5. The Schwarzschild solution

The Einstein eqs. (4.8) are set of 10 coupled, non-linear 2nd order partial differential equations for $g_{\mu\nu}$ $\Rightarrow$ very hard to solve in general.
The problem simplifies significantly if we concentrate on spacetimes with certain symmetries. The strategy is to make an Ansatz for $g_{\mu\nu}$ consistent with the symmetries and then substitute it into (4.8).

5.1 The Schwarzschild metric

The solution of Einstein equations for an empty spacetime surrounding a spherically symmetric object was found by K. Schwarzschild in 1916. The Schwarzschild sol. is of key importance: it describes the spacetime surrounding stars and it describes the simplest black holes.

Let us derive the Schwarzschild solution. Assume the following:

1) Empty spacetime outside the object $T_{\mu\nu} = 0$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \Leftrightarrow R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

$$T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$$

2) Static system: $\exists$ crd's where $\partial_0 g_{\mu\nu} = 0$ & $g_{0i} = 0$

if $g_{0i} \neq 0$, $ds^2 \supset dt dx$ which is not invariant under $t \to -t$ $\Rightarrow$ spacetime cannot be static

3) Spherical symmetry: $\exists$ crd's where θ, ϕ enter ds^2 through $(d\theta^2 + \sin^2\theta d\phi^2)$, i.e. no terms of type $dr d\theta, dr d\phi, d\theta d\phi$

1,2,3 $\Rightarrow$ Metric must be of the form:

$$(5.1) \quad ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2)$$

The functions $A(r), B(r), C(r)$ are found by substituting the Ansatz (5.1) into the Einstein eqs. (4.8). Before doing this, we can however simplify the form of (5.1) a bit.

Rescale the radial crd:

$$\tilde{r} \equiv \sqrt{C(r)} \qquad d\tilde{r} = \frac{C'(r)}{2\sqrt{C(r)}} dr$$

$$ds^2 = -\underbrace{A(r(\tilde{r}))}_{\equiv \tilde{A}(\tilde{r})} dt^2 + \underbrace{B(r(\tilde{r}))\frac{4C(r(\tilde{r}))}{C'(r(\tilde{r}))^2}}_{\equiv \tilde{B}(\tilde{r})} d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)$$

In order to have the right signature, $ds^2 < 0$ for timelike curves, we need to have:

$$\tilde{A}, \tilde{B} > 0 \Rightarrow \tilde{A}(\tilde{r}) = e^{2\alpha(\tilde{r})} \qquad (\alpha = \tfrac{1}{2}\ln \tilde{A} \in \mathbb{R})$$
$$\tilde{B}(\tilde{r}) = e^{2\beta(\tilde{r})}$$

Rename $\tilde{r} \equiv r$:

$$(5.2) \quad ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The connection coefficients and Ricci tensor of this metric are given by (exercise):

$$(5.3) \quad \Gamma^0_{01} = \alpha'(r) \quad \Gamma^1_{00} = e^{2(\alpha(r)-\beta(r))}\alpha'(r) \quad \Gamma^2_{12} = \frac{1}{r} \quad \Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^1_{11} = \beta'(r) \quad \Gamma^2_{33} = -\sin\theta\cos\theta \quad \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}$$

$$\Gamma^1_{22} = -e^{-2\beta(r)}$$

$$\Gamma^1_{33} = -re^{-2\beta(r)}\sin^2\theta$$

$$(5.4) \quad R_{00} = e^{2(\alpha-\beta)}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2\alpha'}{r}\right) \quad R_{11} = -\alpha'' - \alpha'^2 + \alpha'\beta' + \frac{2\beta'}{r}$$

$$R_{22} = e^{-2\beta}(r(\beta'-\alpha')-1)+1 \quad R_{33} = \sin^2\theta\, R_{22}$$

The Einstein eqs. read:

$$R_{\mu\nu} = (8\pi G)\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$$

Outside the object $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

Using (5.4) $R_{\mu\nu} = 0$ implies:

$$e^{-2(\alpha-\beta)}R_{00} + R_{11} = 0 \qquad \text{as } R_{00} = 0,\ R_{11} = 0$$

$$\cancel{\alpha'' + \alpha'^2 - \alpha'\beta'} + \frac{2\alpha'}{r} \cancel{- \alpha'' - \alpha'^2 + \alpha'\beta'} + \frac{2\beta'}{r} = 0$$

$$\alpha' = -\beta'$$

$$\alpha(r) = -\beta(r) + c_1 \quad \leftarrow \text{const.}$$

Rescale $\tilde{t} = e^{c_1}t$:

$$e^{2\alpha}dt^2 = e^{2(-\beta+c_1)}dt^2 = e^{-2\beta}d\tilde{t}^2$$

Rename $\tilde{t} = t$:

$$(5.5) \quad ds^2 = -e^{-2\beta}dt^2 + e^{2\beta}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Then solve for $\beta(r)$ using $R_{22}=0$:

$$R_{22} = \underbrace{e^{-2\beta}(r\,2\beta'-1)+1}_{=\frac{d}{dr}\left(-re^{-2\beta}\right)} = 0 \quad \Leftrightarrow \quad \frac{d}{dr}\left(r e^{-2\beta}\right) = 1$$

$$\int d\left(re^{-2\beta(r)}\right) = \int dr$$

$$re^{-2\beta(r)} = r - R_s$$

const. of integration

$$e^{-2\beta} = 1-\frac{R_s}{r}$$

Substituting this into (5.5) we get:

$$(5.6) \quad ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

What is the physical role of the integration constant R_s? In the limit $R_s \to 0$ we recover the Minkowski metric. The limit $r \gg R_s$ corresponds to the Newtonian limit where we have shown that:

$$g_{00} = -(1+2\Phi) = -\left(1-\frac{R_s}{r}\right) \quad , \quad r \gg R_s$$

$$\Rightarrow R_s = -2\Phi r$$

In Newton's gravity the solution outside a spherical body of mass M is:

$$\Phi = -\frac{GM}{r} \quad \Rightarrow \quad R_s = 2GM$$

Therefore, the Schwarzschild metric (5.6) becomes:

$$(5.7) \quad \boxed{ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)}$$

The combination $2GM$ defines the Schwarzschild radius of mass M:

$$(5.8) \quad \boxed{R_s = 2GM}$$

This describes the spacetime outside a spherically symmetric static object of mass M.

