + \frac{1}{2} h^i_j x^j = (x^1, x^2, x^3) + \frac{\alpha}{2} \cos(k(t - x^3))(x^1, -x^2, 0)$$

$\Rightarrow$ $\tilde{\zeta}^3 = \text{const}$, $\tilde{\zeta}^1$ and $\tilde{\zeta}^2$ oscillate:




This causes the shape of the test particle cloud around the reference point to oscillate.

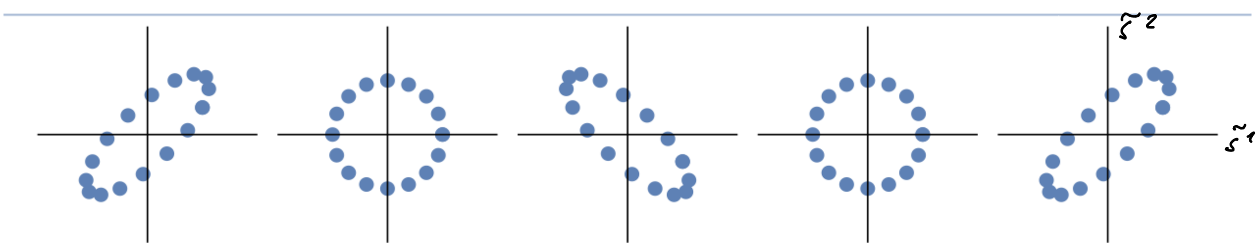



$t = x^3$ $\longrightarrow$ $t$ grows

Consider then the case $\alpha = 0$, and choose $\beta > 0$: $\times$ polarisation

$$h_{\mu\nu} = \mathrm{Re}\left(\beta e^{\times}_{\mu\nu} e^{ik_\sigma x^\sigma}\right) = \begin{pmatrix} & & & \\ & & \beta & \\ & \beta & & \\ & & & \end{pmatrix} \cos(k(-x^0 + x^3))$$

$$\tilde{s}^i = s^i + \frac{1}{2} h^i_j s^j = (s^1, s^2, s^3) + \frac{\beta}{2}\cos(k(x^0 - x^3))(s^2, s^1, 0)$$




$x_0 = x_3$ $\qquad$ $x_0$ grows $\longrightarrow$

A general GW is a superposition of $+$ and $\times$ polarisations according to eq (7.29).

## 7.4 Production of gravitational waves

So far we have been discussing the vacuum solution $T_{\mu\nu}=0$ for which eq.(7.15) becomes:

$$\Box \bar{h}_{\mu\nu} = 0 \quad , \quad \partial^\mu \bar{h}_{\mu\nu} = 0$$

In this case the solutions are plane waves and we were able to fix the residual gauge freedom ($\partial^\mu \bar{h}_{\mu\nu}=0$ unaffected by gauge transf. $\Box \xi^\mu = 0$) by going to the transverse traceless gauge which can be defined by

$$h^\mu{}_\mu = 0 \quad , \quad h_{0i} = 0 \quad , \quad \partial^\mu h_{\mu\nu} = 0$$

The last 2 conditions imply $\partial^\mu h_{\mu 0} = \partial^0 h_{00} + \underbrace{\partial^i h_{i0}}_{=0} = 0 \Rightarrow h_{00} = \text{const.}$

By doing a further gauge transf. $h_{00} \to h_{00} - 2\partial_0 \xi_0 = 0$ we can set $h_{00}=0$ without affecting the TT conditions (exercise).

If we now return to eq. (7.15) in the presence of a source $T_{\mu\nu} \neq 0$

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu} \quad , \quad \partial^\mu \bar{h}_{\mu\nu} = 0$$

we see that trying to impose the TT gauge cond. would imply $T_{00} = \rho = 0$ *, i.e. vanishing energy density. Therefore, in we cannot use the TT gauge in the regime where $T_{\mu\nu} \neq 0$

*Newtonian limit $T_{\mu\nu} = \rho u_\mu u_\nu$

The strategy then is the following

1) In the regime $T_{\mu\nu} \neq 0$ use the Lorenz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ and solve for $\bar{h}_{\mu\nu}$ from

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

2) Far away from the source $T_{\mu\nu} = 0$ and we can convert the solution $\bar{h}_{\mu\nu}$ into the TT gauge $h^{TT}_{\mu\nu}$ where we know how GW affects test bodies

The general solution is of $\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$ , $\partial_\sigma \bar{h}_\mu{}^\sigma = 0$ can be expressed as

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi G \int d^4y \, G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) \tag{7.32}$$

where $G(x-y)$ is the Green function of the operator $\Box$. The Green function is defined as the solution of

$$\Box G(x^\mu - y^\mu) = \left(-\frac{\partial^2}{\partial t^2} + \delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\right) G(x^\mu - y^\mu) = \delta^{(4)}(x^\mu - y^\mu) \tag{7.33}$$

with appropriate boundary conditions. The Green function splits into the advanced part proportional to $\theta(y^0 - x^0)$ and the retarded part proportional to $\theta(x^0 - y^0)$ which lie respectively on the future and past light cone of $x^\sigma$. Due to causality, only the past light cone contributes to $\bar{h}_{\mu\nu}(x^\sigma)$ and we therefore need the retarded Green function.

From the definition (7.33) it follows that $\Box \bar{h}_{\mu\nu}$ of (7.32) reads:

$$\Box_x \left(-16\pi G \int d^4y \, G(x^\sigma - y^\sigma) \, T_{\mu\nu}(y^\sigma)\right) = -16\pi G \int d^4y \, \Box_x G(x^\sigma - y^\sigma) \, T_{\mu\nu}(y^\sigma)$$

$$\Box_x \equiv -\frac{\partial^2}{\partial (x^0)^2} + \frac{\partial^2}{\partial \bar{x}^2}$$

$$= -16\pi G \int d^4y \, \delta(x^\sigma - y^\sigma) \, T_{\mu\nu}(y^\sigma)$$

$$= -16\pi G \, T(x^\sigma) \quad ,$$

so that (7.32) indeed is the solution of (7.15).

