4. Gravitation

In the previous chapters we have developed machinery to measure curvature in terms of tensors.
We will see that Einstein eqs. are given by:

$$G_{\mu\nu} = 8\pi G \; T_{\mu\nu}$$

energy-momentum tensor, describes matter

Newton's constant

Einstein tensor (3.23), measures curvature

This is a set of 10 partial differential eqs. of the form

$$\text{“} \; \partial^2 g = T(g, p) \; \text{”}$$

geometry   matter

Before motivating the form of Einstein eqs. we need to discuss the energy-momentum tensor on the RHS.

4.1 Energy momentum tensor

The energy and pressure (= momentum) and their losses due to dissipation can be represented by the energy momentum tensor $T_{\mu\nu}$. Its components are related to thermodynamical quantities and transport equations derived from kinetic theory (energy/particle number (non-) conservation, Euler equation etc.) can be obtained from its properties.

We do not present a systematical derivation here but just give the form of $T_{\mu\nu}$.

Consider ideal fluid for which entropy is conserved along the fluid flow; $\frac{dS}{d\tau}=0$ along the integral curves $x^\mu(\tau)$ of the 4-velocity of the fluid $u^\mu = \frac{dx^\mu}{d\tau}$. This means that there are no dissipative terms and the fluid is locally in thermal equilibrium.

For ideal fluid:

$$(4.1) \qquad T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}$$

ρ = energy density in the rest frame $\frac{dx^i}{d\tau}=0$

p = pressure —— // ——

Using that $u^\mu u_\mu = -1$, we have

$$(4.2) \qquad \rho = u^\mu u^\nu T_{\mu\nu}$$

$$(4.3) \qquad p = \frac{1}{3}(g^{\mu\nu} + u^\mu u^\nu) T_{\mu\nu}$$

Properties of the ideal fluid are defined by the equation of state:

$$(4.5) \qquad w(\rho) \equiv \frac{p}{\rho}$$

Common cases are: non-relativistic non-interacting particles (dust) $p=0$

relativistic non-interacting particles (radiation) $p = \frac{1}{3}$

For a generic matter component described by the Lagrangian $\mathcal{L}_{matter}$, the energy-momentum tensor is determined by:

$$(4.4) \qquad T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g_{\mu\nu}} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g}\, \mathcal{L}_{matter}$$

In the generic case (4.4) we use (4.2) and (4.3) to define the energy density and pressure.

In GR all quantities are geometrical and there is no a priori preferred frame with a specific physical significance. Mathematically a manifold M and tensor fields T on it are equivalent if they are related by a smooth, topology preserving map i.e. a diffeomorphism (see e.g. Nakahara: Spacetime, geometry and physics). This means that also the matter action $S_{matter}(g_{\mu\nu}, \underset{\uparrow \text{matter fields}}{\phi_i})$ must be invariant under a generic diffeomorphism:

$$\delta S_m = \int d^4x \left(\frac{\delta S_m}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S_m}{\delta \phi_i} \delta \phi_i \right) = 0 \quad ,$$

where $\delta g_{\mu\nu}$, $\delta\phi_i$ are variations of the components under the diffeomorphism. A diffeom. is generated by a vector field, call it V^μ, and the variation of the metric under it (see Carroll: Appendix B or Nakahara) is given by:

$$\delta g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu$$

Hence

$$\delta S_m = 0 \Rightarrow \int d^4x \frac{\delta S_m}{\delta g_{\mu\nu}} \nabla_\mu V_\nu = \underbrace{\cancel{\int d^4x \sqrt{-g}\, \nabla_\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}} V_\nu \right)}}_{0 \text{ boundary term, see page...}} - \int d^4x \sqrt{-g}\, V_\nu \nabla_\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}} \right) = 0$$

$$\Rightarrow \int d^4x \sqrt{-g}\, V_\nu \nabla_\mu T^{\mu\nu} = 0 \quad \text{for any } V_\nu \text{ that generates a diffeomorphism.}$$

This yields the energy-momentum (non-) conservation:

$$(4.5) \quad \boxed{\nabla_\mu T^{\mu\nu} = 0} \quad \text{for } T^{\mu\nu} \text{ defined by eq. (4.4)}$$

