3. Curvature

3.1 Covariant derivative and connection

In the flat Minkowski space we showed that partial derivatives of tensors form new higher rank tensors:

$$\partial : \underbrace{T_p^* \times \ldots \times T_p^*}_{m \text{ copies}} \times \underbrace{T_p \times \ldots \times T_p}_{n \text{ copies}} \longrightarrow \underbrace{T_p^* \times \ldots \times T_p^*}_{m \text{ copies}} \times \underbrace{T_p \times \ldots \times T_p}_{n+1 \text{ copies}} \quad \text{when acting on } (m,n) \text{ tensors}$$

$\partial_\mu T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}$ is a $(m, n+1)$ tensor in Minkowski

We already noted that in a general curved spacetime $\partial_\mu T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}$ is not a tensor.

We want to generalise the partial derivative into something $\partial \to \nabla$ that is a tensor also in a curved spacetime.

Define the covariant derivative ∇ as a map:

$$\nabla : \underbrace{T_p^* \times \ldots \times T_p^*}_{m \text{ copies}} \times \underbrace{T_p \times \ldots \times T_p}_{n \text{ copies}} \longrightarrow \underbrace{T_p^* \times \ldots \times T_p^*}_{m \text{ copies}} \times \underbrace{T_p \times \ldots \times T_p}_{n+1 \text{ copies}} \quad \text{when acting on } (m,n) \text{ tensors}$$

and impose the following conditions (which hold for ∂ in Minkowski)

1. linearity: $\nabla(T+S) = \nabla T + \nabla S$ $\quad$ (T, S any tensors)
2. Leibnitz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

   which holds also when indices are contracted.
3. For a scalar $f : M \to \mathbb{R}$: $\nabla f \equiv df = \partial_\mu f \, dx^\mu \equiv \nabla_\mu f \, dx^\mu$

   $$\Rightarrow \nabla_\mu f = \partial_\mu f$$

   covariant derivative of a scalar is just the partial derivative

Consider now e.g. a covariant derivative of a vector $V \in T_p$ which should give a $(1,1)$ tensor $\nabla V = (\nabla_\nu V^\mu) dx^\nu \otimes \partial_\mu$

$$\nabla V = \nabla (V^\mu \partial_\mu) = (\nabla V^\mu)\partial_\mu + V^\mu \nabla \partial_\mu$$

$V^\mu(p) = \dfrac{dx^\mu(p)}{dx}$ defines a function (in a given crd set)

Use cond. 3) above: $\nabla V^\mu = dV^\mu = \partial_\nu V^\mu dx^\nu$

$$= \partial_\nu V^\mu dx^\nu \otimes \partial_\mu + V^\sigma \nabla \partial_\sigma$$

covariant derivative of the basis vector

$$\nabla \partial_\sigma = dx^\nu \otimes (\nabla_\nu \partial_\sigma)$$

Now define: connection coefficients

$$\nabla_\nu \partial_\sigma \equiv \Gamma^\mu_{\nu\sigma} \partial_\mu$$

$$\nabla V = (\partial_\nu V^\mu + \Gamma^\mu_{\nu\sigma} V^\sigma) dx^\nu \otimes \partial_\sigma \equiv (\nabla_\nu V^\mu) dx^\nu \otimes \partial_\sigma$$

Thus we get that the components of the covariant derivative of a vector are given by:

$$\boxed{\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\sigma_{\nu\mu} V^\mu} \qquad (3.01)$$

(components of) (1,1) tensor — not a tensor — not a tensor

Let us see how the connection coefficients Γ transform under $x^\mu \to x^{\mu'}$

$$\nabla_{\nu'} V^{\mu'} = \frac{\partial x^\alpha}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^\beta} \nabla_\alpha V^\beta = \frac{\partial x^\alpha}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^\beta} (\partial_\alpha V^\beta + \Gamma^\beta_{\alpha\gamma} V^\gamma)$$

(1,1) tensor — eq. (3.01)

On the other hand:

$$\nabla_{\nu'} V^{\mu'} = \partial_{\nu'} V^{\mu'} + \Gamma^{\mu'}_{\nu'\sigma'} V^{\sigma'} = \partial_{\nu'}\left(\frac{\partial x^{\mu'}}{\partial x^\alpha} V^\alpha\right) + \Gamma^{\mu'}_{\nu'\sigma'} \frac{\partial x^{\sigma'}}{\partial x^\alpha} V^\alpha$$

transf. of V^μ

$$\Rightarrow \frac{\partial x^\alpha}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^\beta}\left(\cancel{\partial_\alpha V^\beta} + \Gamma^\beta_{\alpha\gamma}V^\gamma\right) = \partial_{\nu'}\left(\frac{\partial x^{\mu'}}{\partial x^\alpha}V^\alpha\right) + \Gamma^{\mu'}_{\nu'\sigma'}\frac{\partial x^{\sigma'}}{\partial x^\alpha}V^\alpha$$

$$= \cancel{\frac{\partial x^\beta}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^\alpha}\partial_\beta V^\alpha} + \frac{\partial x^\beta}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\alpha \partial x^\beta}V^\alpha + \Gamma^{\mu'}_{\nu'\sigma'}\frac{\partial x^{\sigma'}}{\partial x^\alpha}V^\alpha$$

$$\Rightarrow V^\gamma \frac{\partial x^\alpha}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^\beta}\Gamma^\beta_{\alpha\gamma} = \left(\frac{\partial x^\beta}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\gamma} + \Gamma^{\mu'}_{\nu'\sigma'}\frac{\partial x^{\sigma'}}{\partial x^\gamma}\right)V^\gamma$$

$$\Rightarrow \Gamma^{\mu'}_{\nu'\sigma'} = \underbrace{\frac{\partial x^\gamma}{\partial x^{\sigma'}}\frac{\partial x^\alpha}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^\beta}\Gamma^\beta_{\alpha\gamma}}_{\text{tensorial part}} - \underbrace{\frac{\partial x^\gamma}{\partial x^{\sigma'}}\frac{\partial x^\beta}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\gamma}}_{\text{non-tensorial part that cancels against the non-tensorial part of } \partial_\mu V^\nu \text{ in (3.01)}}$$