The retarded Green function of $\Box$ is given by (exercise)

(7.34) $$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\bar{x}-\bar{y}|} \, \delta(|\bar{x}-\bar{y}| - (x^0 - y^0)) \, \theta(x^0 - y^0)$$

where $|\bar{x}-\bar{y}| = (\delta_{ij}(x^i - y^i)(x^j - y^j))^{1/2}$

Substituting this into (7.32) we get

$$\bar{h}_{\mu\nu}(x^\sigma) = 4G \int d^4y \, \frac{\delta(|\bar{x}-\bar{y}| - (x^0 - y^0)) \, \theta(x^0 - y^0)}{|\bar{x}-\bar{y}|} \, T_{\mu\nu}(y^\sigma)$$

$$= 4G \int \frac{d\bar{y}}{|\bar{x}-\bar{y}|} \int dy^0 \, \delta(y^0 - x^0 + |\bar{x}-\bar{y}|) \, T_{\mu\nu}(y^\sigma) \, \theta(x^0 - y^0)$$

(7.35) $$\bar{h}_{\mu\nu}(x^\sigma) = 4G \int d\bar{y} \, \frac{T_{\mu\nu}(x^0 - |\bar{x}-\bar{y}|, \bar{y})}{|\bar{x}-\bar{y}|}$$

time events $y^0 = x^0 - |\bar{x}, \bar{y}|$ lie on the past light cone of the point $x^\sigma$

Take the Fourier transform of (7.35) wrt $x^0$:

$$\tilde{h}_{\mu\nu}(\omega, x^i) \equiv \int_{-\infty}^{\infty} dx^0 e^{i\omega x^0} \bar{h}_{\mu\nu}(x^0, x^i) \qquad , \quad \bar{h}_{\mu\nu}(x^0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x^0} \tilde{h}_{\mu\nu}(\omega, x^i)$$

$$= 4G \int d\bar{y} \, \frac{1}{|\bar{x}-\bar{y}|} \underbrace{\int_{-\infty}^{\infty} dx^0 e^{i\omega x^0} T_{\mu\nu}(x^0 - |\bar{x}-\bar{y}|, \bar{y})}_{u = x^0 - |\bar{x}-\bar{y}|}$$

$$= e^{i\omega|\bar{x}-\bar{y}|} \int_{-\infty}^{\infty} du \, e^{i\omega u} T_{\mu\nu}(u, \bar{y}) = e^{i\omega|\bar{x}-\bar{y}|} \tilde{T}_{\mu\nu}(\omega, \bar{y})$$

$$= 4G \int_{|\bar{y}| \le L} d\bar{y} \, e^{i\omega|\bar{x}-\bar{y}|} \frac{\tilde{T}_{\mu\nu}(\omega, \bar{y})}{|\bar{x}-\bar{y}|}$$

Assume the source is localised in the region $|\bar{y}| \le L$ and $T_{\mu\nu}(y) = 0$ for $|\bar{y}| > L$

We are interested in the GW solution far away from the source:

$$|\bar{x}| \gg L \Rightarrow \frac{e^{i\omega|\bar{x}-\bar{y}|}}{|\bar{x}-\bar{y}|} \simeq \frac{e^{i\omega|\bar{x}|}}{|\bar{x}|}$$

and we get:

$$(7.36) \quad \tilde{h}_{\mu\nu}(\omega, x^i) \simeq 4G \frac{e^{i\omega|\bar{x}|}}{|\bar{x}|} \int d\bar{y} \, \tilde{T}_{\mu\nu}(\omega, \bar{y})$$

Recall that we are working in the Lorenz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$. Using the gauge condition, we can express $\tilde{h}_{0\mu}$ in terms of $\tilde{h}_{ij}$ and it suffices to solve for $\tilde{h}_{ij}$ only.

Indeed, taking the Fourier transform of $\partial_\sigma \bar{h}_\mu{}^\sigma = 0$ we get:

$$\partial_\sigma \bar{h}_\mu{}^\sigma = \int \frac{d\omega}{(2\pi)}\, e^{-i\omega x^0}(-i\omega)\,\bar{h}_\mu{}^0(\omega, x^i) + \int \frac{d\omega}{2\pi}\, e^{-i\omega x^0}\,\partial_i \bar{h}_\mu{}^i(\omega, x^i) = 0$$

$$\Rightarrow \quad \bar{h}_\mu{}^0(\omega, x^i) = \frac{i}{\omega}\,\partial_k \bar{h}_\mu{}^k(\omega, x^i)$$

From this we get:

$$(7.37) \qquad \begin{cases} \bar{h}_{00} = \dfrac{-i}{\omega}\,\partial_k \bar{h}_0{}^k \\[2mm] \bar{h}_{j0} = -\dfrac{i}{\omega}\,\partial_k \bar{h}_j{}^k \end{cases}$$

Let us now work out the expression for $\bar{h}^{ij}$ using (7.36):

$$\bar{h}^{ij}(\omega, x^i) = \frac{4G\, e^{i\omega|\bar{x}|}}{|\bar{x}|}\underbrace{\int d\bar{y}\, T^{ij}(\omega, \bar{y})}$$

