# 6. Schwarzschild black holes

Up to this point we investigated the Schwarzschild spacetime outside the Schwarzschild radius $r > 2GM$ where the metric is

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\underbrace{(d\theta^2+\sin^2\theta\, d\phi^2)}_{\equiv d\Omega^2}$$

In these coordinates there is an apparent singularity:

$$g_{rr}\Big|_{r=2GM} = \infty$$

This is however a coordinate effect which can be avoided by switching to another set of crd's. Indeed for example the quantity:

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\Big|_{r=2GM} = 48\frac{G^2M^2}{r^6}\Big|_{r=2GM} = \frac{3}{4}(GM)^{-4} \quad \text{finite}$$

On the other hand:

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \to \infty \quad \text{as} \quad r \to 0$$

This is a crd-independent result and hence the spacetime has a true singularity at $r=0$.

# Extending the Schwarzschild solution

Let us attempt to construct coordinates in which the coordinate singularity at $r = 2GM$ is removed. This task divides into several steps.

**Step 1:**

Consider radial null curves:
$$dt = \pm \frac{dr}{1-\frac{2GM}{r}}$$

$$t = \pm \int \frac{r-2GM+2GM}{r-2GM}\, dr$$

$$t = \pm\left(r + 2GM \ln\left(\frac{r}{2GM}-1\right) + \underbrace{\text{const.}}_{=0}\right)$$

Define:
$$r_*(r) = r + 2GM \ln\left(\frac{r}{2GM}-1\right) \qquad (6.1)$$

Radial light rays correspond to $r_* = \pm t$

Now rewrite the metric in $(t, r_*)$ crd's:

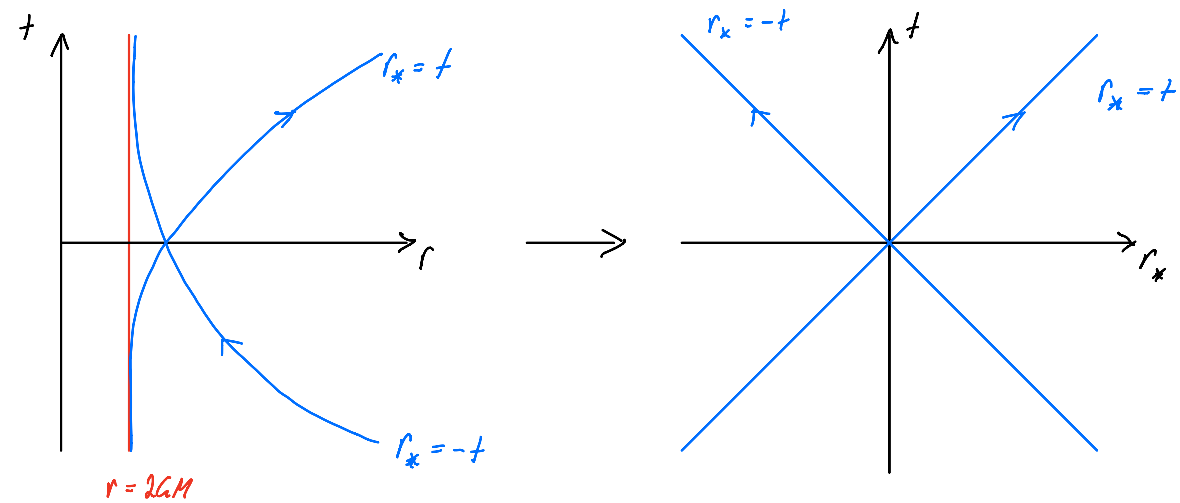
$$(6.2) \quad ds^2 = -\left(1-\frac{2GM}{r}\right)\left(dt^2+dr_*^2\right) + r^2 d\Omega^2$$

since by definition $dr_*^2 = \frac{dr^2}{\left(1-\frac{2GM}{r}\right)^2}$

In these crd's:

$r_*(2GM) = -\infty$ so that
$$\begin{matrix} 2GM < r < \infty \\ -\infty < t < \infty \end{matrix} \longrightarrow \begin{matrix} -\infty < r_* < \infty \\ -\infty < t < \infty \end{matrix}$$

$r=2GM$ is still a singularity in $(t, r_*)$ crd's but the causal structure is easier to analyse in the new crd system.