There is an apparent divergence at $r = 2GM$. However this is just a coordinate effect no component of $R^{\sigma}{}_{\mu\rho\nu}$ diverges at $r = 2GM$. The Schwarzschild radius still has a physical role. If the radius R_0 of the object of mass M is smaller than its Schwarzschild radius $R_0 < 2GM$, the object is a black hole. In this case not even light can escape from the regime $r < 2GM$, the Schwarzschild radius determines the horizon of the black hole. How large is R_s for familiar bodies:

Earth: $M_{earth} \simeq 6 \cdot 10^{24}\,kg \Rightarrow R_s \simeq 0{,}9\,cm$

Sun: $M_\odot \simeq 2{,}0 \cdot 10^{30}\,kg \Rightarrow R_s \simeq 3\,km$

$\Rightarrow$ The Schwarzschild radius of the Earth and the Sun much smaller than their size $R_s \lll R_0$.

When $R_0 > R_s$ we always have $r > R_s$ and the coordinates (5.7) can be used throughout the analysis. In this section we will discuss Schwarzschild solutions where $R_0 > R_s$ and return to black holes in the next section.

5.2 Birkhoff's theorem

It can be shown that the Schwarzschild metric (5.7) is the unique spherically symmetric vacuum solution. This is the Birkhoff's theorem, see e.g. Wald's book for the proof.

Due to the Birkhoff's th. we could have dropped the assumption of static spacetime above; Einstein eqs. impose this. Moreover, this implies that the gravitational field of a collapsing spherically symmetric object is static outside the object and described by the Schwarzschild solution.

5.3 Distances, time intervals, red- and blueshifts

Distances:

On $t=$const. surfaces the line element (5.7) reads:

$$ds^2 = \left(1-\frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Due to sph. symm. we can rotate the crd's s.t. any two points (on t=const. surface) lie on the plane θ = const., ϕ = const.

$$ds^2 = \left(1-\frac{2GM}{r}\right)^{-1} dr^2 > dr^2$$

Proper distance between two points r_1 and r_2 is then:

$$(5.9) \quad s_{12} = \int_{r_1}^{r_2} dr \left(1-\frac{2GM}{r}\right)^{-1/2} = \left(r_2(r_2+2GM)\right)^{1/2} - \left(r_1(r_1-2GM)\right)^{1/2} + 2GM \ln\left(\frac{\sqrt{r_2}+\sqrt{r_2-2GM}}{\sqrt{r_1}+\sqrt{r_1-2GM}}\right)$$

For $r \gg 2GM$ this gives $s_{12} \simeq (r_2 - r_1)$. Therefore, Δr corresponds to distances in the asymptotic flat space limit. Close to the object, gravity stretches the distances since $\Delta s^2 > \Delta r^2$.

Time intervals:

Consider a stationary observer $dr = d\theta = d\phi = 0$. The time measured by her is her proper time

$$d\tau^2 = -ds^2 = \left(1-\frac{2GM}{r}\right) dt^2$$

$$(5.10) \quad d\tau = \sqrt{1-\frac{2GM}{r}}\, dt < dt$$

Again $d\tau \to dt$ as $r \gg 2GM \Rightarrow t$ is the time of asymptotic flat space. Close to the object time slows down $d\tau < dt$, gravity makes clocks tick slower.

Gravitational red- and blueshift

Consider a light ray emitted at $(t_1, r_1, \theta_1, \phi_1)$ and received at $(t_2, r_2, \theta_2, \phi_2)$, where the emitter and receiver are stationary $\frac{dr_i}{dt} = \frac{d\theta_i}{dt} = \frac{d\phi_i}{dt} = 0$.

Light rays move along null curves:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + g_{ij}dx^i dx^j = 0$$

$$dt = \sqrt{\left(1 - \frac{2GM}{r}\right)^{-1} g_{ij} \frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}\, d\sigma$$

receiving time

$$\int_{t_1}^{t_2} dt = \int_{\sigma_1}^{\sigma_2} \sqrt{\left(1 - \frac{2GM}{r}\right)^{-1} g_{ij} \frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}\, d\sigma$$

emission time

$$t_2 - t_1 = \int_{\sigma_1}^{\sigma_2} \sqrt{\left(1 - \frac{2GM}{r}\right)^{-1} g_{ij} \frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}\, d\sigma$$

The paths $x^i(\sigma)$ are the same for any emission/receiving time since we assume stationary source and observer and since the spacetime is static.

Consider two signals emitted at t_1 and $t_1' = t_1 + \Delta t_1$:

$$t_2' - t_1' = t_2 - t_1 \Rightarrow \Delta t_2 = \Delta t_1$$

But this concerns the coordinate time.