0 boundary term, source vanishes for $|\bar{y}| > L$

$$= \int d\bar{y}\left(\cancel{\partial_k\left(y^i T^{kj}\right)} - y^i \partial_k T^{kj}\right)$$

$$= -\int d\bar{y}\, y^i \partial_k T^{kj}(\omega, \bar{y})$$

$$\partial_\mu T^{\mu\nu} = \partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$$\Rightarrow -i\omega T^{0\nu}(\omega, \bar{y}) + \partial_i T^{i\nu}(\omega, \bar{y}) = 0$$

$$\Rightarrow +i\omega T^{0j}(\omega, \bar{y}) = \partial_i T^{ij}(\omega, \bar{y})$$

$$= -i\omega \int d\bar{y}\, y^i T^{0j}(\omega, \bar{y})$$

$$= -\frac{i\omega}{2}\int d\bar{y}\underbrace{\left(y^i T^{0j}(\omega, \bar{y}) + y^j T^{0i}(\omega, \bar{y})\right)}_{= \partial_k\left(y^i y^j T^{0k}\right) - y^i y^j \partial_k T^{0k}}$$

our starting point is symmetric in $ij$ so this must be so as well

0 boundary term

$$= -\frac{i\omega}{2}\int d\bar{y}\left(\cancel{\partial_k\left(y^i y^j T^{0k}\right)} - y^i y^j \underbrace{\partial_k T^{0k}}_{= i\omega T^{00}}\right)$$

(from $\partial_\mu T^{\mu\nu} = 0$)

$$= -\frac{\omega^2}{2}\int d\bar{y}\, y^i y^j T^{00}$$

$$\Rightarrow \quad \bar{h}^{ij}(\omega,\bar{x}) = -\frac{2G\omega^2 e^{i\omega|\bar{x}|}}{|\bar{x}|}\int d\bar{y}\, y^i y^j T^{00}(\omega,\bar{y}) \qquad (7.38)$$

The quantity on the RHS is the quadrupole moment tensor

$$(7.39) \qquad \tilde{I}^{ij}(\omega) \equiv \int d\bar{y}\, y^i y^j T^{00}(\omega,\bar{y})$$

$$I^{ij}(t) \equiv \int d\bar{y}\, y^i y^j T^{00}(t,\bar{y})$$

Recall that the energy density: $\rho \equiv u^\mu u^\nu T_{\mu\nu}$

For slowly moving particles: $u^\mu \simeq (1,\bar{0}) \Rightarrow \rho \simeq T_{00} = T^{00}$

Taking the inverse Fourier transformation of (7.38) we finally get:

$$\bar{h}_{ij}(t,x^i) = -2G\int \frac{d\omega}{2\pi} e^{-i\omega t}\, \omega^2 \frac{e^{i\omega|\bar{x}|}}{|\bar{x}|}\tilde{I}^{ij}(\omega)$$

$$= -\frac{2G}{|\bar{x}|}\int \frac{d\omega}{2\pi} e^{-i\omega(t-|\bar{x}|)}\,\omega^2 \tilde{I}^{ij}(\omega)$$

$$= \frac{2G}{|\bar{x}|}\frac{d^2}{dt^2}\underbrace{\int \frac{d\omega}{2\pi} e^{-i\omega(t-|\bar{x}|)}\tilde{I}^{ij}(\omega)}_{= I^{ij}(t-|\bar{x}|)}$$

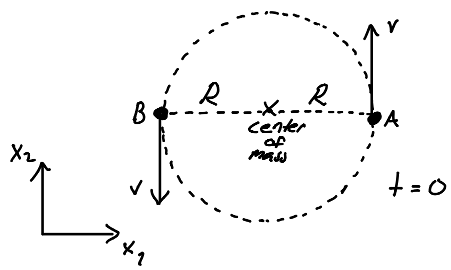
$$(7.40) \qquad \boxed{\bar{h}_{ij}(t,x^i) = \frac{2G}{|\bar{x}|}\frac{d^2}{dt^2} I_{ij}(t-|\bar{x}|), \quad |\bar{x}| \gg L \sim \omega^{-1}}$$

characteristic frequency of the source $\omega \sim L^{-1}$

Thus we learn that GW are sourced by the quadrupole momentum (not dipole like EM waves) and diluted as $|\bar{x}|^{-1}$ far away from the source ( not $|\bar{x}|^{-2}$ as one might have expected).

## Example: GW from a binary star

Consider two stars of mass $M$ on a circular orbit on the $x^1, x^2$-plane.




$M_A = M_B = M$

Use Newtonian gravity to describe the motion (i.e. neglect GR corrections here).

$$\frac{GM^2}{(2R)^2} = \frac{Mv^2}{R} \quad \Rightarrow \quad v = \left(\frac{GM}{4R}\right)^{1/2}$$

The angular frequency is: $\omega = \frac{2\pi}{T} = \frac{2\pi v}{2\pi R} = \left(\frac{GM}{4R^3}\right)^{1/2}$

Trajectories of the stars A & B are (see the figure):

$$x_A^1 = R\cos\omega t \qquad x_B^1 = -R\cos\omega t$$
$$x_A^2 = R\sin\omega t \qquad x_B^2 = -R\sin\omega t$$

Assuming the motion is slow $v \ll 1$, we can approximate $T^{00} \simeq \rho$. Using this we get:

$$T^{00}(t,x^i) = M\,\delta(x^3)\left(\delta(x^1 - x_A^1)\delta(x^2 - x_A^2) + \delta(x^1 - x_B^1)\delta(x^2 - x_B^2)\right)$$

Substituting this into (7.39) we find:

$$I_{11} = \int d\bar{y}\, y^1 y^1\, T^{00} = M x_A^1 x_A^1 + M x_B^1 x_B^1 = 2MR^2\cos^2\omega t$$

$$I_{22} = 2MR^2\sin^2\omega t$$

$$I_{12} = 2MR^2\cos\omega t\sin\omega t = I_{21}$$

$$I_{i3} = 0$$

Using that
$$\frac{d^2}{dt^2}(\cos^2\omega t) = \frac{d^2}{dt^2}\left(\frac{1}{2}(1+\cos 2\omega t)\right) = -2\omega^2\cos 2\omega t$$
$$\frac{d^2}{dt^2}(\sin^2\omega t) = \frac{d^2}{dt^2}\left(\frac{1}{2}(1-\cos 2\omega t)\right) = 2\omega^2\cos 2\omega t$$
$$\frac{d^2}{dt^2}(\cos\omega t\sin\omega t) = \frac{d^2}{dt^2}\left(\frac{1}{2}\sin 2\omega t\right) = -2\omega^2\sin 2\omega t$$

we find:
$$\frac{d^2 I_{ij}}{dt^2} = 4MR^2\omega^2\begin{pmatrix} -\cos 2\omega t & -\sin 2\omega t & 0 \\ -\sin 2\omega t & \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Substituting this into (7.40) we then get:
$$\bar{h}_{ij}(t,r) = -\frac{8GMR^2\omega^2}{r}\begin{pmatrix} \cos 2\omega t_r & \sin 2\omega t_r & 0 \\ \sin 2\omega t_r & -\cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
where $t_r = t - r$ and $r \gg \omega^{-1}$

This describes GW with an angular frequency $2\omega$.