By contracting this with u_ν we get:

$$\begin{aligned}
0 &= u^\mu \nabla_\nu T^\nu{}_\mu \\
&= u^\mu \nabla_\nu \left((\rho+p) u^\nu u_\mu + p\, g^\nu{}_\mu \right) \\
&= \underbrace{-u^\nu \nabla_\nu (\rho+p)}_{\substack{= u^\nu \partial_\nu (\rho+p) \\ = \frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\nu}(\rho+p) = \frac{d}{d\tau}(\rho+p)}} + (\rho+p)\left(-\nabla_\nu u^\nu + u^\nu \underbrace{u^\mu \nabla_\nu u_\mu}_{= \frac{1}{2}\nabla_\nu(u^\mu u_\mu) = \frac{1}{2}\nabla_\nu(-1) = 0} \right) + u^\nu \nabla_\nu p \\
&= -\frac{d}{d\tau}(\rho+p) - (\nabla_\nu u^\nu)(\rho+p) + \frac{dp}{d\tau}
\end{aligned}$$

This yields the continuity equation for the energy density ρ:

(4.6) $$\frac{d\rho}{d\tau} + (\nabla_\mu u^\mu)(\rho + p) = 0$$

Example

Homogeneous & isotropic universe $ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j$

The fluid must be at rest in comoving crd's, otherwise homogeneity & isotropy not preserved:

$$u^\mu = (1,0,0,0)$$

$$\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\nu} u^\nu$$
$$= \partial_0 u^0 + \Gamma^\mu_{\mu 0} u^0 \qquad \Gamma^\mu_{\mu 0} = \Gamma^i_{i0} = \frac{\dot a}{a} \quad \text{here } \dot a \equiv \frac{da}{dt}$$
$$= 3\frac{\dot a}{a}$$
sum over $i = 1,2,3$

The continuity eq. (4.6) becomes:

$$\dot\rho + 3\frac{\dot a}{a}(\rho + p) = 0 \qquad d\tau = dt \text{ in the comoving frame}$$

$\dot\rho \neq 0$ even for an ideal fluid in the fluid rest frame if $\dot a \neq 0$!

The above example demonstrates that energy is not in general conserved in curved spacetime.

Flat spacetime: $\partial_\mu T^{\mu\nu} = 0$

$t \to t + \Delta t$ symmetry $\Rightarrow$ conserved charge $E = \int T^{00} d^3x$

$$\frac{dE}{dt} = 0$$

$x^i \to x^i + \Delta x^i$ symmetry $\Rightarrow$ conserved charge $p^i = \int T^{0i} d^3x$

$$\frac{dp^i}{dt} = 0$$

Curved spacetime: $\nabla_\mu T^{\mu\nu} = 0$

$$t \to t + \Delta t \quad \text{not a symmetry} \quad \Rightarrow \quad \frac{dE}{dt} \neq 0$$

$$x^i \to x^i + \Delta x^i \quad \text{not a symmetry} \quad \Rightarrow \quad \frac{dp^i}{dt} \neq 0$$

These results concern the energy and momentum of matter fields $\neq$ gravity. In a dynamical system of matter + gravity it is not surprising that the matter energy and momentum alone are not conserved (c.f. harmonic oscillator with a time-dep. mass $\to$ energy not conserved). Defining the energy of gravitational degrees of freedom in GR for a generic spacetime is a non-trivial problem. See e.g. Carroll pages 120, 137 and 252-253.

4.2 Newtonian limit

Define the Newtonian limit by:

1) Particles move slowly $v \ll 1$ $(c = 1)$
2) Weak gravitational fields
3) Static gravitational fields

In this limit we know from observations that Newtonian gravity works well. Any theory of gravity should therefore reduce to Newton in the limit 1-3.

Let us consider the motion of freely falling test particles. The strong equivalence priciple states that they move along geodesics:

$$a^\mu = u^\nu \nabla_\nu u^\mu = 0$$

Let us check if this reduces to

$$\bar{a} = -\nabla \Phi$$

in the Newtonian limit.