In terms of the observer/emitter proper time (5.10):

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \frac{\sqrt{1-\frac{2GM}{r_2}}}{\sqrt{1-\frac{2GM}{r_1}}}\frac{\Delta t_2}{\Delta t_1} = \left(\frac{1-\frac{2GM}{r_2}}{1-\frac{2GM}{r_1}}\right)^{1/2} \quad \text{as } \Delta t_2 = \Delta t_1$$

Correspondingly the relation between the observed and emitted wavelengths and frequencies are given by:

(5.11) $$\frac{\lambda_2}{\lambda_1} = \frac{\Delta\tau_2}{\Delta\tau_1} = \left(\frac{1-\frac{2GM}{r_2}}{1-\frac{2GM}{r_1}}\right)^{1/2}$$

Gravitational red-/blueshift

$r_2 > r_1 \Rightarrow \lambda_2 > \lambda_1$
$r_2 < r_1 \Rightarrow \lambda_2 < \lambda_1$

(5.12) $$\frac{f_2}{f_1} = \frac{\Delta\tau_1}{\Delta\tau_2} = \left(\frac{1-\frac{2GM}{r_1}}{1-\frac{2GM}{r_2}}\right)^{1/2}$$

Photons climb up a "gravitational potential" $\rightarrow$ loose energy $\rightarrow$ redshift

In the asymptotic limit $r \gg 2GM$ these yield:

$$\frac{\lambda_2}{\lambda_1} \simeq 1 + GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad r \gg 2GM$$

$$\frac{f_2}{f_1} \simeq 1 - GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad r \gg 2GM$$

5.4 Geodesics & the Schwarzschild space

To discuss the motion of objects we need to find the geodesics. The Schwarzschild space is spherically symmetric and stationary: the metric is independent of t and ϕ. This independence leads to the existence of two constants of motion related to the symmetries $t \to t + \Delta t$ and $\phi \to \phi + \Delta\phi$.

To find the geodesics we write down the Euler-Lagrange equations for the metric $\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ as we did in section 3.

For the Schwarzschild metric (5.7):

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(-\left(1 - \frac{2GM}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2GM}{r}} + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right),$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$ and τ is the proper time (consider timelike geodesics, i.e. trajectories of massive objects)

Geodesic eq. $\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$

t-component: $\frac{\partial \mathcal{L}}{\partial t} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left(1 - \frac{2GM}{r}\right)\dot{t} \equiv -k$

$$\frac{d}{d\tau}\left(-\left(1 - \frac{2GM}{r}\right)\dot{t}\right) = 0 \Leftrightarrow \frac{dk}{d\tau} = 0 \quad \text{constant of motion}$$

ϕ-component:

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \sin^2\theta\,\dot{\phi} \equiv h$$

$$\frac{d}{d\tau}\left(r^2 \sin^2\theta\,\dot{\phi}\right) = \frac{dh}{d\tau} = 0 \quad \text{constant of motion}$$

θ-component:

$$\frac{\partial \mathcal{L}}{\partial \theta} = r^2 \sin\theta\cos\theta\,\dot{\phi}^2 \quad , \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\dot{\theta}$$

$$(r^2\dot{\theta})^{\cdot} - r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} - r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\theta} + \frac{2\dot{r}}{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

r-component:

$$\frac{\partial \mathcal{L}}{\partial r} = -\frac{GM}{r^2}\dot{t}^2 - \frac{1}{2}\left(1-\frac{2GM}{r}\right)^{-2}\left(+\frac{2GM}{r^2}\right)\dot{r}^2 + r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\dot{r}}{1-\frac{2GM}{r}}$$

$$\frac{\ddot{r}}{1-\frac{2GM}{r}} - \frac{\dot{r}^2}{\left(1-\frac{2GM}{r}\right)^2}\frac{2GM}{r^2} + \frac{GM}{r^2}\dot{t}^2 + \frac{GM}{r^2}\frac{\dot{r}^2}{\left(1-\frac{2GM}{r}\right)^2} - r\left(\dot{\theta}^2 + \sin^2\phi\,\dot{\phi}^2\right) = 0$$

$$\ddot{r} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\dot{t}^2 - \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 - r\left(1-\frac{2GM}{r}\right)\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = 0$$

At any given time we can rotate the coordinates s.t. $\theta = \frac{\pi}{2}$ due to the rotational symmetry of the system. Then:

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\frac{\pi}{2}\cos\frac{\pi}{2}\,\dot{\phi}^2 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0 \implies \theta = \text{const.}$$

Therefore we can choose the coordinates s.t. $\theta(\tau) = \frac{\pi}{2}$ and the above set of geodesic eqs. becomes:

$$(5.13) \qquad \left(1-\frac{2GM}{r}\right)\dot{t} = k = \text{const.}$$

$$(5.14) \qquad r^2\dot{\phi} = h = \text{const.}$$

$$(5.15) \qquad \ddot{r} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\dot{t}^2 - \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 - r\left(1-\frac{2GM}{r}\right)\dot{\phi}^2 = 0$$

In addition to the geodesic eqs. there is a relation that follows directly from the definition of the proper time:

$$d\tau^2 = -ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)^{-1}dr^2 - r^2 d\phi^2$$

$$\Rightarrow \quad \left(1-\frac{2GM}{r}\right)\dot{t}^2 - \left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 1 \qquad (5.16)$$

For massive test particles we can define:

(5.16) $E \equiv mk = \left(1-\frac{2GM}{r}\right)m\dot{t} \equiv$ conserved energy

These definitions are justified below.