What about the TT gauge? For an observer on the $x^3$ axis, $x^i = (0,0,r)$, the result is already in the TT-gauge and we can write:
$$\bar{h}_{ij}(t,r) = -\frac{8GMR^2\omega^2}{r}\, Re\Big[\underbrace{\left(e^+_{ij} + i e^\times_{ij}\right)}_{\equiv\sqrt{2}e^R_{ij}\ \text{circular polarisation}} e^{-i2\omega(t-r)}\Big]$$

For an observer not on the $x^3$ axis we need to rotate the crd system by a constant angle to go to the TT gauge.

# 7.5 Energy loss due to gravitational radiation

As we have already mentioned, there is no universally good way of defining the gravitational energy in GR where gravity is not treated as a force. In the weak field limit we can however define the energy momentum tensor for the metric fluctuations, here the GW's. The procedure is complicated by the gauge dependence, i.e. arbitrariness in splitting the metric into background (not included in $T_{\mu\nu}$) and fluctuation. Here we will use the TT gauge throughout, it can be shown that the final result is gauge-invariant althought this is not obvious in the intermediate steps. For further discussion see Carroll Chapter 7.6, Wald Chapter 4.4b and Misner, Thorne and Wheeler Chapters 35 and 36. The presentation here is a mixture of Carroll and Wald.

## Expanding Einstein eqs. to second order

To find the $T_{\mu\nu}$, we need to go to second order in perturbations:

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\delta h_{\mu\nu}}_{\text{1st. order perturbation}} + \underbrace{\delta^2 h_{\mu\nu}}_{\text{2nd. order perturbation}}$$

Here we introduce temporarily the notation where $\delta \ldots$ and $\delta^2 \ldots$ refer to first and second order perturbations respectively.

The Ricci tensor is expanded similarly as:

$$R_{\mu\nu} = 0 + \delta R_{\mu\nu} + \delta^2 R_{\mu\nu} \tag{7.41}$$

In the TT gauge (Note that this specifies the gauge only to 1st. order)

(7.42) $\delta h^\mu{}_\mu = 0$ , $\delta h_{0i} = 0$ , $\partial^\mu \delta h_{\mu\nu} = 0$

we get from (7.7)

(7.43) $\delta R_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial^\sigma \delta h_{\nu\sigma} + \partial_\nu \partial^\sigma \delta h_{\mu\sigma} - \partial_\mu \partial_\nu \delta h - \Box \delta h_{\mu\nu}\right) \overset{\text{TT gauge}}{=} -\frac{1}{2}\Box \delta h_{\mu\nu}$

and by expanding $R_{\mu\nu}$ to second order (exercise):

(7.44)
$$\delta^2 R_{\mu\nu} = \frac{1}{2}\delta h^{\rho\sigma}\partial_\mu\partial_\nu \delta h_{\rho\sigma} + \frac{1}{4}(\partial_\mu \delta h_{\rho\sigma})\partial_\nu \delta h^{\rho\sigma} + (\partial^\sigma \delta h^\rho{}_\nu)\partial_{[\sigma}\delta h_{\rho]\mu}$$
$$- \delta h^{\rho\sigma}\partial_\rho \partial_{(\mu}\delta h_{\nu)\sigma} + \frac{1}{2}\delta h^{\rho\sigma}\partial_\sigma\partial_\rho \delta h_{\mu\nu}$$
$$+ \frac{1}{2}\left(\partial_\mu \partial^\sigma \delta^2 h_{\nu\sigma} + \partial_\nu \partial^\sigma \delta^2 h_{\mu\sigma} - \partial_\mu\partial_\nu \delta^2 h - \Box \delta^2 h_{\mu\nu}\right)$$

(Recall: $\partial_{[\sigma}h_{\rho]\mu} = \frac{1}{2}\partial_\sigma h_{\rho\mu} - \frac{1}{2}\partial_\rho h_{\sigma\mu}$
$\partial_{(\mu}h_{\nu)\sigma} = \frac{1}{2}\partial_\mu h_{\nu\sigma} + \frac{1}{2}\partial_\nu h_{\mu\sigma}$ )

The first order perturbation $\delta h_{\mu\nu}$ satisfies the equation of motion:

(7.45) $\Box \delta h_{\mu\nu} = 0 \;\Rightarrow\; \delta R_{\mu\nu} = 0$

and hence

(7.46) $R_{\mu\nu} = \delta^2 R_{\mu\nu}$

To proceed, let us introduce some notation and define:

(7.47) $\underset{\text{function}}{R^{(1)}_{\mu\nu}}(\underset{\text{argument with 2 indices}}{u_{\alpha\beta}}) \equiv \frac{1}{2}\left(\partial_\mu \partial^\sigma u_{\nu\sigma} + \partial_\nu \partial^\sigma u_{\mu\sigma} - \partial_\mu \partial_\nu u^\alpha{}_\alpha - \Box u_{\mu\nu}\right)$

$$(7.48) \quad R^{(2)}_{\mu\nu}(u_{\alpha\beta}) = \frac{1}{2} u^{\rho\sigma}\partial_\mu\partial_\nu u_{\rho\sigma} + \frac{1}{4}(\partial_\mu u_{\rho\sigma})\partial_\nu u^{\rho\sigma} + (\partial^\sigma u^\rho{}_\nu)\partial_{[\sigma} u_{\rho]\mu}$$
$$- u^{\rho\sigma}\partial_\rho\partial_{(\mu} u_{\nu)\sigma} + \frac{1}{2} u^{\rho\sigma}\partial_\sigma\partial_\rho u_{\mu\nu}$$