Step 2:

Next switch to light cone crd's:

(6.3) $\quad v = t + r_* \qquad r_* = v - t \qquad v = \text{const} \Leftrightarrow \text{ingoing ray}$

$\quad u = t - r_* \qquad r_* = -u + t \qquad u = \text{const} \Leftrightarrow \text{outgoing ray}$

and our original $(t,r)$ patch maps to:

$$\begin{matrix} 2GM < r < \infty \\ -\infty < t < \infty \end{matrix} \longrightarrow \begin{matrix} -\infty < u < \infty \\ -\infty < v < \infty \end{matrix}$$

Using $\quad 2r_* = v - u \Rightarrow dr_* = \frac{1}{2}(dv - du)$

$\quad 2t = u + v \Rightarrow dt = \frac{1}{2}(du + dv)$

we get:

(6.4) $$ds^2 = -\left(1-\frac{2GM}{r}\right) du\,dv + r^2 d\Omega^2 \qquad \text{where } r(u,v) = r_*^{-1}\left(\frac{1}{2}(v-u)\right)$$

## Step 3:

In the $(u,v)$ crd's: $t=\infty,\ r=2GM \longrightarrow v=0,\ u=\infty \qquad (r_* = -t)$

$t=-\infty,\ r=2GM \longrightarrow u=0,\ v=-\infty \qquad (r_* = t)$

Next map these to finite values & preserve the light cone structure by defining new crd's $(u',v')$:

$$(6.5) \qquad v' = e^{v/4GM} \qquad \text{s.t.} \quad v=0,\ u=\infty \to v'=1,\ u'=0$$

$$u' = -e^{-u/4GM} \qquad\qquad u=0,\ v=-\infty \to v'=0,\ u'=-1$$

$$dv' = \frac{dv}{4GM} e^{v/4GM} = \frac{v'}{4GM} dv$$

$$du' = \frac{1}{4GM} e^{-u/4GM} du = -\frac{u'}{4GM} du$$

$$dudv = -\frac{16G^2M^2}{u'v'} du'dv'$$

$$= (4GM)^2 e^{(u-v)/4GM} du'dv' \qquad \text{from (6.3)} \quad u-v = -2r_*$$

$$= (4GM)^2 e^{-r_*/2GM} du'dv' \qquad r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

$$= (4GM)^2 e^{-\frac{r}{2GM}} \frac{2GM}{r}\left(1 - \frac{2GM}{r}\right)^{-1} du'dv'$$

In $(u',v')$ crd's the metric reads

$$(6.6) \qquad ds^2 = -\frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} du'dv' + r^2 d\Omega^2$$

where $r(u',v')$ determined by $u'v' = -e^{r_*/2GM}$

$$= -e^{r/2GM}\left(\frac{r}{2GM} - 1\right)$$

## Step 4

Finally, the metric (6.6) can be brought back to diagonal form by switching to the so called Kruskal coordinates:

$$(6.7) \qquad T = \frac{1}{2}(v' + u')$$

$$R = \frac{1}{2}(v' - u')$$

Using eqs. (6.5), (6.3) and (6.1) we can relate $(T,R)$ to $(t,r)$ as:

$$T = \frac{1}{2}\left(e^{v/4GM} - e^{-u/4GM}\right) \qquad \begin{aligned} v &= t + r_* \\ u &= t - r_* \end{aligned}$$

$$= \frac{1}{2} e^{r_*/4GM}\left(e^{t/4GM} - e^{-t/4GM}\right)$$

$$= e^{r_*/4GM} \sinh(t/4GM) \qquad r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

$$(6.8) \qquad \begin{aligned} T &= e^{r/4GM}\left(\frac{r}{2GM} - 1\right)^{1/2} \sinh(t/4GM) \\ R &= e^{r/4GM}\left(\frac{r}{2GM} - 1\right)^{1/2} \cosh(t/4GM) \end{aligned}$$

Using that $\begin{aligned} T + R &= v' \\ T - R &= u' \end{aligned} \Rightarrow dv'du' = dT^2 - dR^2$ we can rewrite the metric (6.6) as

$$(6.9) \qquad ds^2 = \frac{32 G^3 M^3 e^{-r/2GM}}{r}\left(-dT^2 + dR^2\right) + r^2 d\Omega^2 \, ,$$

where $r = r(T,R)$ is determined by (6.8)

$$(6.10) \qquad T^2 - R^2 = u'v' = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$$

It is already quite evident that (6.9) remains non-singular at $r=2GM$. Let us study the causal structure closer.

Denote the original $(t,r)$ patch by:

Region I: $\quad 2GM < r < \infty$

$\quad -\infty < t < \infty$

How does this map into $(T,R)$ crd's?