From 1) it follows that: $\quad \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} = \mathcal{O}(1)$

From 2+3) it follows that: $\exists$ crd's which cover the entire spacetime and

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1 \quad , \quad \dot{h}_{\mu\nu} = 0$$

To first order in $h_{\mu\nu}$ the inverse metric is just:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \quad , \quad h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$$

$$g^{\mu\alpha}g_{\alpha\nu} = \eta^{\mu\alpha}\eta_{\alpha\nu} + \eta^{\mu\alpha}h_{\alpha\nu} - h^{\mu\alpha}\eta_{\alpha\nu} + \mathcal{O}(h^2)$$

$$= \delta^\mu{}_\nu + \eta^{\mu\alpha}h_{\alpha\nu} - \underbrace{\eta^{\mu\beta}\eta^{\alpha\gamma}h_{\beta\gamma}\eta_{\alpha\nu}}_{= \delta^\gamma{}_\nu \eta^{\mu\beta}h_{\beta\gamma} = \eta^{\mu\alpha}h_{\alpha\nu}} + \mathcal{O}(h^2)$$

$$= \delta^\mu{}_\nu + \mathcal{O}(h^2)$$

The geodesic eq. $a^\mu = u^\nu \nabla_\nu u^\mu = 0$ then becomes (linearize in the small quantities)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2(1+\mathcal{O}(v)) = 0 \quad , \quad \Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}(\underbrace{\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0}}_{=0} - \partial_\lambda g_{00})$$

$$\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2}\eta^{\mu\lambda}(\partial_\lambda h_{00})\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda h_{00}$$

$$= -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}$$

$\mu = 0$:

$$\frac{d^2t}{d\tau^2} - \frac{1}{2}\eta^{00}\underbrace{\partial_0 h_{00}}_{=0 \text{ because of } 3)}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \Rightarrow \frac{d^2t}{d\tau^2} = 0 \quad (*)$$

$\mu = i$:

$$\underbrace{\frac{d^2x^i}{d\tau^2}} - \frac{1}{2}\eta^{ij}\partial_j h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$= \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx^i}{dt}\right) = \underbrace{\frac{d^2t}{d\tau^2}}_{=0 \; (*)}\frac{dx^i}{dt} + \left(\frac{dt}{d\tau}\right)^2\frac{d^2x^i}{dt^2}$$

$$\left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x^i}{dt^2} - \frac{1}{2}\eta^{ij}\partial_j h_{00}\right) = 0$$

$$\Rightarrow \frac{d^2x^i}{dt^2} = \frac{1}{2}\delta^{ij}\partial_j h_{00} \quad , \quad \eta^{ij} = \delta^{ij}$$

Comparing to the Newtonian result

$$\frac{d^2x^i}{dt^2} = -\partial^i\Phi$$

we see that these are the same provided that we identify:

$h_{00} = -2\Phi$ $\Rightarrow$ perturbation around the Minkowski metric plays the role of the Newtonian gravitational potential.

$g_{00} = -(1+2\Phi)$

This confirms that gravitational effects indeed can be associated to geometry in the Newtonian limit, at least what comes to freely falling objects.

4.3 Einstein equations

In Newton's gravity, the gravitational potential Φ is determined by the Poisson equation. Now we want to find a corresponding eq. for GR which in the Newtonian limit reduces to

$$\nabla^2 \Phi = 4\pi \rho_m$$

Fluid rest frame: $g_{00} = -1 - 2\Phi$, $T_{00} = \rho = \rho_m$

$\nabla^2 \Phi = 4\pi G \rho_m \Rightarrow -\nabla^2 g_{00} = 8\pi G\, T_{00}$

Need sth. tensorial $\sim \partial^2 g_{\mu\nu}$

$\nabla^\sigma \nabla_\sigma g_{\mu\nu}$ not OK since $\nabla_\sigma g_{\mu\nu} = 0$

$\rho_m = \frac{mN}{V}$ mass density, not a tensor as V changes under Lorentz

The energy density $\rho = u^\mu u^\nu T_{\mu\nu}$ is a scalar, why not include also other components of $T_{\mu\nu}$?

Try: $\quad R_{\mu\nu} = \mathcal{K}\, T_{\mu\nu}$ does not work $\quad \nabla_\mu T^{\mu\nu} = 0$

($\mathcal{K}$: constant) $\quad \nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R \neq 0$

Take insted $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu}$ on the LHS and postulate:

(4.7) $\quad G_{\mu\nu} = \mathcal{K}\, T_{\mu\nu} \qquad \nabla^\mu G_{\mu\nu} = \mathcal{K}\, \nabla^\mu T_{\mu\nu} = 0 \quad$ ok

Check if this gives the correct Newtonian limit. In the Newtonian limit $v \ll 1$ which means that particles are non-relativistic $|\vec{p}| \ll m$:

$$T_{\mu\nu} = \rho u^\mu u^\nu \quad , \; p = 0 \text{ pressureless dust}$$

For $\rho = 0$: $G_{\mu\nu} = 0 \Rightarrow g_{\mu\nu} = \eta_{\mu\nu}$ is a solution.