What about dual vectors $\omega \in T_p^*$? By definition $\nabla\omega$ should be a $(0,2)$ tensor and proceeding as above we get:

another set of connection coefficients

$$\nabla\omega = \nabla\omega_\mu dx^\mu = \underbrace{(\nabla\omega_\mu)}_{=\partial_\nu\omega_\mu dx^\nu} \otimes dx^\mu + \omega_\mu \underbrace{\nabla dx^\mu}_{\text{define}} \qquad \nabla_\nu dx^\mu \equiv \tilde{\Gamma}^\mu_{\sigma\nu}dx^\sigma$$

$$= \partial_\nu\omega_\mu dx^\nu \otimes dx^\mu + \omega_\mu \tilde{\Gamma}^\mu_{\sigma\nu} dx^\sigma \otimes dx^\nu$$

rename indices

$$= \left(\partial_\nu\omega_\mu + \tilde{\Gamma}^\sigma_{\nu\mu}\omega_\sigma\right)dx^\nu \otimes dx^\mu$$

The coefficients $\tilde{\Gamma}$ can be related to Γ using the conditions 2 and 3 above:

$$\nabla_\nu(\omega_\mu V^\mu) \underset{\text{cond. 2}}{=} (\nabla_\nu\omega_\mu)V^\mu + \omega_\mu\nabla_\nu V^\mu \underset{\text{cond. 3 } (\omega_\mu V^\mu \text{ is a scalar})}{=} \partial_\nu(\omega_\mu V^\mu)$$

$$(\partial_\nu\omega_\mu + \tilde{\Gamma}^\sigma_{\nu\mu}\omega_\sigma)V^\mu + \omega_\mu(\partial_\nu V^\mu + V^\sigma\Gamma^\mu_{\nu\sigma}) = \omega_\mu\partial_\nu V^\mu + V^\mu\partial_\nu\omega_\mu$$

$$\tilde{\Gamma}^\sigma_{\nu\mu}\omega_\sigma V^\mu + \Gamma^\sigma_{\nu\mu}\omega_\sigma V^\mu = 0$$

$$\Rightarrow \tilde{\Gamma}^\sigma_{\nu\mu} = -\Gamma^\sigma_{\nu\mu} \qquad (\Rightarrow \nabla_\mu dx^\nu = -\Gamma^\nu_{\sigma\mu}dx^\sigma)$$

Hence we get :

$$\nabla\omega = (\partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda)\, dx^\mu \otimes dx^\nu \qquad \text{i.e.} \quad \boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda}$$

Now we know how to compute covariant derivatives of vectors and duals. Using the conditions 1, 2, 3 we can compute the covariant derivative of an arbitrary rank tensor. The (components of) covariant derivative of (m,n) tensor T is given by

$$(3.1) \quad \boxed{\begin{aligned}\nabla_\mu T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} = \partial_\mu T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} + \Gamma^{\alpha_1}_{\mu\lambda} T^{\lambda \alpha_2 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} + \dots + \Gamma^{\alpha_m}_{\mu\lambda} T^{\alpha_1 \dots \alpha_{m-1}\lambda}{}_{\beta_1 \dots \beta_n} \\ - \Gamma^\lambda_{\mu\beta_1} T^{\alpha_1 \dots \alpha_m}{}_{\lambda \beta_2 \dots \beta_n} - \dots - \Gamma^\lambda_{\mu\beta_n} T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_{n-1}\lambda}\end{aligned}}$$

For reference, let us rewrite separately the results for scalars, vectors and duals :

$$(3.2) \quad \boxed{\begin{aligned}&\nabla_\mu \phi = \partial_\mu \phi \\ &\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \\ &\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda\end{aligned}}$$

In flat spacetime we can find a coordinate system over the entire M where $\Gamma^\mu_{\nu\sigma} = 0$. So the connection coefficients must somehow carry information about the curvature. But how to compute them and are the uniquely defined in the first place? The answer to the latte question is no.

The conditions 1–3 do not uniquely define the connection and hence the covariant derivative. Consider another set of connection coefficients $\hat{\Gamma}^\nu_{\mu\lambda}$ which define another covariant derivative $\hat{\nabla}$:

$$\hat{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \hat{\Gamma}^\nu_{\mu\lambda} V^\lambda \qquad (1,1) \text{ tensor}$$

$$\nabla_\mu V^\nu - \hat{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda - \partial_\mu V^\nu - \hat{\Gamma}^\nu_{\mu\lambda} V^\lambda$$

$$\underbrace{\nabla_\mu V^\nu - \hat{\nabla}_\mu V^\nu}_{(1,1)\text{ tensor}} = (\Gamma^\nu_{\mu\lambda} - \hat{\Gamma}^\nu_{\mu\lambda}) \underset{\substack{\uparrow \\ (1,0)\text{ tensor}}}{V^\lambda}$$

$$\Rightarrow \Gamma^\nu_{\mu\lambda} - \hat{\Gamma}^\nu_{\mu\lambda} \equiv C^\nu_{\mu\nu} \quad \text{is a } (1,2) \text{ tensor}$$

The connection $\Gamma^\nu_{\mu\lambda}$ is not a tensor but the difference btw two connections is a tensor. From any set of connection coefficients $\Gamma^\nu_{\mu\lambda}$ we get a new one by adding an arbitrary $(1,2)$ tensor

We also get a new connection by switching the lower indices $\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

The difference btw $\Gamma^\lambda_{\mu\nu}$ and $\Gamma^\lambda_{\nu\mu}$ is a $(1,2)$ tensor called the torsion tensor of the connection $\Gamma^\lambda_{\mu\nu}$

$$(3.3) \qquad T^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\,\Gamma^\lambda_{[\mu\nu]}$$

In GR we will set

$$T^\lambda_{\mu\nu} = 0 \Rightarrow \Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$$

Christoffel connection

In general, the metric $g_{\mu\nu}$ and the connection $\Gamma^{\lambda}_{\mu\nu}$ are independent degrees of freedom and, as the above discussion shows, the connection for a given spacetime is not uniquely defined (there are several possible choices).