With this notation we then have:

$$(7.49) \quad \delta R_{\mu\nu} = R^{(1)}_{\mu\nu}(\delta h_{\alpha\beta}) = 0 \quad (\text{as in } 7.45)$$

$$(7.50) \quad \delta^2 R_{\mu\nu} = \underbrace{R^{(2)}_{\mu\nu}(\delta h_{\alpha\beta})}_{\text{Quadratic in } \delta h_{\alpha\beta}} + \underbrace{R^{(1)}_{\mu\nu}(\delta^2 h_{\alpha\beta})}_{\text{Linear in } \delta^2 h_{\alpha\beta}}$$

The expansion for the Ricci scalar becomes:

$$R = g^{\mu\nu}R_{\mu\nu} = (\eta^{\mu\nu} - \delta h^{\mu\nu} + O(\delta^2))(0 + \delta R_{\mu\nu} + \delta^2 R_{\mu\nu})$$
$$R = \eta^{\mu\nu}\delta R_{\mu\nu} + \eta^{\mu\nu}\delta^2 R_{\mu\nu} - \delta h^{\mu\nu}\delta R_{\mu\nu} + O(\delta^3)$$
$$\equiv \delta R + \delta^2 R$$

$$\Rightarrow \quad \delta R = \delta R^\mu{}_\mu = R^{(1)\mu}{}_\mu(\delta h_{\alpha\beta}) \overset{(7.45)}{=} 0$$

$$\delta^2 R = \delta^2 R^\mu{}_\mu - \delta h^{\mu\nu}\delta R_{\mu\nu} \qquad (7.51)$$
$$= R^{(1)\mu}{}_\mu(\delta^2 h_{\alpha\beta}) + R^{(2)\mu}{}_\mu(\delta h_{\alpha\beta}) - \delta h^{\mu\nu}\underbrace{R^{(1)\mu}{}_\mu(\delta h_{\alpha\beta})}_{=0}$$
$$= R^{(1)\mu}{}_\mu(\delta^2 h_{\alpha\beta}) + R^{(2)\mu}{}_\mu(\delta h_{\alpha\beta})$$

Finally using the definition $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and eqs. (7.50 - 7.51) we find the expression for the perturbed Einstein tensor to second order:

$$\delta^2 G_{\mu\nu} = \delta^2 R_{\mu\nu} - \frac{1}{2} \delta h_{\mu\nu} \underbrace{\delta R}_{=0} - \frac{1}{2} \eta_{\mu\nu} \delta^2 R$$

(7.52) $$\delta^2 G_{\mu\nu} = R^{(1)}_{\mu\nu}(\delta^2 h_{\alpha\beta}) - \frac{1}{2} \eta_{\mu\nu} R^{(1)\mu}{}_{\mu}(\delta^2 h_{\alpha\beta}) + R^{(2)}_{\mu\nu}(\delta h_{\alpha\beta}) - \frac{1}{2} \eta_{\mu\nu} R^{(2)\mu}{}_{\mu}(\delta h_{\alpha\beta})$$

The vacuum Einstein eq. expanded to second order then yields:

(7.53) $$\delta^2 G_{\mu\nu} = 0$$

$$\Leftrightarrow \quad R^{(1)}_{\mu\nu}(\delta^2 h_{\alpha\beta}) - \frac{1}{2} \eta_{\mu\nu} R^{(1)\mu}{}_{\mu}(\delta^2 h_{\alpha\beta}) = -\left(R^{(2)}_{\mu\nu}(\delta h_{\alpha\beta}) - \frac{1}{2} \eta_{\mu\nu} R^{(2)\mu}{}_{\mu}(\delta h_{\alpha\beta})\right)$$

where $\Box \delta h_{\mu\nu} = 0$

(Since $G_{\mu\nu} = 0 = 0 + \delta G_{\mu\nu} + \delta^2 G_{\mu\nu}$ and $\delta G_{\mu\nu} = 0$ for $\Box \delta h_{\mu\nu} = 0$)

The LHS has the form of the first order perturbation $\delta G_{\mu\nu}$ but with $\delta^2 h_{\mu\nu}$ instead of $\delta h_{\mu\nu}$ as the argument. The RHS is quadratic in $\delta h_{\mu\nu}$ and acts as the source for $\delta^2 h_{\mu\nu}$. The second order part $\delta^2 h_{\mu\nu}$ can be thought to arise as a backreaction of the first order part $\delta h_{\mu\nu}$ to the spacetime dynamics. Since we are in vacuum this is purely through gravitational self interactions (terms $(\delta h_{\mu\nu})^2$ arise from $(\delta h_{\mu\nu})^3$ in the action).



Defining:

(7.54) $$G^{(1)}_{\mu\nu}(u_{\alpha\beta}) \equiv R^{(1)}_{\mu\nu}(u_{\alpha\beta}) - \frac{1}{2}\eta_{\mu\nu} R^{(1)\mu}{}_{\mu}(u_{\alpha\beta})$$

(7.55) $$8\pi G t_{\mu\nu} \equiv -\left(R^{(2)}_{\mu\nu}(\delta h_{\alpha\beta}) - \frac{1}{2}\eta_{\mu\nu} R^{(2)\mu}{}_{\mu}(\delta h_{\alpha\beta})\right)$$

We can recast eq. (7.53) into the form:

(7.56) $$G^{(1)}_{\mu\nu}(\delta^2 h_{\mu\nu}) = 8\pi G t_{\mu\nu}$$

Here we identify $t_{\mu\nu}$ the RHS as the energy momentum tensor of gravitational waves.