(5.17) $L \equiv mh = mr^2\dot{\phi} \equiv$ conserved angular momentum

Note that $E \neq -u^\mu p_\mu = E_{obs}$ ← energy measured by an observer with the 4-velocity u^μ :

Take a stationary observer $u^i = 0$: $d\tau^2 = -g_{00}dt^2$

$$u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{-g_{00}}}$$

$$u_0 = g_{00}u^0 = -\sqrt{-g_{00}} = -\sqrt{1-\frac{2GM}{r}}$$

$$E_{obs} = -u_\mu p^\mu = -u_0 p^0 = \sqrt{1-\frac{2GM}{r}}\, m\dot{t} \neq E \quad , \quad p^\mu = m\frac{dx^\mu}{d\tau}$$

E_{obs} ↑ includes only non-gravitational energy = mass + kinetic energy

E ↑ includes all energy

Using the conserved quantities E and L, we can rewrite eq. (5.16) as:

$$\frac{E^2}{m^2} - \dot{r}^2 - \left(1-\frac{2GM}{r}\right)\frac{L^2}{m^2r^2} = \left(1-\frac{2GM}{r}\right) \quad \Big/ \cdot \frac{m}{2}$$

$$(5.18) \qquad \underbrace{\frac{E^2}{2m}}_{\equiv \tilde{E}} = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{GML^2}{mr^3} + \frac{m}{2}}_{\equiv V(r)}$$

Written in this way (5.18) looks just like the energy conservation equation of Newtonian physics for a particle with energy $\tilde{E}$. This is an analogy that can be used to understand the motion. The effective potential $V(r)$ differs from the Newtonian result by a trivial constant $\frac{m}{2}$ and by the term $-\frac{GML^2}{mr^3}$. This attractive term is a pure GR effect and it gives rise to precession of planetary orbits.

For massless test particles $m=0$, we choose to parametrise the geodesics by λ normalised such that:

$$p^\mu = \frac{dx^\mu}{d\lambda} \qquad \left(\text{dimensions } [\lambda] = m^{-2}\right)$$

The equations for null geodesics are then given by:

$$\text{(5.19)} \qquad \left(1-\frac{2GM}{r}\right)\dot{t} \equiv E = \text{const.} \qquad\qquad \dot{} \equiv \frac{d}{d\lambda}$$

$$\text{(5.20)} \qquad r^2\dot{\phi} \equiv L = \text{const.}$$

$$\text{(5.21)} \qquad \ddot{r} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\dot{t}^2 - \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 - r\left(1-\frac{2GM}{r}\right)\dot{\phi}^2 = 0$$

and $ds^2 = 0$ yields:

$$\text{(5.22)} \qquad \left(1-\frac{2GM}{r}\right)\dot{t}^2 - \left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

which using (5.19) and (5.20) can be recast as:

$$E^2 - \dot{r}^2 - \left(1-\frac{2GM}{r}\right)r^2\dot{\phi}^2 = 0$$

$$E^2 - \dot{r}^2 - \left(1-\frac{2GM}{r}\right)\frac{L^2}{r^2} = 0$$

$$E^2 = \dot{r}^2 + \frac{L^2}{r^2} - \frac{2GML^2}{r^3}$$

5.5 Motion in the Schwarzschild space

Using the geodesic eqs. we can now investigate the motion of massive and massless test particles. We concentrate on a few special cases.

Vertical fall

For vertical fall $\phi = $ const. Concentrating on massive test particles and setting $\dot{\phi} = 0$ in eq. (5.16) we get:

$$\left(1-\frac{2GM}{r}\right)\dot{t}^2 - \left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 = 1$$

$$\underbrace{\left(1-\frac{2GM}{r}\right)^2\dot{t}^2}_{= k^2 \text{ (eq. 5.13)}} = 1-\frac{2GM}{r} + \dot{r}^2$$

$$k^2 = \underset{\text{kinetic part}}{\dot{r}^2} + \underset{\text{potential part}}{\left(1-\frac{2GM}{r}\right)}$$

From eq. (5.16): $E = mk$ $\qquad k < 1 \Rightarrow E < m$ bound particle

$\qquad k \geq 1 \Rightarrow E \geq m$ unbound particle

Consider vertical fall starting from rest: $r(\tau_0) = r_0$, $\dot{r}(\tau_0) = 0$

$$k^2 = \dot{r}^2 + 1 - \frac{2GM}{r} = 1 - \frac{2GM}{r_0} < 1 \qquad \text{bound particle}$$

$$\dot{r}^2 = 1 - \frac{2GM}{r_0} - \left(1 - \frac{2GM}{r}\right)$$

$$\frac{1}{2}\dot{r}^2 = GM\left(\frac{1}{r} - \frac{1}{r_0}\right) \qquad \Big| \frac{d}{d\tau}$$

$$\ddot{r} = -\frac{GM}{r^2}$$

looks like the Newtonian equation but recall that $\dot{} \equiv \frac{d}{d\tau}$ and $d\tau \neq dt$.

Compute the proper time that the observer measures when falling down to some

$r < r_0$:

$$d\tau = \frac{d\tau}{dr}\, dr = \frac{dr}{\dot r}$$

$$\int_{\tau_0}^{\tau_r} d\tau = \int_{r_0}^{r} \frac{dr}{\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)}}$$

$$\tau_r - \tau_0 = \int_{r_0}^{r} \frac{dr}{\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)}}$$

What is the proper time required to reach the Schwarzschild radius* $r = 2GM$?