We can however define a specific connection which is fully determined by the metric by imposing some extra conditions in addition to 1–3.

Require:

4. Torsion-free: $T^{\lambda}_{\mu\nu} = 0 \Leftrightarrow \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$
5. Metric compatibility: $\nabla_\lambda g_{\mu\nu} = 0$

The conditions 1–5 define a unique connection, the Christoffel connection.

The explicit form of the Christoffel connection can be found as follows. Write out the condition $\nabla_\lambda g_{\mu\nu} = 0$ explicitly for different permutations and subtract:

$$
\begin{array}{lll}
+ & \nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \cancel{\Gamma^{\lambda}_{\sigma\mu} g_{\lambda\nu}} - \cancel{\Gamma^{\lambda}_{\sigma\nu} g_{\mu\lambda}} = 0 \\
- & \nabla_\mu g_{\nu\sigma} = \partial_\mu g_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} - \cancel{\Gamma^{\lambda}_{\mu\sigma} g_{\nu\lambda}} = 0 \\
- & \nabla_\nu g_{\mu\sigma} = \partial_\nu g_{\mu\sigma} - \Gamma^{\lambda}_{\nu\mu} g_{\lambda\sigma} - \cancel{\Gamma^{\lambda}_{\nu\sigma} g_{\mu\lambda}} = 0 \\
\hline
 & \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} - \partial_\nu g_{\sigma\mu} + 2\Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} = 0 & \quad | \cdot g^{\sigma\rho}
\end{array}
$$

$$
\boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)} \qquad (3.4)
$$

Christoffel connection

In GR it turns out that even if we start from $g_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu}$ as independent degrees of freedom, the equations of motion set $\Gamma^{\rho}_{\mu\nu}$ equal to (3.4). Therefore, in GR the connection is always the Christoffel connection. (Recall that GR is a classical theory, all physics is on-shell, i.e. obeys classical eqs of motion.) In more general theories of gravity this is not true and we get different results if we vary $g_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu}$ independently or set $\Gamma^{\rho}_{\mu\nu}$ equal to (3.4). This is something that you should keep in mind but in this course we discuss GR only and hence the connection is given by (3.4).

It can be shown (exercise) that the connection coefficients (3.4) (also called Christoffel symbols) satisfy:

$$(3.5) \qquad \Gamma^{\mu}_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\right) \qquad \text{recall } g \equiv \det(g_{\mu\nu})$$

This yields:

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^{\mu}_{\mu\nu} V^\nu$$
$$= \partial_\mu V^\mu + \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\right) V^\nu$$

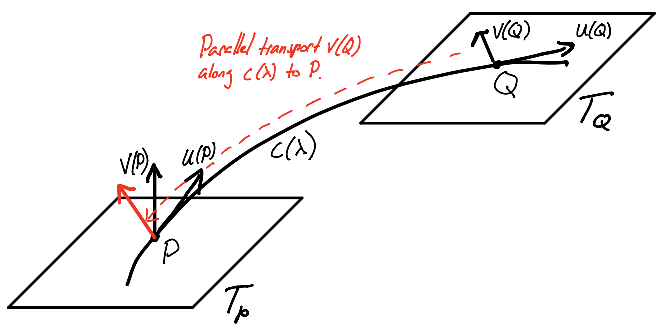
$$(3.6) \qquad \nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, V^\mu\right) \qquad \text{which is sometimes a useful relation.}$$

In going from SR to GR we basically replace $\partial_\mu \to \nabla_\mu$ in all non-gravitational expressions. The gravitational sector is the non-trivial part which we discuss later. This is just the strong equivalence principle in action. At any point P_0 we can go to the local Lorentz frame where physics is SR and $\Gamma^{\nu}_{\mu\lambda} = 0$ at P_0. So in these crd's at P_0 we have $\partial_\mu = \nabla_\mu$. But eqs written in terms of ∇_μ are manifestly tensorial and can be directly rewritten in terms of any other crds. So in practise $\partial_\mu \to \nabla_\mu$ and that's all.

3.2 Parallel transports

In curved spacetime there is no a priori unique way to compare tensors at different points, say $V(P)$ and $V(Q)$ where $P \neq Q$. This because they live in different tensor spaces $T_P \neq T_Q$. To compare the tensors, we need to define a curve which connects P and Q and can be used to map an object of T_Q to T_P. The outcome will depend on the chosen curve, or mapping.

The parallel transport of a vector gives us the concept of mapping a vector $V(Q)$ to $V(P)$ "without changing its direction":



Let $c(\lambda): \mathbb{R} \to M$ be a curve and $U = U^\mu \partial_\mu = \frac{dx^\mu(c(\lambda))}{d\lambda} \frac{\partial}{\partial x^\mu}$ its tangent vector. $V(P) \in T_p(M)$ is a vector field defined over the entire M (or at least over points on the curve $c(\lambda)$).

The parallel transport of $V(c(\lambda))$ along the curve $c(\lambda)$ is defined as the solution of:

$$(3.7) \qquad U^\mu \nabla_\mu V^\nu = \frac{dx^\mu(c(\lambda))}{d\lambda} \nabla_\mu V^\nu = 0$$

sol. gives the components of the parallel transport of V^μ.