## Radiated energy, part I

For a general energy momentum tensor $T_{00}$ defines the energy density in the fluid rest frame (= no bulk motion) and $T_{0i}$ is the flux of energy in the $x^i$ direction (momentum density).

Taking (7.55) as the definition of GW energy momentum tensor, the total radiated energy can be defined as:

(7.57) $$E = \int_S t_{i0}\, dS^i$$

where $S$ is a timelike surface at $r \to \infty$ define such that all future oriented null rays cross through it (see Wald Chapter 11 for details).

The area element $dS^i$ in (7.57) can be written as:

(7.58) $$dS^i = n^i r^2 d\Omega dt \,, \qquad d\Omega = \sin\theta d\theta d\phi$$

where $n^i$ is the unit normal vector of the surface ($n^i n_i = 1$). We rotate (globally) the crd's s.t. in spherical coordinates $(r, \theta, \phi)$ $n^i$ points along $\hat{e}_r$, the radial unit basis vector:

$$n^i = (1,0,0)$$

Let us investigate separately the various terms entering in (7.57). We start with:

$$\int dS^i R^{(2)}_{0i}(\delta h_{\alpha\beta}) = \int dS^i \Big( \frac{1}{2} \delta h^{\rho\sigma} \partial_0 \partial_i \delta h_{\rho\sigma} + \frac{1}{4} (\partial_0 \delta h_{\rho\sigma}) \partial_i \delta h^{\rho\sigma} + (\partial^\sigma \delta h^\rho{}_i) \partial_{[\sigma} \delta h_{\rho]0}$$
$$- \delta h^{\rho\sigma} \partial_\rho \partial_{(0} \delta h_{i)\sigma} + \frac{1}{2} \delta h^{\rho\sigma} \partial_\sigma \partial_\rho \delta h_{0i} \Big)$$

This can be simplified by doing partial integrations and using that $\delta h_{\mu\nu} \propto \frac{1}{r}$ (see eq. 7.40), e.g.:

$$\int dS^i \frac{1}{4} (\partial_0 \delta h_{\rho\sigma}) \partial_i \delta h^{\rho\sigma} = \frac{1}{4} \Big( \underbrace{\underset{\sim r^2}{\int dS^i} \underset{\sim \frac{1}{r}}{\partial_i} \big( \underset{\sim \frac{1}{r}}{\delta h^{\rho\sigma}} \underset{\sim \frac{1}{r}}{\partial_0 \delta h_{\rho\sigma}} \big)}_{\sim \frac{1}{r} \to 0 \text{ as } r \to \infty} - \int dS^i \delta h^{\rho\sigma} \partial_i \partial_0 \delta h_{\rho\sigma} \Big)$$

$$= -\frac{1}{4} \int dS^i \delta h^{\rho\sigma} \partial_i \partial_0 \delta h_{\rho\sigma}$$

In the same way all other boundary terms vanish and we get:

$$\int dS^i R^{(2)}_{0i}(\delta h_{\alpha\beta}) = \int dS^i \Big( \frac{1}{2}\delta h^{\rho\sigma}\partial_0\partial_i\,\delta h_{\rho\sigma} - \frac{1}{4}\delta h_{\rho\sigma}\partial_0\partial_i\delta h^{\rho\sigma} - \frac{1}{2}\delta h^{\rho}{}_{i}\underbrace{\partial^\sigma\partial_\sigma\,\delta h_{\rho 0}}_{\substack{=0 \text{ as} \\ \Box\,\delta h_{\mu\nu}=0}}$$

$$+ \frac{1}{2}\delta h^{\rho}{}_{i}\underbrace{\partial^\sigma\partial_\rho\,\delta h_{\sigma 0}}_{=\partial_\rho\partial^\sigma\delta h_{\sigma 0}=0 \text{ because } \partial^\mu\delta h_{\mu\nu}=0} - \frac{1}{2}\delta h^{\rho\sigma}\partial_\rho\partial_i\,\delta h_{0\sigma}$$

$$+ \frac{1}{2}\delta h^{\rho\sigma}\partial_\sigma\partial_\rho\,\delta h_{0i}\Big)$$

$$\int dS^i R^{(2)}_{0i}(\delta h_{\alpha\beta}) = \int dS^i \Big( \frac{1}{4}\delta h^{\rho\sigma}\partial_0\partial_i\,\delta h_{\rho\sigma} - \frac{1}{2}\delta h^{\rho\sigma}\partial_\rho\partial_i\,\delta h_{0\sigma}$$

$$+ \frac{1}{2}\delta h^{\rho\sigma}\partial_\sigma\partial_\rho\,\delta h_{0i}\Big)$$

By further doing double partial integrations in the last two terms we get:

$$\int dS^i R^{(2)}_{0i}(\delta h_{\alpha\beta}) = \int dS^i \Big( \frac{1}{4}\delta h^{\rho\sigma}\partial_0\partial_i\,\delta h_{\rho\sigma} - \frac{1}{2}\delta h_{0\sigma}\,\partial_i\underbrace{\partial_\rho\,\delta h^{\rho\sigma}}_{=0}$$

$$+ \frac{1}{2}\delta h_{0i}\underbrace{\partial_\sigma\partial_\rho\,\delta h^{\rho\sigma}}_{=0}\Big)$$

$$= \frac{1}{4}\int dS^i\,\delta h^{\rho\sigma}\partial_0\partial_i\,\delta h_{\rho\sigma}$$

By doing one final partial integration we finally get:

(7.59) $$\int dS^i R^{(2)}_{0i}(\delta h_{\alpha\beta}) = -\frac{1}{4}\int dS^i (\partial_i \delta h^{\gamma\sigma})(\partial_0 \delta h_{\gamma\sigma})$$

In a similar way we find (exercise):

(7.60) $$\int dS^i R^{(2)}(\delta h_{\alpha\beta}) = 0$$

Using these in (7.57) we then arrive at the result

$$E = \int_S t_{i0}\, dS^i = \frac{-1}{8\pi G}\int dS^i R^{(2)}_{0i}(\delta h_{\alpha\beta})$$

$$E = \frac{1}{32\pi G}\int dS^i (\partial_i \delta h^{\mu\nu})(\partial_0 \delta h_{\mu\nu})$$

From now on we return to our original notation $h_{\mu\nu} = \delta h_{\mu\nu}$ and rewrite the above result as:

(7.61) $$\boxed{E = \frac{1}{32\pi G}\int dS^i (\partial_i h^{\mu\nu})(\partial_0 h_{\mu\nu})}$$ TT gauge

This the main result of this technical part I.