Using $k^2 = 1 - \frac{2GM}{r_0} \Rightarrow r_0 = \frac{2GM}{1-k^2}$ we can rewrite:

$$\tau_r - \tau_0 = \int_{r_0}^{r} \frac{dr}{\sqrt{\underbrace{\frac{2GM}{r} - 1 + k^2}_{\to\, 1-1+k^2 \text{ as } r \to 2GM}}} < \infty \quad \text{as} \quad r \to 2GM$$

$\Rightarrow$ It takes a finite proper time to reach the Schwarzschild radius

But the corresponding coordinate time t = time measured by an observer at $r = \infty$ diverges:

$$dt = \frac{dt}{d\tau}\, d\tau$$

eq. (5.13): $\left(1 - \frac{2GM}{r}\right)\dot t = k$

$$dt = \frac{k}{1 - \frac{2GM}{r}}\, d\tau$$

$k^2 = 1 - \frac{2GM}{r_0}$

$$= \frac{\sqrt{1 - \frac{2GM}{r_0}}}{1 - \frac{2GM}{r}}\, \frac{d\tau}{dr}\, dr$$

$\frac{1}{2}\dot r^2 = GM\left(\frac{1}{r} - \frac{1}{r_0}\right) \Rightarrow \dot r = -\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)}$

$\frac{dr}{d\tau} < 0$

$$= -\frac{\sqrt{1 - \frac{2GM}{r_0}}}{1 - \frac{2GM}{r}}\, \frac{dr}{\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)}}$$

* We assume the Schwarzschild radius is outside the object, so the object is a black hole.

$$\int_{t_0}^{t} dt = -\int_{r_0}^{r} \frac{\sqrt{r_0 - 2GM}}{\sqrt{2GM}} \, \frac{r^{3/2}}{(r-2GM)(r_0-r)^{1/2}} \, dr$$

Set $r = 2GM(1+\epsilon)$, $\epsilon \to 0$ as the integration limit

$$t - t_0 = -\sqrt{\frac{r_0}{2GM} - 1} \int_{r_0}^{2GM(1+\epsilon)} dr \, \frac{r^{3/2}}{(r-2GM)(r_0-r)^{1/2}}$$

Consider the upper limit:

$$\int^{2GM(1+\epsilon)} dr \, \frac{r^{3/2}}{(r-2GM)(r_0-r)^{1/2}} \xrightarrow[\epsilon \to 0]{} \underbrace{\frac{(2GM)^{3/2}}{r_0^{1/2}}}_{(r_0 \gg 2GM)} \int^{2GM(1+\epsilon)} \frac{dr}{r-2GM} \propto \ln(2GM\epsilon) \to \infty \text{ as } \epsilon \to 0$$

$\Rightarrow$ $t \to \infty$ as $r \to 2GM$; an asymptotic observer never sees the object reaching $R_s = 2GM$.

The signal becomes infinitely redshifted as the object approaches $R_s = 2GM$:

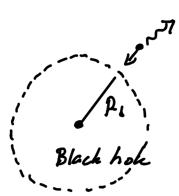
$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{1 - \frac{2GM}{r_{obs}}}{1 - \frac{2GM}{r_{em}}} = \frac{1}{1 - \frac{2GM}{r_{em}}} \to \infty \quad \text{as } r_{em} \to 2GM$$

$r_{obs} = \infty$

$r_{obs} \ggg 2GM$

$\lambda_{obs} \to \infty$

$f_{obs} \to 0$ signals received at infinitely long intervals



Circular orbit

For circular orbits $r = \text{const}$ eq. (5.15) becomes:

$$\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\dot{t}^2 - r\left(1-\frac{2GM}{r}\right)\dot{\phi}^2 = 0$$

$$\frac{GM}{r^3}\dot{t}^2 = \dot{\phi}^2$$

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r^3} = \text{const.}$$

$$\Rightarrow \quad \phi(t) = \sqrt{\frac{GM}{r^3}}\, t$$

Therefore the coordinate time to complete an orbit $\Delta\phi = 2\pi$ is:

$$\Delta t = \sqrt{\frac{r^3}{GM}}\, 2\pi$$

$$(\Delta t)^2 = \frac{4\pi^2 r^3}{GM}$$

Again this looks like the Newtonian result (Kepler's law) but differences arise due to $\Delta t \neq \Delta\tau$

The orbit time measured by an observer at fixed r_0 (not a freely falling observer, force needed to stay stationary at $r=r_0$):

$$\Delta\tau_{r_0} = \sqrt{1-\frac{2GM}{r_0}}\,\Delta t < \Delta t$$

$$\Delta\tau_{r_0} = \sqrt{1-\frac{2GM}{r_0}}\left(\frac{4\pi^2 r^3}{GM}\right)^{1/2}$$

The orbit time measured by an observer living on the orbiting planet:

$$d\tau^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \underbrace{r^2 d\phi^2}_{=\frac{GM}{r^3} r^2 dt^2} = \left(1-\frac{3GM}{r}\right)dt^2$$

$$\Delta\tau_{obs} = \left(1-\frac{3GM}{r}\right)^{1/2}\left(\frac{4\pi^2 r^3}{GM}\right)^{1/2} = 2\pi\left(\frac{r^3}{GM}\left(1-\frac{3GM}{r}\right)\right)^{1/2} \longrightarrow 0 \quad \text{as } r \to 3GM$$

Circular orbits of freely falling objects not possible for $r < 3GM$. A circular path with $r < 3GM$ requires powered flight.

It is also instructive to compare the orbit times measured by an observer on a planet with the orbit radius r_0 (freely falling object):

$$\Delta\tau_{obs} = 2\pi\left(\frac{r_0^3}{GM}\left(1-\frac{3GM}{r_0}\right)\right)^{1/2}$$

and an astronaut hovering at $r = r_0$, $\phi = \phi_0$ (powered flight)

$$\Delta\tau_a = \sqrt{1-\frac{2GM}{r_0}}\left(\frac{4\pi^2 r_0^3}{GM}\right)^{1/2} = 2\pi\left(\frac{r_0^3}{GM}\left(1-\frac{2GM}{r_0}\right)\right)^{1/2} > \tau_{obs}$$

$\Rightarrow$ the freely falling observer measures a shorter time, geodesics not necessarily global maximals of the proper time.