This is the curved space generalization of $\frac{dx^\mu}{d\lambda}\frac{\partial V^\nu}{\partial x^\mu} = \frac{dV^\nu}{d\lambda}$. To stress this Carroll denotes $U^\mu \nabla_\mu V^\nu \equiv \frac{D}{d\lambda} V^\nu$.

The same definition applies for a general rank (m,n) tensor T:

(3.8) $\quad u^\mu \nabla_\mu T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} = 0$ sol. gives the components of the parallel transport of $T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}$.

Using (3.2) the parallel transport equation (3.7) becomes:

$$\frac{dx^\mu}{d\lambda}\left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda\right) = 0$$

(3.9) $$\frac{dV^\nu}{d\lambda} + \Gamma^\nu_{\mu\lambda} V^\lambda \frac{dx^\mu}{d\lambda} = 0$$

Given the vector V^μ at a point $c(\lambda_0)$ the solution of (3.9) gives the parallel transport at any other point on the curve.

Note that because of the metric compatibility $\nabla_\lambda g_{\mu\nu} = 0$ we get:

$$u^\lambda \nabla_\lambda g_{\mu\nu} = 0 \quad \text{for any } u \in T_p(M)$$

The parallel transport of the metric is just the metric itself. From this it follows that:

$$\begin{aligned} u^\mu \nabla_\mu (V_\nu W^\nu) &= u^\mu \nabla_\mu (g_{\sigma\nu} V^\sigma W^\nu) \\ &= g_{\sigma\nu} u^\mu \nabla_\mu (V^\sigma W^\nu) \\ &= V_\nu (u^\mu \nabla_\mu W^\nu) + W_\nu (u^\mu \nabla_\mu V^\nu) \end{aligned}$$

Therefore, if W^μ and V^μ are parallel transported, their inner product remains unchanged under paralle transport:

$$u^\mu \nabla_\mu W^\nu = 0,\ u^\mu \nabla_\mu V^\nu = 0 \quad \Rightarrow \quad u^\mu \nabla_\mu (V_\nu W^\nu) = 0$$

$\Rightarrow$ The parallel transport preserves lengths and angles.

3.3 Geodesics

Geodesics are special curves which parallel transport their own tangent vector. The geodesics are the generalisation of straight lines of Euclidean space to curved spacetimes.

The defining condition of geodesics curves $c(\lambda)$ are that the tangent vector $u = \frac{dx^\mu}{d\lambda}\partial_\mu$ is parallel transported along $c(\lambda)$:

(3.10) $\quad u^\mu \nabla_\mu u^\nu = 0 \quad$ Geodesic equation

Writing this out in explicit form we get:

$$\frac{dx^\mu}{d\lambda}\left(\partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda} u^\lambda\right) = 0$$

$$\underbrace{\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}}_{=\frac{d}{d\lambda}}\left(\frac{dx^\nu}{d\lambda}\right) + \Gamma^\nu_{\mu\lambda} u^\mu u^\lambda = 0$$

(3.11) $$\boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = 0}$$ solutions $x^\mu(\lambda)$ are geodesics

Consider the Minkowski limit $\Gamma^\mu_{\sigma\nu} = 0$. In this case the geodesic eq. 3.11) reduces to:

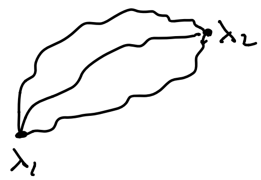
$$\frac{d^2 x^\mu}{d\lambda^2} = 0 \quad \Rightarrow \quad x^\mu(\lambda) = c^\mu \lambda + d^\mu$$

in flat space geodesics = straight lines.

In the flat space, straight lines minimise the distance between two points P and Q. The geodesics maximise the proper time (for timelike curves $ds^2 < 0$).

Consider the proper time between two spacetime points along timelike curves

$$\tau_{12} = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$ find the curve that extremises τ_{12}



Vary $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$ with fixed endpoints $x^\mu(\lambda_1) = x_1^\mu$, $x^\mu(\lambda_2) = x_2^\mu$.

The variation of τ_{12} is given by:

$$\delta\tau_{12} = \int_{\lambda_1}^{\lambda_2} d\lambda \,\frac{1}{2}\Big(\underbrace{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}_{=\frac{d\tau^2}{d\lambda^2}}\Big)^{-1/2}\Big(-\delta g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\Big)$$

$$= \int_{\lambda_1}^{\lambda_2} \frac{1}{2}\underbrace{d\lambda\,\frac{d\lambda}{d\tau}}_{=d\tau\frac{d\lambda^2}{d\tau^2}}\Big(-\underbrace{\delta g_{\mu\nu}}_{=(\partial_\sigma g_{\mu\nu})\delta x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\Big)$$

$$= \int_{\lambda_1}^{\lambda_2}\frac{d\tau}{2}\Big(-\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma - 2\cdot\underbrace{g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{d\delta x^\mu}{d\tau}}\Big)$$

$$= \frac{d}{d\tau}\Big(g_{\mu\nu}\frac{dx^\nu}{d\tau}\delta x^\mu\Big) - \underbrace{\frac{dg_{\mu\nu}}{d\tau}}_{=\frac{dx^\sigma}{d\tau}\partial_\sigma g_{\mu\nu}}\frac{dx^\nu}{d\tau}\delta x^\mu - g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\delta x^\mu$$

Denote $\frac{dx^\mu}{d\tau} \equiv \dot{x}^\mu$

$$= \int_{\lambda_1}^{\lambda_2}\frac{d\tau}{2}\Big(2g_{\mu\nu}\ddot{x}^\nu\delta x^\mu + \overbrace{2\partial_\sigma g_{\mu\nu}\dot{x}^\nu\dot{x}^\sigma\delta x^\mu - \partial_\sigma g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\delta x^\sigma}^{=(\partial_\sigma g_{\nu\mu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu})\dot{x}^\nu\dot{x}^\sigma\delta x^\mu \text{ just rename sum indices}}\Big)$$

$$- \cancel{\int_{\lambda_1}^{\lambda_2} d\tau\frac{d}{d\tau}\Big(g_{\mu\nu}\frac{dx^\nu}{d\tau}\delta x^\mu\Big)} \to 0 \text{ as } \delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$$

$$\delta\tau_{12} = \int_{\lambda_1}^{\lambda_2} d\tau \left(g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\left(\partial_\sigma g_{\nu\mu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}\right)\dot{x}^\nu \dot{x}^\sigma \right)\delta x^\mu \quad (*)$$