1) Radial null curves $ds^2=0$ correspond to $T=\pm R + \text{const.}$

Incoming ray: $v'=\text{const.}$, $\quad T = v' - R$

Outgoing ray: $u'=\text{const.}$, $\quad T = u' + R$

2) The Schwarzschild radius corresponds to:

$r = 2GM \Rightarrow T = \pm R$ (using (6.10))

3) Surfaces $r=\text{const.}$ correspond to:

$r = \text{const.} \Rightarrow T^2 - R^2 = \text{const.}$ (using (6.10))

4) Surfaces $t=\text{const.}$ correspond to:

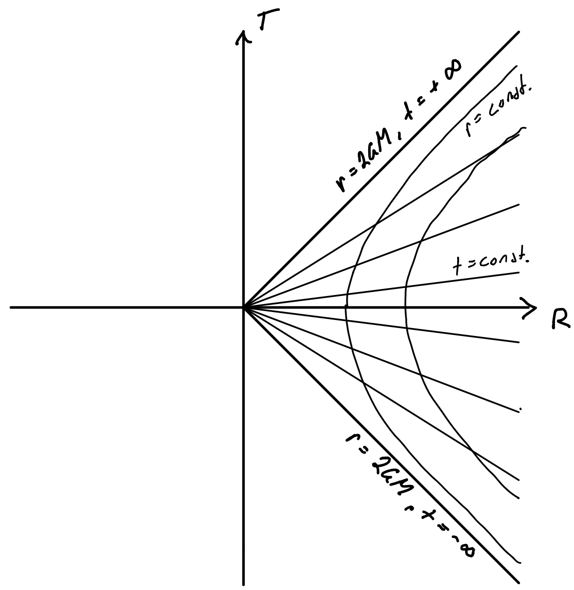
$$\frac{T}{R} = \frac{\sinh(t/4GM)}{\cosh(t/4GM)} = \tanh(t/4GM) \quad \text{(using (6.8))}$$

From 4): $t = \pm\infty \rightarrow T = \pm R$
2): $r = 2GM \rightarrow T = \pm R$

$\Rightarrow$ $r = 2GM,\ t = \pm\infty \rightarrow T = \pm R$
$r = 2GM,\ t < \infty \rightarrow T = 0,\ R = 0$

$\Rightarrow$ Region I maps to

$$-R \leq T \leq R$$
$$R > 0$$




Kruskal diagram of Region I

The metric in $(T,R)$ coordinates is non-singular at $r \to 2GM$ corresponding to $T \to \pm R$. Therefore, there is no reason to expect that the spacetime would end at $T = \pm R$. Thus we can use (6.9) to define the maximal extension of the Schwarzschild solution as:

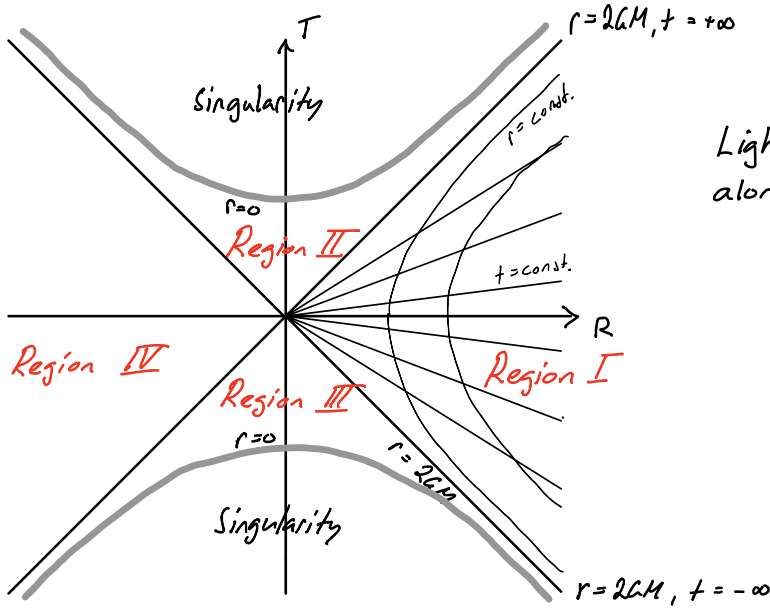
(6.11) $$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}\left(-dT^2 + dR^2\right) + r^2 d\Omega^2$$

with $$R^2 - T^2 = \left(\frac{r}{2GM} - 1\right)e^{r/2GM} > -1$$

and $$-\infty < R < \infty$$

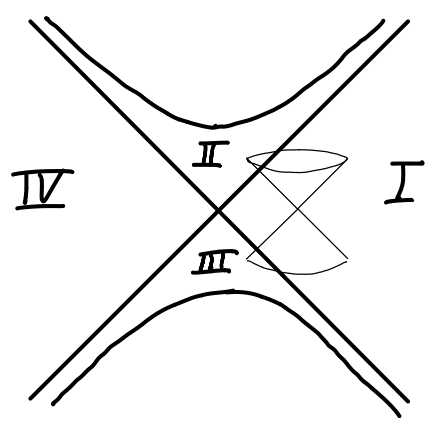
There is a true singularity at $R^2 - T^2 = -1$ corresponding to $r = 0$