Consider photons on a circular orbit $r = \text{const}$. The null condition $ds^2=0$ given by eq. (5.22) yields:

$$\left(1-\frac{2GM}{r}\right)\dot{t}^2 = r^2\dot{\phi}^2 \qquad \dot{} \equiv \frac{d}{d\lambda} \; ; \; p^\mu = \frac{dx^\mu}{d\lambda}$$

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{1}{r^2}\left(1-\frac{2GM}{r}\right)$$

The geodesic eq. (5.21) gives

$$\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\dot{t}^2 = r\left(1-\frac{2GM}{r}\right)\dot{\phi}^2$$

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r^3}$$

Combining these two we get:

$$\frac{GM}{r^3} = \frac{1}{r^2}\left(1-\frac{2GM}{r}\right)$$

$$\frac{GM}{r} = 1-\frac{2GM}{r} \quad \Rightarrow r = 3GM$$

Photons orbit at $r = 3GM$, extreme bending light.

For radially moving photons $\dot{\phi}=0$ and eq. (5.22) yields:

$$\left(1-\frac{2GM}{r}\right)\dot{t}^2 = \left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2$$

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2GM}{r}\right) \Rightarrow \frac{dr}{dt} = 1-\frac{2GM}{r} \longrightarrow 0 \text{ as } r \to 2GM$$

(outward motion $\frac{dr}{dt} > 0$)

A radial line of sight never reaches $r = 2GM$, cannot see inside $r = 2GM$

General orbits

In Newtonian gravity, planetary orbits are ellipses, parabolas or hyperbola. In the Schwarzschild solution this is no longer true but there will be GR modifications to the orbits.

To facilitate the discussion of general orbits, let us recollect the the geodesic eqs. together with the $ds^2 < 0$ or $ds^2 = 0$ condition of massive and massless test particles, respectively:

(5.23) $$\left(1-\frac{2GM}{r}\right)\dot{t} = k \qquad \dot{} \equiv \frac{d}{d\lambda} \qquad \begin{cases} \lambda = \tau & m \neq 0 \\ \frac{dx^\mu}{d\lambda} = p^\mu & m = 0 \end{cases}$$

(5.24) $$r^2\dot{\phi} = h$$

(5.25) $$\left(1-\frac{2GM}{r}\right)\dot{t}^2 - \left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \begin{cases} 1 & , m \neq 0 \\ 0 & , m = 0 \end{cases}$$

Using the first two in the last one we get:

$$k^2 - \dot{r}^2 - r^2\left(1-\frac{2GM}{r}\right)\frac{h^2}{r^4} = \begin{cases} 1-\frac{2GM}{r} \\ 0 \end{cases} \qquad \dot{r} = \frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \frac{dr}{d\phi}\dot{\phi} = \frac{dr}{d\phi}\frac{h}{r^2}$$

$$k^2 - \left(\frac{dr}{d\phi}\right)^2\frac{h^2}{r^4} - \frac{h^2}{r^2}\left(1-\frac{2GM}{r}\right) = \begin{cases} 1-\frac{2GM}{r} \\ 0 \end{cases}$$

$$\frac{k^2 r^4}{h^2} - r^2\left(1-\frac{2GM}{r}\right) - \frac{r^4}{h^2}\begin{Bmatrix} 1-\frac{2GM}{r} \\ 0 \end{Bmatrix} = \left(\frac{dr}{d\phi}\right)^2$$

Define a new variable $u = \frac{1}{r}$, $du = -\frac{dr}{r^2} = -u^2 dr \Rightarrow dr = -\frac{du}{u^2}$

$$\left(\frac{du}{d\varphi}\right)^2 = u^4\left(\frac{k^2}{h^2u^4} - \frac{1}{u^2}(1-2GMu) - \frac{1}{u^4h^2}\begin{Bmatrix}1-2GMu\\0\end{Bmatrix}\right)$$

$$= \frac{k^2}{h^2} - u^2(1-2GMu) - \frac{1}{h^2}\begin{Bmatrix}1-2GMu\\0\end{Bmatrix}$$

Massive case $m \neq 0$:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{k^2}{h^2} - \left(u^2+\frac{1}{h^2}\right)(1-2GMu)$$

$$(5.26)\qquad \left(\frac{du}{d\varphi}\right)^2 + u^2 = \underset{\substack{\uparrow\\ \text{const.}}}{\frac{k^2-1}{h^2}} + \frac{2GMu}{h^2} + 2GMu^3 \equiv \underbrace{A + \frac{2GMu}{h^2}}_{\text{Newtonian terms}} + \underbrace{2GMu^3}_{\text{GR correction}}$$

Massless case $m = 0$:

$$(5.27)\qquad \left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{k^2}{h^2} + 2GMu^3 \equiv F + 2GMu^3$$

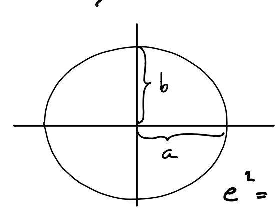
Perihelion precession

For $A < 0$ the Newtonian eq.

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = A + \frac{2GM}{h^2}u$$

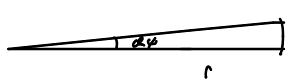
has the elliptic solution (bound system):

$$u = \frac{GM}{h^2}\left(1 + e\cos(\varphi-\varphi_0)\right), \quad e = 1 + \frac{Ah^2}{G^2M^2} \quad \text{eccentricity of the orbit}$$



$$e^2 = 1 - \frac{b^2}{a^2}$$

Planets in the Solar system are moving slowly $v \ll 1$:



$\ell = r\,d\varphi$ $\qquad v^2 \sim r^2\varphi'^2 \ll 1$

The GR term in (5.36) is a small correction:

$$\delta \equiv \frac{2GMu^3}{\left(\frac{2GMu}{h^2}\right)} = h^2u^2 = \frac{h^2}{r^2} = r^2\dot{\varphi}^2 \sim V^2 \ll 1$$