# Radiated energy, part II

The next step is to substitute (7.40) into (7.61) to obtain the expression for GW energy in terms of the source properties encoded in the quadrupole moment $I_{ij}$ in (7.40). Recall that (7.61) is written in the TT. Therefore we need to convert (7.40) into the TT gauge as well (in the vacuum regime $r \gg L$). To this end we define the projection operator

(7.62) $\quad P_{ij} \equiv \delta_{ij} - n_i n_j \quad$ where $n^i$ points along the GW propagation direction and is normal to the surface $S$ (7.58).

Far away from the source, $r \gg L$, the solution of (7.40) is a plane wave:

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left(A_{\mu\nu}\, e^{-i n_\sigma x^\sigma}\right), \quad \underbrace{\partial^\mu \bar{h}_{\mu\nu} = \mathrm{Re}\left(-i n^\mu A_{\mu\nu}\, e^{-i n_\sigma x^\sigma}\right) = 0}_{\text{Lorenz gauge cond.}}$$

Defining: $\quad \hat{h}_{ij} \equiv \left(P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl}\right) \bar{h}_{kl}$

we get a quantity satisfying (exercise)

$$\partial^i \hat{h}_{ij} = 0, \quad \hat{h}^i{}_i = 0$$

which using (7.37) can be shown to be equivalent to the TT gauge conditions (7.42).

Therefore, we seet that the the TT gauge perturbation can be projected out of $\bar{h}_{\mu\nu}$ simply by:

$$(7.63) \qquad h_{ij}^{TT} = (P_i{}^k P_j{}^l - \tfrac{1}{2} P_{ij} P^{kl}) \bar{h}_{kl}$$

We can then rewrite the Lorenz gauge equation (7.40) in the TT gauge by simply contracting it with $(P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl})$:

$$(7.64) \qquad h_{ij}^{TT}(t,r) = \frac{2G}{r} \frac{d^2}{dt^2} I_{ij}^{TT}(t-r) \quad , \quad r \gg L$$

where now:

$$(7.65) \qquad I_{ij}^{TT} = (P_i{}^k P_j{}^l - \tfrac{1}{2} P_{ij} P^{kl}) I_{ij}$$

In the TT gauge $(I^{TT})^i{}_i = 0$ and instead of $I_{ij}$ we can equivalently use the reduced quadrupole moment

$$(7.66) \qquad J_{ij} \equiv I_{ij} - \tfrac{1}{3} \delta_{ij} \delta^{kl} I_{kl} \qquad\qquad J_{ij}^{TT} = I_{ij}^{TT}$$

The point is that $J_{ij}$ is easier to connect to real life applications, it appears as a coefficient in the (multipole) expansion of the Newtonian potential:

$$\Phi(\bar{r}) = -\frac{GM}{r} - \frac{G}{r^3} \underset{\uparrow}{D_i} x^i - \frac{3G}{2r^5} J_{ij} x^i x^j + \dots$$

dipole $\int T_{00} \, x^i d^3x$

Thus we can recast (7.64) as

$$(7.67) \qquad h_{ij}^{TT}(t,r) = \frac{2G}{r}\frac{d^2}{dt^2} J_{ij}^{TT}(t-r) \quad ,$$

which can be directly substituted into (7.61):

$$E = \frac{G}{8\pi}\int_S d\Omega\, dt\, r^2 \underset{\substack{\uparrow \\ n^i = (0,1,0,0) \\ \uparrow \\ \text{choice of crds, GW} \uparrow\uparrow \hat{e}_r}}{n^i} \partial_i\left(\frac{1}{r}\frac{d^2}{dt^2} J_{TT}^{lm}(t-r)\right)\partial_0\left(\frac{1}{r}\frac{d^2}{dt^2} J_{lm}^{TT}(t-r)\right)$$

$$= \frac{G}{8\pi}\int d\Omega\, dt\, r^2 \underbrace{\partial_r\left(\frac{1}{r}\frac{d^2}{dt^2} J_{TT}^{lm}(t-r)\right)}_{= -\frac{1}{r^2}\frac{d^2}{dt^2} J^{lm} + \frac{1}{r}\underbrace{\frac{\partial}{\partial r}\frac{d^2}{dt^2} J^{lm}(t-r)}_{= -\frac{d^3}{dt^3} J^{lm}(t-r) \text{ as } \frac{\partial}{\partial r} f(t-r) = -\frac{\partial}{\partial t} f(t-r)}} \frac{1}{r}\frac{d^3}{dt^3} J_{lm}^{TT}(t-r)$$

$$= \frac{G}{8\pi}\int d\Omega\, dt\, r^2 \underbrace{\left(-\frac{1}{r^2}\frac{d^2}{dt^2} J^{lm} - \frac{1}{r}\frac{d^3}{dt^3} J^{lm}\right)}_{\simeq -\frac{1}{r}\frac{d^3}{dt^3} J^{lm} \quad \text{as } r \to \infty} \frac{1}{r}\frac{d^3}{dt^3} J_{lm}$$

$$= -\frac{G}{8\pi}\int d\Omega\, dt\, r^2 \frac{1}{r^2}\left(\frac{d^3}{dt^3} J_{TT}^{lm}(t-r)\right)\left(\frac{d^3}{dt^3} J_{lm}^{TT}(t-r)\right)$$

$$= -\frac{G}{8\pi}\int_S d\Omega\, dt \left(\frac{d^3}{dt^3} J_{TT}^{lm}(t-r)\right)\left(\frac{d^3}{dt^3} J_{lm}^{TT}(t-r)\right)$$