(13)

Extremals:

$$\delta\tau_{12} = \int_{\lambda_1}^{\lambda_2} d\tau \left(g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\left(\partial_\sigma g_{\nu\mu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}\right)\dot{x}^\nu \dot{x}^\sigma \right)\delta x^\mu = 0 \qquad \forall\, \delta x^\mu$$

$$\Rightarrow \qquad g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\left(\partial_\sigma g_{\nu\mu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}\right)\dot{x}^\nu \dot{x}^\sigma = 0 \quad | \cdot g^{\lambda\mu}$$

$$\ddot{x}^\lambda + \underbrace{\frac{1}{2} g^{\lambda\mu}\left(\partial_\sigma g_{\nu\mu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}\right)}_{= \Gamma^\lambda_{\sigma\nu} \text{ Christoffel connection (3.4)}} \dot{x}^\sigma \dot{x}^\nu = 0$$

So we found that

$$\delta\tau_{12} = 0 \quad \Rightarrow \quad \ddot{x}^\mu + \Gamma^\mu_{\sigma\nu}\dot{x}^\sigma \dot{x}^\nu = 0$$ This is just the geodesic equation (3.11)

Therefore, we see that timelike geodesics connecting two points extremise the proper time between the points. The extrema are actually maxima so the geodesics maximise the proper time btw. different points.

Note that above in step $(*)$ the integrand can can be written as

$$\left(g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\left(\partial_\sigma g_{\nu\mu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}\right)\dot{x}^\nu \dot{x}^\sigma \right)\delta x^\mu = \left(\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu}\right)\delta x^\mu$$

where $\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$

Check:

$$\left(\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu}\right)\delta x^\mu = \left((g_{\mu\nu}\dot{x}^\nu)^{\cdot} - \frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)\delta x^\mu$$

$$= \left(g_{\mu\nu}\ddot{x}^\nu + \dot{x}^\nu \frac{dx^\sigma}{d\tau}\partial_\sigma g_{\mu\nu} - \frac{1}{2}\dot{x}^\alpha \dot{x}^\beta \partial_\mu g_{\alpha\beta}\right)\delta x^\mu$$

$$= \left(g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\left(\partial_\sigma g_{\nu\mu}\dot{x}^\nu \dot{x}^\sigma + \partial_\nu g_{\mu\sigma}\dot{x}^\nu \dot{x}^\sigma - \partial_\mu g_{\sigma\nu}\dot{x}^\sigma \dot{x}^\nu\right)\right)\delta x \qquad \text{ok}$$

Therefore, we find that extremising the action:

$$S = \int d\tau \, \mathcal{L}(x, \dot{x}) = \int d\tau \, \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

yields the geodesic equations:

$$\frac{\delta S}{\delta x^\mu} = 0 \iff$$

$$(3.12) \quad \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \iff \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \qquad \mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

This gives an alternative way to compute the Christoffel connection for a given metric. Variation of $\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ gives the Euler-Lagrange equations which according to (3.12) are the geodesic eqs. for the metric $g_{\mu\nu}$. The connection coefficients can then be read off from these. Using (3.12) is often an easier way to get $\Gamma^\mu_{\alpha\beta}$ than the definition (3.4). On the other hand, (3.4) can be readily implemented on computer.

Example:

Find the connection coefficients for $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$.

Could use (3.4) but we will here demonstrate using (3.12).

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$= -\frac{1}{2}\dot{t}^2 + \frac{a^2}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = -\frac{1}{2}\dot{t}^2 + \frac{a^2}{2}\delta_{ij}\dot{x}^i \dot{x}^j$$

$$\delta \int d\tau \, \mathcal{L}(x, \dot{x}) = 0 \Rightarrow \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \qquad \text{E-L}$$

$\mu = 0$ component:

$$\frac{\partial \mathcal{L}}{\partial t} = a(t) \underbrace{\frac{\partial a}{\partial t}}_{\equiv a'} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = a a' (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = a a' \delta_{ij} \dot{x}^i \dot{x}^j$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{d}{d\tau}\left(-\dot{t}\right) = -\ddot{t}$$

<u>E-L:</u> $-\ddot{t} - aa'\delta_{ij}\dot{x}^i\dot{x}^j = 0$

$$\ddot{t} + aa'\delta_{ij}\dot{x}^i\dot{x}^j = \ddot{x}^0 + \Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

$$\Rightarrow \begin{cases} \Gamma^0_{ij} = aa'\delta_{ij} \\ \Gamma^0_{0i} = \Gamma^0_{00} = 0 \end{cases}$$

<u>i-components:</u>

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = a^2\dot{x}^j\delta_{ij}$$

$$\frac{\partial \mathcal{L}}{\partial x^i} = 0$$

E-L: $\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) = 0$

$$a^2\ddot{x}^j\delta_{ij} + \frac{dt}{d\tau}\frac{\partial}{\partial t}\left(a^2\dot{x}^j\delta_{ij}\right) = 0$$

$$a^2\ddot{x}^j\delta_{ij} + 2aa'\dot{x}^j\dot{t}\,\delta_{ij} = 0 \quad /\cdot\delta^{ik}$$

$$\ddot{x}^k + \frac{2a'}{a}\dot{x}^0\dot{x}^k = 0$$

$$\ddot{x}^k + \Gamma^k_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

note $\Gamma^k_{0k}\dot{x}^0\dot{x}^k + \Gamma^k_{k0}\dot{x}^k\dot{x}^0 = 2\Gamma^k_{0k}\dot{x}^0\dot{x}^k$

$$\Rightarrow \begin{cases} \Gamma^k_{0k} = \frac{a'}{a} \\ \Gamma^k_{00} = \Gamma^k_{ij} = 0 \end{cases}$$