Thus we can expand (5.36) around the Newtonian solution:

$$\left(\frac{du}{d\varphi}\right)^2 = \underbrace{A + \frac{2GMu}{h^2} + 2GMu^3 - u^2}$$

$$= 2GM(u-u_1)(u-u_2)(u-u_3) \quad \text{where } u_i \text{ are roots of}$$

$$= 2GMu^3 - \underbrace{2GM(u_1+u_2+u_3)}_{=1}u^2 + \underbrace{2GM(u_1u_2+u_2u_3+u_3u_1)}_{=\frac{2GM}{h^2}}u \underbrace{-2GMu_1u_2u_3}_{=-A}$$

Here u_i are roots of $\left(\frac{du}{d\varphi}\right)^2 = A + \frac{2GMu}{h^2} + 2GMu^3 - u^2 = 0$, i.e. they correspond to local extremals of the distance $r = \frac{1}{u}$.

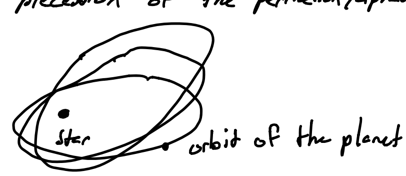
In the Newtonian limit there are only two roots:

$$\left(\frac{du}{d\varphi}\right)^2 = \underbrace{A + \frac{2GMu}{h^2} - u^2}_{= 2GM(u-\bar{u}_1)(u-\bar{u}_2)} = 0$$

which are the perihelion (furthest distance) and aphelion (closest distance) and the orbits are closed ellipses. Since the GR effects are small we can approximate:

$$\underset{\substack{\uparrow \\ \text{perihelion}}}{u_1} = \bar{u}_1(1+\mathcal{O}(\delta)), \quad \underset{\substack{\uparrow \\ \text{aphelion}}}{u_2} = \bar{u}_2(1+\mathcal{O}(\delta))$$

However, now $\left.\left(\frac{du}{d\varphi}\right)^2\right|_{\bar{u}_i} = \mathcal{O}(\delta)$ which causes slight precession of the perihelion/aphelion:



$$\left(\frac{du}{d\varphi}\right) = \left(2GM(u-u_1)(u-u_2)(u-u_3)\right)^{1/2}$$

$$= \left((u-u_1)(u-u_2)\right)^{1/2}\left(2GM\left(u-\left(\frac{1}{2GM}-u_1-u_2\right)\right)\right)^{1/2}, \qquad u_1 \le u \le u_2 \Leftrightarrow r_2 \le r \le r_1$$

$$\left(\frac{du}{d\varphi}\right) = \sqrt{(u-u_1)(u_2-u)}\sqrt{1 - \underbrace{2GM(u+u_1+u_2)}_{= \mathcal{O}\left(\frac{2GM}{r}\right) \ll 1}}$$

$$\simeq \sqrt{(u-u_1)(u_2-u)}\left(1 - GM(u+u_1+u_2)\right)$$

$$\Rightarrow \quad \frac{d\varphi}{du} = \frac{1 + GM(u+u_1+u_2)}{\sqrt{(u-u_1)(u_2-u)}}$$

Integrate this from an aphelion u_2 to the successive perihelion u_1: (Exercise)

$$\varphi_{12} = \int_{u_1}^{u_2} du \, \frac{1+GM(u+u_1+u_2)}{\sqrt{(u-u_1)(u_2-u)}} = \pi + \frac{3\pi}{2} GM(u_1+u_2)$$

The perihelion precession = the deviation of successive perihelia from 2π

$$\Delta\varphi = 2\varphi_{12} - 2\pi = 2\pi + 3\pi GM(u_1+u_2) - 2\pi = 3\pi GM\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

For the planet Mercury we get: $\quad \Delta\varphi = \frac{43{,}0''}{\text{century}}$ (arcsecs)

This matches well with the observed value (from which the $\mathcal{O}(10)$ bigger effects due to other planets are first subtracted).

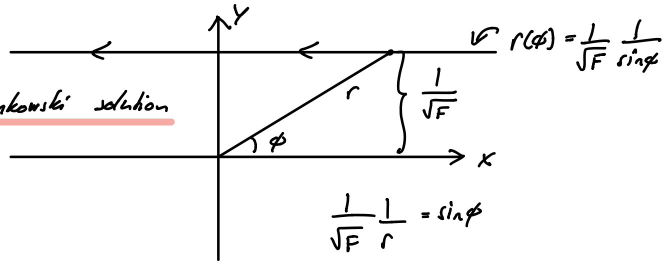
Bending of light

Taking first $M=0$ in eq. (5.27) we get:

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = F \Rightarrow u = \sqrt{F}\sin\phi$$

$$r = \frac{1}{\sqrt{F}\sin\phi}$$

A straight line, the Minkowski solution



GR generates deviations from the $M=0$ solution, causing bending of the light rays.

In the limit $r \gg 2GM$, GR terms are a small correction:

$$\frac{2GMu^3}{u^2} = \frac{2GM}{r} \ll 1$$

small correction

$$\left(\frac{du}{d\phi}\right)^2 = F - u^2(1 - 2GMu)$$

Denote by u_0 the point where the distance is extremised:

$$\left.\left(\frac{du}{d\phi}\right)^2\right|_{\phi_0} = F - u_0^2 + 2GMu_0^3 = 0 \quad \Rightarrow F = u_0^2(1 - 2GMu_0)$$

We can rewrite (5.27) as:

$$\left(\frac{du}{d\phi}\right)^2 = u_0^2(1-2GMu_0) - u^2(1-2GMu)$$

Since $2GMu \ll 1$, the solution should be close to the $M=0$ case.