In the Newtonian limit: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$\Rightarrow \rho$ is a first order perturbation

$$T_{\mu\nu} = \rho(\bar{u}^\mu + \delta u^\mu)(\bar{u}^\nu + \delta u^\nu) \qquad \bar{u}^\mu = (1, 0) \quad \bar{u}_\mu = (-1, 0) \quad \text{background quantities}$$

$$= \rho \bar{u}^\mu \bar{u}^\nu + \mathcal{O}(\delta^2) \qquad \rho = \mathcal{O}(\delta)$$

To first order in the small quantities:

$$T_{00} = \rho \ , \quad T_{ij} = T_{0i} = 0 \ , \quad T = g^{\mu\nu} T_{\mu\nu} = \eta^{00} T_{00} = -\rho$$

$= \rho$ first order perturbation

$$G_{\mu\nu} = \varkappa T_{\mu\nu} \iff R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \varkappa T_{\mu\nu} \quad \Big| \cdot g^{\mu\nu}$$

$$\Rightarrow \quad R - \frac{4}{2} R = \varkappa T$$

$$-R = \varkappa T$$

Therefore, $G_{\mu\nu} = \varkappa T_{\mu\nu}$ can be rewritten as:

$$R_{\mu\nu} = \varkappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R$$

$$= \varkappa \left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right)$$

Using the above results for $T_{\mu\nu}$ we get:

$$R_{00} = \varkappa \left(T_{00} - \frac{1}{2} T g_{00} \right)$$

$$= \varkappa \left(\rho - \frac{1}{2} (-\rho) \eta_{00} \right)$$

$$= \frac{\varkappa \rho}{2}$$

Now express R_{00} in terms of the metric:

$$R_{00} = R^{\mu}{}_{0\mu 0} \qquad (R^{0}{}_{000} = 0 \text{ by symmetries})$$

$$= R^{i}{}_{0i0}$$

$$= \partial_i \Gamma^i_{00} - \underbrace{\partial_0 \Gamma^i_{i0}}_{=0 \text{ as } \partial_0 h_{\mu\nu} = 0 \text{, static limit}} + \mathcal{O}(\delta^2)$$

$$= \partial_i \frac{1}{2} g^{i\sigma} \big(\underbrace{\partial_0 g_{0\sigma} + \partial_0 g_{\sigma 0}}_{=0} - \partial_\sigma g_{00} \big)$$

$$= -\frac{1}{2} \left(\partial_i g^{ij} \partial_j g_{00} \right)$$

$$= -\frac{1}{2} \eta^{ij} \partial_i \partial_j h_{00}$$

$$= -\frac{1}{2} \nabla^2 h_{00} \qquad , \nabla^2 \equiv \delta^{ij} \partial_i \partial_j$$

Thus:

$$R_{00} = \varkappa \frac{\rho}{2} \iff -\frac{1}{2}\nabla^2 h_{00} = \varkappa \frac{\rho}{2}$$

$$\nabla^2 h_{00} = -\varkappa \rho$$

Earlier we found that in the Newtonian limit:

$$h_{00} = -2\Phi \implies \nabla^2 \Phi = \frac{\varkappa}{2}\rho$$

This is just the Poisson eq. $\nabla^2\Phi = 4\pi G\rho$ provided that we choose

$$\frac{\varkappa}{2} = 4\pi G \iff \underline{\varkappa = 8\pi G}$$

Therefore, (4.7) becomes:

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \iff R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}} \tag{4.8}$$

These are the Einstein equations (10 coupled eqs.) which are the GR equations of motion for the metric $g_{\mu\nu}$ given matter described by $T_{\mu\nu}$.

The Einstein eqs. have the following properties:

1) $G_{\mu\nu} = 0$ for a flat spacetime
2) $G_{\mu\nu}$ constructed from $R^{\rho}{}_{\mu\sigma\nu}$ and $g_{\mu\nu}$ only
3) $G_{\mu\nu}$ linear in $R^{\rho}{}_{\mu\sigma\nu}$
4) $G_{\mu\nu}$ symmetric and 2nd order in derivatives
5) $\nabla^{\mu} G_{\mu\nu} = 0$ satisfied identically

It turns out that GR is the simplest theory where gravity = curvature. By dropping 2) – 4) one can construct alternative theories of gravity. They are constrained both by the Newtonian limit and by cosmological observations. So far all observations are consistent with GR so any modification of it must reduce to GR in appropriate limits.