Defining the power $P$ of the radiated GW's as

$$E \equiv \int P \, dt \tag{7.68}$$

we find from above

$$\boxed{P = -\frac{G}{8\pi} \int_S d\Omega \left( \frac{d^3}{dt^3} J^{lm}_{TT}(t-r) \right) \left( \frac{d^3}{dt^3} J^{TT}_{lm}(t-r) \right)} \tag{7.69}$$

As the final step, let us express this in terms of our original Lorenz gauge quantity $J_{ij}$ instead of $J^{TT}_{ij}$.

From the relations (exercise):

$$J^{TT}_{ij} = \left( P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl} \right) J_{kl}$$

$$P_i{}^k P_j{}^l J^{ij}_{TT} = J^{kl}_{TT}$$

$$P_{ij} J^{ij}_{TT} = 0$$

it follows that:

$$\begin{aligned}
J^{ij}_{TT} J^{TT}_{ij} &= J^{kl}_{TT} J_{kl} \\
&= \left( P^{ki} P^{lj} - \frac{1}{2} P^{kl} P^{ij} \right) J_{ij} J_{kl} \\
&= J^{kl} J_{kl} - J^k{}_j J_{kl} n^l n^j - J_i{}^l J_{kl} n^k n^i + n^k n^i n^l n^j J_{ij} J_{kl} - \frac{1}{2} P^{kl} J_{kl} P^{ij} J_{ij} \\
&= J^{kl} J_{kl} - 2 J_i{}^l J_{kl} n^k n^i + n^k n^i n^l n^j J_{ij} J_{kl} - \frac{1}{2} \left( J_k{}^k - n^k n^l J_{kl} \right) \left( J^i{}_i - n^i n^j J_{ij} \right) \\
&= J^{kl} J_{kl} - 2 J_i{}^l J_{kl} n^k n^i - \frac{1}{2} J^2 + J n^i n^j J_{ij} + \frac{1}{2} n^i n^j n^k n^l J_{kl} J_{ij} \\
&= J^{kl} J_{kl} - 2 J_i{}^l J_{kl} n^k n^i + \frac{1}{2} n^i n^j n^k n^l J_{kl} J_{ij}
\end{aligned} \tag{7.70}$$

where in the last step we used $J \equiv J^j{}_j = 0$ which follows from the definition (7.66).

(7.70) Holds equally for $\frac{d^3}{dt^3}\ddot{J}^{ij}_{TT}$ $\frac{d^3}{dt^3}J^{TT}_{ij}$ and we can the directly recast (7.69) as

$$(7.71) \quad P = -\frac{G}{8\pi}\int_S d\Omega \left( \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J_{ij}}{dt^3} - 2\frac{d^3 J_i{}^l}{dt^3}\frac{d^3 J_{kl}}{dt^3} n^k n^i + \frac{1}{2} n^i n^j n^k n^l \frac{d^3 J_{kl}}{dt^3}\frac{d^3 J_{ij}}{dt^3} \right)$$

Here all $J_{ij} = J_{ij}(t-r)$ so they come out of the $\int d\Omega$ integrals, e.g.

$$\int d\Omega \left( \frac{d^3 J_{ij}(t-r)}{dt^3} \frac{d^3 J_{ij}(t-r)}{dt^3} \right) = \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J_{ij}}{dt^3}\int d\Omega$$

$$= 4\pi \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J_{ij}}{dt^3}$$

For the next two terms we use cartesian crd's where the spatial components of $n^i$ are given by $n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and:

$$\int d\Omega\, n_i n_j \propto \delta_{ij} \quad , \quad i=j=3: \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \cos^2\theta = 2\pi \cdot \frac{2}{3} = \frac{4\pi}{3}$$

and others give the same by symmetry

$$\Rightarrow \int d\Omega\, n_i n_j = \frac{4\pi}{3}\delta_{ij}$$

Similarly one finds:

$$\int d\Omega \, n_i n_j n_k n_l = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

so that

$$\int d\Omega \, \frac{d^3 J_i{}^l}{dt^3} \frac{d^3 J_{kl}}{dt^3} n^k n^i = \frac{4\pi}{3} \frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3}$$

$$\left.\int d\Omega \, n^i n^j n^k n^l \frac{d^3 J_{kl}}{dt^3} \frac{d^3 J_{ij}}{dt^3}\right) = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \frac{d^3 J^{kl}}{dt^3} \frac{d^3 J^{ij}}{dt^3}$$

$$= \frac{4\pi}{15}\left(\underbrace{\frac{d^3 J}{dt^3}\frac{d^3 J}{dt^3}}_{=0 \text{ as } J=0} + 2\frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3}\right)$$

Using these in (7.71) we finally arrive at the result:

$$P = \underbrace{-\frac{G}{8\pi}\left(4\pi - \frac{2}{3}\cdot 4\pi + \frac{1}{2}\frac{2}{15}4\pi\right)}_{} \frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3}$$

$$= \frac{G}{2}\left(\frac{1}{3} + \frac{1}{15}\right) = \frac{G}{2}\frac{6}{15} = \frac{G}{5}$$

(7.72)

$$\boxed{P = -\frac{G}{5}\frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3} \qquad J^{ij} = J^{ij}(t-r), \quad r \to \infty}$$

This is the main result of this section, it gives the power of gravitational wave emission far away from the source characterised by the reduced quadrupole moment $J_{ij}$.