$\Rightarrow \Gamma^0_{ij} = aa'\delta_{ij}$, $\Gamma^i_{i0} = \frac{a'}{a}$, all other components are zero

Reminder of variational calculus:

Consider an action $S[\phi(t)] = \int_{t_1}^{t_2} dt\, \mathcal{L}(\phi(t), \dot\phi(t))$

S is a functional = function of the function $\phi(t)$

Vary the form of $\phi(t)$ keeping the endpoints fixed: $\phi(t) \to \phi(t) + \delta\phi(t)$

$\delta\phi(t_1) = \delta\phi(t_2) = 0$

Linearize everything in the small variation $\delta\phi(t)$:

$$\delta S \equiv S[\phi(t) + \delta\phi(t)] - S[\phi(t)] = \int_{t_1}^{t_2} dt \Big(\underbrace{\mathcal{L}(\phi + \delta\phi, (\phi+\delta\phi)^{\cdot})}_{= \mathcal{L}(\phi,\dot\phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \dot\phi}\delta\dot\phi + \dots} - \mathcal{L}(\phi,\dot\phi)\Big)$$

$$= \int_{t_1}^{t_2} dt \Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot\phi}\delta\dot\phi}_{= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\delta\phi\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right)\delta\phi} \Big)$$

$$= \int_{t_1}^{t_2} dt \Big(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot\phi}\Big)\Big)\delta\phi + \underbrace{\Big/_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot\phi}\delta\phi}_{=0 \text{ as } \delta\phi(t_2) = \delta\phi(t_1) = 0}$$

Hence, requiring that $\delta S = 0\ \forall\, \delta\phi$ we get the Euler-Lagrange equations:

$$\delta S = 0\ \forall\, \delta\phi \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot\phi}\Big) = 0$$

This can be compared to the differential of a function $df(x) = \frac{\partial f}{\partial x}dx$. Indeed for a functional $S[\phi(\,)]$ we can define the functional derivative which acts like an ordinary partial derivative except that:

$$\frac{\delta\phi(t)}{\delta\phi(t')} = \delta(t-t') \quad , \text{ compare to a discrete set of variables } \{\phi_i\}_{i=1}^{n} : \frac{\partial\phi_i}{\partial\phi_j} = \delta_{ij}$$

$$\frac{\delta S[\phi]}{\delta\phi(t)} = \frac{\delta}{\delta\phi(t)} \int dt'\, \mathcal{L}(\phi(t'), \dot\phi(t'))$$

$$\frac{\delta S[\phi]}{\delta \phi(t)} = \int dt' \left(\frac{\partial \mathcal{L}(\phi(t'),\dot{\phi}(t'))}{\partial \phi(t')} \underbrace{\frac{\delta \phi(t')}{\delta \phi(t)}}_{=\delta(t'-t)} + \frac{\partial \mathcal{L}(\phi(t'),\dot{\phi}(t'))}{\partial \dot{\phi}(t')} \underbrace{\frac{\delta \dot{\phi}(t')}{\delta \phi(t)}}_{\substack{= \frac{d}{dt'} \frac{\delta \phi(t')}{\delta \phi(t)} \\ = \frac{d}{dt'} \delta(t'-t)}} \right)$$

$$= \int dt' \frac{\partial \mathcal{L}(\phi(t'),\dot{\phi}(t'))}{\partial \phi(t')} \delta(t'-t) + \cancelto{0}{\int dt' \frac{d}{dt'} \left(\frac{\partial \mathcal{L}(\phi(t'),\dot{\phi}(t'))}{\partial \dot{\phi}(t')} \delta(t'-t) \right)}$$

$$- \int dt' \frac{d}{dt'} \left(\frac{\partial \mathcal{L}(\phi(t'),\dot{\phi}(t'))}{\partial \dot{\phi}(t')} \right) \delta(t'-t)$$

$$= \frac{\partial \mathcal{L}(\phi(t),\dot{\phi}(t))}{\partial \phi(t)} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}(\phi(t),\dot{\phi}(t))}{\partial \dot{\phi}(t)} \right)$$

So the Euler-Lagrange eqs. are equivalent to $\frac{\delta S[\phi]}{\delta \phi} = 0$.

Geodesics and freely falling objects

In SR objects which feel no force have constant 4-velocity:

$$a^\mu = u^\nu \partial_\nu u^\mu = 0 \qquad u^\mu = \frac{dx^\mu}{d\tau}$$

Consider a freely falling object (feels only gravity, no other forces) in GR. The rest frame of the object coincides with the local Lorentz frame. Physics looks like SR locally and we can write:

$$a^\mu = u^\nu \partial_\nu u^\mu \underset{\substack{\uparrow \\ \text{local Lorentz frame}}}{=} u^\nu \nabla_\nu u^\mu = 0$$

The latter form is a tensorial equation and therefore freely falling objects in any frame obey:

(3.13) $\quad a^\mu = u^\nu \nabla_\nu u^\mu = 0 \qquad$ ($a^\mu = 0 \rightsquigarrow$ gravity not a force)

This is just the geodesic equation

$$a^\mu = u^\nu \nabla_\nu u^\mu = \frac{dx^\nu}{d\tau}\partial_\nu u^\mu + u^\nu \Gamma^\mu_{\nu\sigma} u^\sigma = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$\Rightarrow$ free particles move along geodesics!