4.3 Classical field theory in curved space

The fundamental object of a physical theory is its action or Lagrangean. The Lagrangean formulation of a classical theory in curved spacetime is in priciple analogous to flat spacetime up to some technical issues related to the Gauss law etc. Let us illustrate the formalism first in the simple case of a scalar field theory before moving to GR.

Scalar field theory in flat space

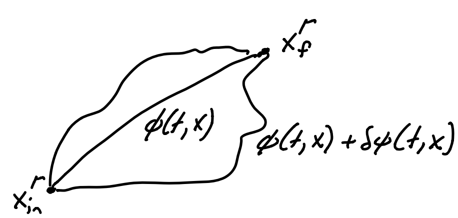
Consider a theory with real scalar fields $\phi(x^\mu)$. Classical trajectories from an initial configuration $\phi(x_{in}^\mu)$ to a final configuration $\phi(x_f^\mu)$ found by extremising the action:

$$S = \int_{x_{in}}^{x_f} d^4x \, \mathcal{L}(\phi(x^\mu), \partial_\nu \phi(x^\mu))$$

This means that we vary the field: $\phi(x^\mu) \to \phi(x^\mu) + \delta\phi(x^\mu)$

$\partial_\nu \phi(x^\mu) \to \partial_\nu \phi(x^\mu) + \partial_\nu \delta\phi(x^\nu)$

s.t. endpoints are kept fixed $\delta\phi(x_{in}^\mu) = \delta\phi(x_f^\mu) = 0$



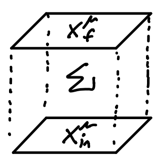
and find the configuration $\phi(x^\mu)$ for which $\delta S[\phi(x)] = 0$ to linear order in $\delta\phi$.

Under $\phi \to \phi + \delta\phi$ the variation of $S[\phi(x^\mu)]$ is:

$$\delta S = \int_{x_i}^{x_f} d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \overbrace{\delta(\partial_\mu \phi)}^{= \partial_\mu \delta\phi}}_{= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi} \right)$$

$$= \int_{\Sigma} d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi + \int_{\Sigma} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$\Sigma \uparrow$ integral volume



The second term is a boundary integral:

$$\int_{\Sigma} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \int_{t_i}^{t_f} dt \, \partial_t \int dx\,dy\,dz \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \delta\phi \right) + \int_{x_i}^{x_f} dx \, \partial_x (\ldots) + \ldots$$

$$= \Big|_{t_i}^{t_f} \int dx\,dy\,dz \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \delta\phi \right) + \Big|_{x_i}^{x_f} \int dt\,dy\,dz (\ldots) + \ldots$$

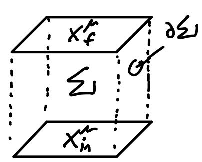
$$= \int_{\partial\Sigma} d^3x \, n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi$$

$\left(n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -n^0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} + \sum_{i=1}^{3} n^i \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \right)$

$\Rightarrow n^0 = -1, \; n^i = 1$

n_μ: unit vector normal to $\partial\Sigma$

$\partial\Sigma$: boundary of Σ



This is just an application of the Gauss law:
$$\int_{\Sigma} d^4x \, \partial_\mu V^\mu = \int_{\partial\Sigma} d^3x \, n_\mu V^\mu$$

$\left(\text{or in 3D} \int dV \, \nabla \cdot \bar{v} = \int d\bar{S} \cdot \bar{v} \right)$

The boundary term vanishes because $\delta\phi = 0$ at the boundary:

$$\int_{\Sigma} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \int_{\partial\Sigma} d^3x \, n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underset{\uparrow}{\delta\phi} = 0$$

$$\delta\phi\Big|_{\partial\Sigma} = 0$$

Therefore we are just left with:

$$\delta S = \int_{\Sigma} d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi \underset{\uparrow}{=} 0 \quad \forall \, \delta\phi$$

This we require because we want to find a configuration $\phi_{cl}(x)$ for which $\left.\frac{\delta S}{\delta \phi}\right|_{\phi_{cl}(x)} = 0$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$ <u>Euler-Lagrange equations</u>

The Euler-Lagrange equations are the classical equations of motion of the theory. Classical trajectories $\phi_{cl}(x)$ are those which extremise the action.