Make an Ansatz:

$$u(\phi) = u_0(\sin\phi + \underbrace{2GMu_0}_{\equiv \epsilon \ll 1} v(\phi))$$

arbitrary function $\mathcal{O}(1)$

$$u'(\phi) = u_0(\cos\phi + \epsilon v'(\phi))$$

Substitute into eq. (5.27) and linearise in ϵ.

$$u_0^2(\cos\phi+\epsilon v')^2 = u_0^2(1-\epsilon) - u_0^2\underbrace{(\sin\phi+\epsilon v)^2(1-\epsilon(\sin\phi+\epsilon v))}$$

$$(\sin^2\phi + 2\sin\phi\,\epsilon v)(1-\epsilon\sin\phi + \mathcal{O}(\epsilon^2))$$

$$= \sin^2\phi + \epsilon(2v\sin\phi - \sin^3\phi) + \mathcal{O}(\epsilon^2)$$

$$(\cos\phi+\epsilon v')^2 = 1-\sin^2\phi + \epsilon(-1-2v\sin\phi+\sin^3\phi) + \mathcal{O}(\epsilon^2)$$

$$\cos^2\phi + 2\epsilon v'\cos\phi = 1-\sin^2\phi + \epsilon(-1-2v\sin\phi+\sin^3\phi)$$

$$\epsilon\, 2v'\cos\phi = \epsilon(\sin^3\phi - 1 - 2v\sin\phi)$$

$$\underbrace{v'\cos\phi + v\sin\phi}_{=\cos^2\phi\frac{d}{d\phi}\left(\frac{v}{\cos\phi}\right)} = \frac{1}{2}(\sin^3\phi - 1)$$

$$\frac{d}{d\phi}\left(\frac{v}{\cos\phi}\right) = \frac{1}{2\cos^2\phi}\underbrace{(\sin^3\phi - 1)}_{=\sin\phi(1-\cos^2\phi)\ -1}$$

$$= \frac{1}{2}\Big(-\sin\phi + \underbrace{\frac{\sin\phi}{\cos^2\phi}}_{=\frac{-\frac{d}{d\phi}\cos\phi}{\cos^2\phi}} - \underbrace{\frac{1}{\cos^2\phi}}_{=\frac{d}{d\phi}\tan\phi}\Big)$$

$$\frac{d}{d\phi}\left(\frac{v}{\cos\phi}\right) = \frac{1}{2}\left(-\sin\phi - \frac{1}{\cos^2\phi}\frac{d}{d\phi}\cos\phi - \frac{d}{d\phi}\tan\phi\right)$$

$$\int d\left(\frac{v}{\cos\phi}\right) = -\int\frac{1}{2}d\phi\,\sin\phi - \frac{1}{2}\int\frac{d\cos\phi}{\cos^2\phi} - \frac{1}{2}\int d\tan\phi$$

$$\frac{v}{\cos\phi} = \frac{1}{2}\cos\phi + \frac{1}{2}\frac{1}{\cos\phi} - \frac{1}{2}\tan\phi + \underset{\uparrow\ \text{constant}}{B}$$

$$v(\phi) = \frac{1}{2}\left(1+\cos^2\phi - \sin\phi + B\cos\phi\right)$$

$$\Rightarrow\ u(\phi) = u_0\sin\phi + GM u_0^2\left(1+\cos^2\phi - \sin\phi + B\cos\phi\right)$$

Initial conditions:

$x=+\infty$ $\quad r(\phi=0)=\infty \Rightarrow u(0) = \frac{1}{\infty} = 0$

$$u(0) = GMu_0^2(2+B) = 0 \Rightarrow B = -2$$

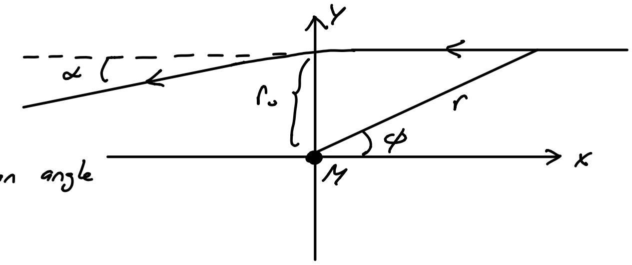
$$\Rightarrow \quad u(\phi) = u_0 (\sin\phi + GMu_0(1+\cos^2\phi - \sin\phi - 2\cos\phi))$$

$$= u_0 (1 - GMu_0)\sin\phi + GMu_0(1-\cos\phi)^2$$

Outcoming light ray:

$X = -\infty$: $\quad r(\pi + \alpha) = \infty$

Small $\alpha \ll 1$ deflection angle



$$u(\pi+\alpha) = u_0(1-GMu_0)\underbrace{\sin(\pi+\alpha)}_{\substack{= \sin\pi + (\cos\pi)\alpha + O(\alpha^2) \\ = -\alpha}} + GMu_0^2(1 - \underbrace{\cos(\pi+\alpha)}_{\substack{= \cos\pi - (\sin\pi)\alpha + O(\alpha^2) \\ = -1}})^2$$

$$= u_0(1-GMu_0)(-\alpha) + GMu_0^2(1+1)^2$$

$$\underbrace{u(\pi+\alpha)}_{\to \frac{1}{\infty} = 0} = -\alpha(1-GMu_0) + 4GMu_0^2 \quad \Rightarrow \quad \alpha = \frac{4GMu_0^2}{1-GMu_0} = \frac{4GM}{r_0^2} + O((GMu_0)^2)$$

Predicted deflection angle for a light ray passing by a star of mass M with the distance r_0.

For a light ray tracing the surface of the Sun $\alpha = 1{,}75''$, first observed by Eddington in 1919.