The causal structure of the maximally extended solution is conveniently described by the Kruskal diagram:

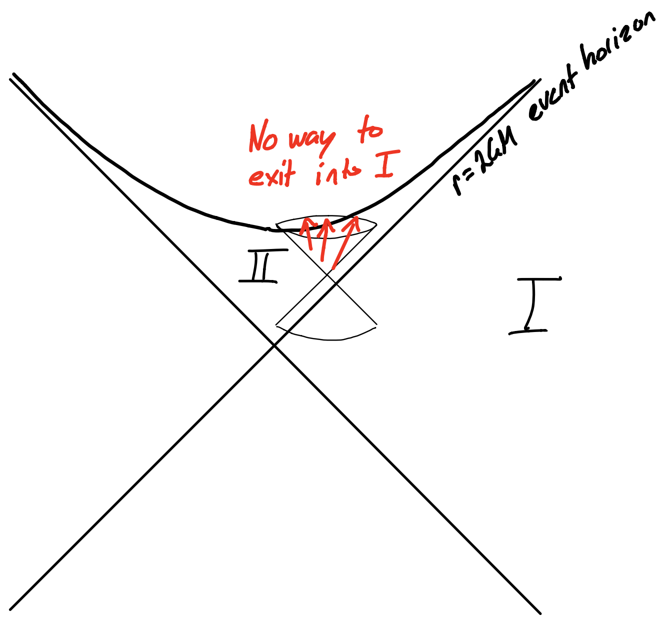



Light rays move along 45° curves

Region I: This is our original patch $r>2GM$, $t\in ]-\infty,\infty[$.
Future null rays reach II and past null rays originate from III
Region IV can be reached only by following spacelike curves.



Region II: This is the black hole interior $r < 2GM$ that can be reached from region I. The Schwarzschild radius $r = 2GM$ is the event horizon. There is no way to exit from region II (except spacelike curves) and all future directed paths will inevitably hit the singularity at $r = 0$. There is no way to escape this fate just like there is no way to prevent yourself moving onwards in time direction (i.e. aging) in the flat space.




By defining:

$$\tilde{t} \equiv 4GM \operatorname{artanh} \frac{T}{R}, \quad \left(\frac{\tilde{r}}{2GM} - 1\right) e^{\tilde{r}/2GM} \equiv R^2 - T^2$$

the metric (6.11) in the region II becomes:

$$ds^2 = -\left(\frac{2GM}{\tilde{r}} - 1\right)^{-1} d\tilde{r}^2 + \left(\frac{2GM}{\tilde{r}} - 1\right) d\tilde{t}^2 + \tilde{r}^2 d\Omega^2,$$

which is the same as (5.7) but with opposite signs for $dr$ and $dt$. Inside the Schwarzschild radius $\tilde{r}=r$ is the temporal crd and $\tilde{t}=t$ spatial crd $\Rightarrow$ spacetime not static.

At $r=0$ there is a true singularity and the curvature ($R_{\alpha\beta\gamma\delta}$) of the spacetime becomes infinity. This is a solution of GR but physically we expect that GR should be modified close to $r=0$ (by quantum gravity effects) so that the singularity gets removed.

Although you cannot come out from inside $r<2GM$, black holes are still not entirely black. If you investigate the behaviour of quantum field theories in classical curved spacetime (gravity not quantized) you find that the presence of the event horizon leads to particle production. Schwarzschild black holes emit black body radiation with the temperature: $T_H = \dfrac{1}{8\pi GM}$ This is called Hawking radiation.

Another this is that the supermassive BH's in the galactic centers are surrounded by accretion disks ($r>2GM$) where the matter falling into the BH's is moving at $v\sim c$. The collisions of the UR particles lead to massive emission of high energy photons and this radiation is seen. The radiation of accretion discs typically exceeds the emission power of the rest of galaxy by orders of magnitude!

Region III : This region is called white hole : all future directed null curves exit this regime but you cannot enter it from outside.

Region IV: This is an asymptotically flat region of spacetime which cannot be reached from I by null or timelike curves.

Regions III and IV exist in the maximally extended Schwarzschild solution where the spacetime is completely empty of matter except for the singularity. In the real world there is always matter sufficiently far away from a star that collapses to form a black hole. Consequently, regions III and IV might just be curious solutions of GR which are not realised anywhere in our universe. However, collapsing spherically symmetric stars do form Schwarzschild black holes and the spacetime around them is described by regions I and II.

The Schwarzschild black hole is the simplest possible case. More generally, black holes can have electric charge and angular momentum, see e.g. Carroll for details