Scalar field theory in curved space

Now repeat the same exercise in curved spacetime. In curved spacetime the Gauss law takes the form:

$$(4.9) \qquad \int_{\Sigma} d^4x \sqrt{-g}\, \nabla_\mu V^\mu = \int_{\partial\Sigma} d^3x \sqrt{-\gamma}\, n_\mu V^\mu \qquad \text{(assuming Christoffel connection)}$$

$\sqrt{-\gamma}$: metric evaluated on $\partial\Sigma$ $\equiv$ induced metric

The scalar field action in curved space is of the form:

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}(\phi, \nabla_\mu \phi) \quad , \quad \text{where } \mathcal{L} \text{ is a scalar}$$

Let the field ϕ vary: $\phi \to \phi + \delta\phi \Rightarrow \nabla_\mu \phi \to \nabla_\mu \phi + \nabla_\mu \delta\phi$

$$\delta S = \int_{\Sigma} d^4x \sqrt{-g} \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \delta \nabla_\mu \phi}_{= \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \delta\phi \right) - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi)} \right) \delta\phi} \right)$$

$$= \int_{\Sigma} d^4x \sqrt{-g} \left(\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \right) \right) \delta\phi + \underbrace{\int_{\Sigma} d^4x \sqrt{-g}\, \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \delta\phi \right)}_{= \int_{\partial\Sigma} d^3x \sqrt{-\gamma}\, n_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \delta\phi \right) = 0 \text{ as } \delta\phi|_{\partial\Sigma} = 0}$$

$$= \int_{\Sigma} d^4x \sqrt{-g} \left(\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \right) \right) \delta\phi$$

Requiring that the variation of S vanishes:

$$\delta S = 0 \;\; \forall\, \delta\phi \;\; \Rightarrow \;\; \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \right) = 0$$

equations of motion in curved space.

Example:

Minimally coupled scalar field:

(non-minimal means adding e.g. $\xi R\phi^2$ or other curvature couplings, such terms generated through loops in general)

$$\mathcal{L} = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)$$

note that for $(+,-,-,-)$ signature we would have $+$ here.

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -V'(\phi) \quad , \quad \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi} = -\nabla^\mu\phi$$

$$\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = -\nabla^\mu\nabla_\mu\phi + V'(\phi) = 0$$

$$\underline{\Box\phi - V'(\phi) = 0} \quad , \quad \Box \equiv \nabla^\mu\nabla_\mu$$

Example:

Energy momentum tensor of a scalar field:

From the definition (4.4):

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$$

Now $\delta S_m = \delta\int d^4x\sqrt{-g}\,\mathcal{L}_m \qquad \mathcal{L}_m = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)$

$$= \int d^4x\left((\delta\sqrt{-g})\mathcal{L}_m - \frac{\sqrt{-g}}{2}(\delta g^{\mu\nu})\nabla_\mu\phi\nabla_\nu\phi\right)$$

$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}$

$\nabla_\mu\phi\nabla_\nu\phi = \partial_\mu\phi\partial_\nu\phi$ no dep. on $g_{\mu\nu}$ here

$$= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L}_m - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right)\delta g^{\mu\nu}$$

$$\frac{\delta S_m}{\delta g^{\alpha\beta}(y)} = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L}_m - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right)\underbrace{\frac{\delta g^{\mu\nu}(x)}{\delta g^{\alpha\beta}(y)}}_{=\delta^\mu{}_\alpha\delta^\nu{}_\beta\delta^4(x^\sigma - y^\sigma)}$$

$$= \sqrt{-g}\left(-\frac{1}{2}g_{\alpha\beta}\mathcal{L}_m - \frac{1}{2}\nabla_\alpha\phi\nabla_\beta\phi\right)$$

$$\Rightarrow T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{1}{2}g_{\mu\nu}\mathcal{L}_m + \nabla_\mu\phi\nabla_\nu\phi = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\phi\nabla^\sigma\phi - g_{\mu\nu}V(\phi)$$

4.4 Einstein-Hilbert action

Now return to GR. The action of GR is given by:

(4.10) $$S_H = \int d^4x \sqrt{-g}\, \frac{R}{16\pi G}$$

Einstein-Hilbert action

(Hilbert was the first one to show that this yields Einstein eqs. The story contains interesting sosiological aspects.)

The GR Lagrangian $\mathcal{L} = \frac{R}{16\pi G}$ is the simplest scalar that contains $\partial^2 g_{\mu\nu}$ and no higher order derivatives. This is why GR is the simplest theory where gravity = curvature.