Example Comoving observers $x^i = \text{const.}$ in the RW space:

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j$$

From page 65, the geodesic eq. for this metric read:

$$\ddot{x}^0 + a a' \delta_{ij} \dot{x}^i \dot{x}^j = 0$$
$$\ddot{x}^i + \frac{2a'}{a}\dot{x}^0 \dot{x}^i = 0$$

Clearly $x^i = \text{const}$, $x^0 = c_1 t + c_2$ are solutions and hence geodesics.

$\Rightarrow$ test particles move along $x^i = \text{const}$ trajectories.

If there is a force f^μ acting on the test particle, we get:

$$a^\mu = \frac{f^\mu}{m} \quad \Rightarrow \quad \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \frac{f^\mu}{m}$$

$\nwarrow$ mass of the particle

Using that $p^\mu = m u^\mu$, the geodesic equation (3.13) can be rewritten as:

$$(3.14) \qquad p^\nu \nabla_\nu p^\mu = 0 \; .$$

This form holds also for massless test particles $m=0$ for which $p^\mu p_\mu = 0$. Massless test particles move along null geodesics.

The choice of curve parameter λ of a geodesic $x^\mu(\lambda)$ is not unique. We are free to make reparameterisations $\tilde{\lambda} = a\lambda + b$, where a, b are constants:

$$\frac{d^2x^\mu}{d\tilde{\lambda}^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tilde{\lambda}}\frac{dx^\beta}{d\tilde{\lambda}} = \frac{1}{a^2}\underbrace{\left(\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)}_{=0} = 0$$

$\Rightarrow x^\mu(\tilde{\lambda})$ is a geodesic too.

For massless particles we usually choose λ s.t.

$$\frac{dx^\mu}{d\lambda} = p^\mu \leftarrow \text{photon 4-momentum}$$

For massive particles we can choose $\lambda = \frac{\tau}{m}$: $\frac{dx^\mu}{d\lambda} = m u^\mu = p^\mu$

Example

Photons in an expanding universe $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$.

Photons move along null geodesics for which $ds^2 = 0$ so we have the equations:

$$(*) \begin{cases} \ddot{t} + a a' \delta_{ij}\dot{x}^i \dot{x}^j = 0 \\ \ddot{x}^i + \frac{2a'}{a}\dot{t}\dot{x}^i = 0 \end{cases} \quad \text{geodesic eqs. from previous examples}$$

$$-\dot{t}^2 + a^2\delta_{ij}\dot{x}^i\dot{x}^j = 0 \quad \text{null condition } ds^2 = 0$$

In general not all solutions of $ds^2=0$ are geodesics.

PARENTHESIS

Consider e.g. circular path on x-y plane

$$x = \cos\omega(t) \qquad \delta_{ij}\dot{x}^i\dot{x}^j = \omega'^2\dot{t}^2(\cos^2\omega+\sin^2\omega) = \omega'^2\dot{t}^2$$
$$y = \sin\omega(t) \qquad -\dot{t}^2+a^2(t)\omega'^2\dot{t}=0 \quad\Rightarrow\quad ds^2=0 \text{ if } \omega'(t)^2=\frac{1}{a^2(t)}$$

But $\ddot{x}^i+\frac{2a'}{a}\dot{t}\dot{x}^i=0$ yields

$$\frac{1}{\dot{x}^i}\frac{d\dot{x}^i}{d\lambda} = -\frac{2a'}{a}\dot{t} = -2\frac{d\ln a}{d\lambda}$$
$$\frac{d\ln\dot{x}^i}{d\lambda} = -2\frac{d\ln a}{d\lambda}$$
$$\ln\dot{x}^i = \ln a^{-2}+\tilde{c}_i$$
$$\dot{x}^i = \frac{c_i}{a^2(t)}$$

Now $\dot{x} = -\omega'\dot{t}\sin\omega(t) = \frac{c_1}{a^2(t)}$, $\omega'=\frac{1}{a}$

$$\dot{t} = \frac{-c_1}{a\sin\omega(t)}$$

$$\dot{y} = \omega'\dot{t}\cos\omega(t) = \frac{c_2}{a^2(t)} \Rightarrow \dot{t} = \frac{c_2}{a\cos\omega(t)} \neq \frac{-c_1}{a\sin\omega(t)}$$

Hence $x=\cos\omega(t)$, $y=\sin\omega(t)$ are NOT geodesics!

However, often it is clear from symmetries which solutions of $ds^2=0$ are geodesics. In practice it is usually easiest to solve $ds^2=0$ (under some assumptions) first and then check that the solution indeed satisfies the geodesic eqs.

In our case the symmetries suggest that geodesics should be straight lines in the 3d subspace (x,y,z) which at any fixed t has a Euclidean geometry.

Rotate coordinates such that the motion is along x-axis. Eqs. $(*)$ become:

(1) $\ddot{t}+aa'\dot{x}^2=0$

(2) $\ddot{x}+\frac{2a'}{a}\dot{x}\dot{t}=0$

(3) $-\dot{t}^2+a^2\dot{x}^2=0$

Solve (3), i.e. $ds^2=0$, to get:

(4) $\frac{dx}{d\lambda} = (+) \frac{1}{a} \frac{dt}{d\lambda}$ (consider motion to $+x$ direction)

Plug (4) into (1):

$$\ddot{t} + \frac{aa'}{a^2} \dot{t}^2 = 0$$

$$\frac{d \ln \dot{t}}{d\lambda} = - \frac{d \ln a}{d\lambda}$$

(5) $\dot{t} = \frac{\omega_0}{a(t)}$, $\omega_0 = \text{const.}$

Check that (4), (5) satisfy (2): $\ddot{x} + \frac{2a'}{a} \dot{x}\dot{t} = \left(\frac{\dot{t}}{a}\right)^{\cdot} + \frac{2a'}{a} \frac{\dot{t}}{a}^2$

$$= \omega_0 \left(\frac{1}{a^2}\right)^{\cdot} + \frac{2a'}{a^2} \frac{\omega_0^2}{a^2}, \quad \dot{a} = a'\dot{t}$$

$$= - \frac{2\omega_0}{a^3} \frac{a'\omega_0}{a} + \frac{2a'\omega_0^2}{a^4} = 0 \quad \underline{\text{ok}}$$

$\Rightarrow$ Null geodesics given by

$$\begin{cases} \frac{dx}{d\lambda} = \frac{\omega_0}{a^2(t)} \\ \frac{dt}{d\lambda} = \frac{\omega_0}{a(t)} \end{cases}$$

these can be integrated to get $t(\lambda), x(\lambda)$ once we know $a(t)$.