Let us now show that setting $\frac{\delta S_H}{\delta g_{\mu\nu}} = 0$ for (4.10) yields the Einstein eqs. (4.8).

It is actually easier to do this by varying $g^{\mu\nu}$ instead of $g_{\mu\nu}$. The two variations are related by:

$$\begin{aligned}
\delta g^{\mu\nu} &= \delta(g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta}) \\
&= (\delta g^{\mu\alpha}) g^{\nu\beta} g_{\alpha\beta} + g^{\mu\alpha}(\delta g^{\nu\beta}) g_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \\
&= \delta^{\nu}{}_{\alpha}\, \delta g^{\mu\alpha} + \delta^{\mu}{}_{\beta}\, \delta g^{\nu\beta} + g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \\
&= 2\delta g^{\mu\nu} + g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}
\end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \delta g^{\mu\nu} &= - g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \\ \delta g_{\mu\nu} &= - g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta} \end{aligned} \qquad (4.11)$$

Vary eq. (4.10):

$$\begin{aligned}
\delta S_H &= \delta \int d^4x \sqrt{-g}\, \frac{g^{\mu\nu} R_{\mu\nu}}{16\pi G} \\
&= \frac{1}{16\pi G} \Big(\underbrace{\int d^4x (\delta\sqrt{-g}) g^{\mu\nu} R_{\mu\nu}}_{\equiv \delta S_1} + \underbrace{\int d^4x \sqrt{-g} (\delta g^{\mu\nu}) R_{\mu\nu}}_{\equiv \delta S_2} + \underbrace{\int d^4x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}}_{\equiv \delta S_3} \Big)
\end{aligned}$$

δS_2 is directly proportional to $\delta g^{\mu\nu}$ so we do not need to do anything for it. The parts δS_1 and δS_3 need some manipulation to find out how they depend on $\delta g^{\mu\nu}$.

Consider δS_1 first:

For any invertible $m\times m$ matrix M:

eigenvalues

$$\det M = \prod_{i=1}^{m} \lambda_i$$
$$\ln \det M = \ln \prod_i \lambda_i$$
$$= \sum_i \ln \lambda_i$$
$$\ln \det M = \mathrm{Tr} \ln M$$
$$\Rightarrow \frac{\delta \det M}{\det M} = \mathrm{Tr}(M^{-1}\delta M)$$

Apply this to the metric components $g_{\mu\nu}$: $\quad (g \equiv \det g_{\mu\nu})$

$$\frac{\delta g}{g} = g^{\nu\mu}\delta g_{\mu\nu}$$
$$\delta g = -g\, g_{\mu\nu}\delta g^{\mu\nu} \quad (\text{using } 4.11)$$
$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-1)\delta g = \frac{g\, g_{\mu\nu}\delta g^{\mu\nu}}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$$

$$\Rightarrow \delta S_1 = \int d^4x \left(-\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}\right) g^{\alpha\beta}R_{\alpha\beta}$$
$$= \int d^4x \sqrt{-g}\left(-\frac{R}{2} g_{\mu\nu}\delta g^{\mu\nu}\right)$$

Then δS_3:

Want to compute $\delta R_{\mu\nu}$.

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\sigma} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

As $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ the connection changes $\Gamma^{\rho}_{\mu\nu} \to \Gamma^{\rho}_{\mu\nu} + \delta\Gamma^{\rho}_{\mu\nu}$

The variation $\delta\Gamma^{\rho}_{\mu\nu}$ is a tensor because it is the difference between two connections. This is enough for us, it is not necessary to work out the explicit form of $\delta\Gamma^{\rho}_{\mu\nu}$.

$$\delta R_{\mu\nu} = \partial_\sigma \delta\Gamma^{\sigma}_{\mu\nu} + \delta\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\sigma}_{\sigma\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - \partial_\nu \delta\Gamma^{\sigma}_{\mu\sigma} - \delta\Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma}$$

$$= \underbrace{\nabla_\sigma \delta\Gamma^{\sigma}_{\mu\nu}}_{\substack{= \partial_\sigma \delta\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\sigma\lambda}\delta\Gamma^{\lambda}_{\mu\nu} \\ - \Gamma^{\lambda}_{\sigma\mu}\delta\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu}\delta\Gamma^{\sigma}_{\mu\lambda}}} + \Gamma^{\lambda}_{\sigma\nu}\delta\Gamma^{\sigma}_{\mu\lambda} - \underbrace{\nabla_\nu \delta\Gamma^{\sigma}_{\mu\sigma}}_{\substack{= \partial_\nu \delta\Gamma^{\sigma}_{\mu\sigma} + \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma} \\ - \Gamma^{\lambda}_{\nu\mu}\delta\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\sigma}_{\mu\lambda}}} - \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\sigma}_{\mu\lambda}$$

$$= \nabla_\sigma \delta\Gamma^{\sigma}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\sigma}_{\mu\sigma}$$

Therefore, we get:

$$\delta S_3 = \int d^4x\sqrt{-g}\, g^{\mu\nu}\left(\nabla_\sigma \delta\Gamma^{\sigma}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\sigma}_{\mu\sigma}\right)$$

$$= \int d^4x\sqrt{-g}\left(\nabla_\sigma\left(g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^{\sigma}_{\sigma\mu}\right)\right) \qquad \text{metric compatibility } \nabla_\sigma g_{\mu\nu} = 0$$

$$= \int_{\partial\Sigma} d^3x\sqrt{-\gamma}\left(n_\sigma g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} - n_\nu g^{\mu\nu}\delta\Gamma^{\sigma}_{\sigma\mu}\right) \qquad \text{using Gauss law}$$

$$= 0 \quad \text{if} \quad \delta\Gamma\Big|_{\partial\Sigma} = 0 \leftarrow \text{this assumes that } \partial_\sigma g_{\mu\nu}\Big|_{\partial\Sigma} = 0$$

Collecting the results we get:

$$\delta S_H = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}$$

$$\Rightarrow \frac{\delta S_H}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\int d^4x'\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\underbrace{\frac{\delta g^{\mu\nu}(x')}{\delta g^{\mu\nu}(x)}}_{=\delta^4(x'-x)}$$

$$= \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\frac{\sqrt{-g}}{16\pi G}$$

Setting $\frac{\delta S_H}{\delta g^{\mu\nu}} = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$, Einstein eqs. in vacuum $T_{\mu\nu} = 0$

Including also matter, the full action is:

(4.12) $S = S_H + S_m$ $\quad$ $S_m = \int d^4x\sqrt{-g}\,\mathcal{L}_m$ matter action

Setting $\frac{\delta S}{\delta g^{\mu\nu}} = 0$ yields:

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta S_H}{\delta g^{\mu\nu}} + \frac{\delta S_m}{\delta g^{\mu\nu}} = 0$$

$$\frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = -\frac{\delta S_m}{\delta g^{\mu\nu}}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\underbrace{\left(\frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\right)}_{\equiv T_{\mu\nu} \quad (4.4)}$$

Now we can understand where the definition comes from. It includes all parts of $\frac{\delta S}{\delta g^{\mu\nu}}$ that are left on the RHS after we organise $G_{\mu\nu}$ on the LHS.

Properties of the Einstein equations

The Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ are a set 10 coupled, non-linear 2nd order partial differential equations for the metric $g_{\mu\nu}$.
In addition to the dynamical equations, there are 4 constraints $\nabla_\mu G^{\mu\nu} = 0$ which follow from the Bianchi identity.

$$\left.\begin{array}{lll} 10 & \text{dynamical equations}: & G_{\mu\nu} = 8\pi G T_{\mu\nu} \\ 4 & \text{constraints}: & \nabla^\mu G_{\mu\nu} = 0 \end{array}\right\} \quad 10-4=6 \text{ dynamical dof's in } g_{\mu\nu}$$

GR is a theory with constraints: $g_{\mu\nu}$ contains 6 dynamical dof's and is subject to 4 constraints. You can think the constraints as conditions that have to be satisfied by physical initial conditions, the dynamical dof's tell how the initial configuration evolves.

Note that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ depends only on the Ricci tensor $R_{\mu\nu}$ which contains 10 of the total 20 dof's of the Riemann tensor. The other 10 are contained in the Weyl tensor $C_{\rho\sigma\mu\nu}$.
In vacuum:

$$T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0 \qquad \text{but} \quad C_{\rho\sigma\mu\nu} \neq 0$$

The solutions $R_{\mu\nu} = 0$, $C_{\rho\sigma\mu\nu} \neq 0$ describe gravitational waves which propagate through a spacetime empty of matter. We will discuss gravitational waves in detail later.