Photon 4-momentum given by:

$$p^\mu = \frac{dx^\mu}{d\lambda} = \left(\frac{\omega_0}{a}, \frac{\omega_0}{a^2}, 0, 0 \right) \qquad p^\mu p_\mu = 0$$

measure energy and momentum but in what frame, how can we connect to observables?

Go to the local inertial frame. The energy E measured by an observer $u^\mu = \frac{dx^\mu}{d\tau}$ given by the SR result (1.34):

$$E_{obs} = - u_\mu p^\mu = - g_{\mu\nu} u^\mu p^\nu$$

Now that we know this is the observed energy, we can compute it in any coordinate system.

Choose the comoving coordinates where $ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j$ and assume for simplicity that our observer is at rest in the comoving frame:

$$u^\mu = \frac{dx^\mu}{d\tau} = (u^0, 0, 0, 0) \qquad u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} {u^0}^2 = -1$$

$$u^0 = \sqrt{\frac{-1}{g_{00}}}$$

Here $g_{00} = -1 \Rightarrow u^0 = 1$

$$E_{obs} = -g_{00} u^0 p^0 = p^0 = \frac{\omega_0}{a}$$

The comoving observer (i.e. freely falling observer) measures photon energy

$$E = \frac{\omega_0}{a(t)}$$

cosmological redshift

Correspondingly, the measured wavelength is

$$\lambda = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0} a(t)$$

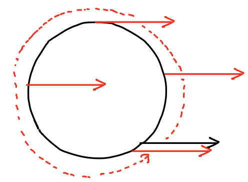
stretching due to expansion of space

3.4 Riemann curvature tensor

We have now enough machinery to determine a tensorial, i.e. coordinate invariant, way to measure curvature = deviation from flat geometry.

In a flat spacetime we have the following properties:

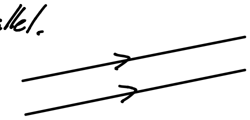
1. Parallel transport around a closed loop is an identity:



2. Covariant derivatives commute:

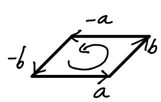
$$\nabla_\mu \nabla_\nu V^\sigma = \nabla_\nu \nabla_\mu V^\sigma$$ because $\exists$ coordinates where $\nabla_\mu = \partial_\mu \ \forall p \in M$

3. Parallel geodesics remain parallel.



None of these properties is true in a generic curved spacetime. Here we consider the first two properties.

1. Consider an infinitesimal loop generated by vectors a, b $(a = dx^\mu \partial_\mu$ etc.)

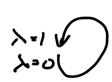


Parallel transport equation for a vector V^μ around the loop:

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma} V^\nu \frac{dx^\sigma}{d\lambda} = 0$$ splits into 4 parts $a^\sigma, b^\sigma, -a^\sigma, -b^\sigma$

Linear in V^μ and a^μ, b^μ:

$$\delta V^\rho \equiv V^\rho(\lambda=1) - V^\rho(\lambda=0) \equiv R^\rho{}_{\sigma\mu\nu} V^\sigma a^\mu b^\nu$$



linear map

Going to opposite direction gives

$$-\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma b^\mu a^\nu \Rightarrow R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$

2. Now $\nabla_\mu \nabla_\nu V^\sigma \neq \nabla_\nu \nabla_\mu V^\sigma$. Consider the commutator:

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu] V^\sigma &= \nabla_\mu \nabla_\nu V^\sigma - \nabla_\nu \nabla_\mu V^\sigma \\
&= \partial_\mu \nabla_\nu V^\sigma + \Gamma^\sigma_{\mu\lambda} \nabla_\nu V^\lambda - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\sigma - (\mu \leftrightarrow \nu) \\
&= \cancel{\partial_\mu \partial_\nu V^\sigma} + \partial_\mu (\Gamma^\sigma_{\nu\lambda} V^\lambda) + \Gamma^\sigma_{\mu\lambda}(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\alpha} V^\alpha) - \Gamma^\lambda_{\mu\nu}(\cancel{\partial_\lambda V^\sigma + \Gamma^\sigma_{\lambda\alpha} V^\alpha}) \\
&\quad - \cancel{\partial_\nu \partial_\mu V^\sigma} - \partial_\nu (\Gamma^\sigma_{\mu\lambda} V^\lambda) - \Gamma^\sigma_{\nu\lambda}(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\alpha} V^\alpha) + \Gamma^\lambda_{\nu\mu}(\cancel{\partial_\lambda V^\sigma + \Gamma^\sigma_{\lambda\alpha} V^\alpha}) \\
&= \partial_\mu \Gamma^\sigma_{\nu\lambda} V^\lambda + \cancel{\Gamma^\sigma_{\nu\lambda} \partial_\mu V^\lambda} + \cancel{\Gamma^\sigma_{\mu\lambda} \partial_\nu V^\lambda} + \Gamma^\sigma_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} V^\alpha \\
&\quad - \partial_\nu \Gamma^\sigma_{\mu\lambda} V^\lambda - \cancel{\Gamma^\sigma_{\mu\lambda} \partial_\nu V^\lambda} - \cancel{\Gamma^\sigma_{\nu\lambda} \partial_\mu V^\lambda} - \Gamma^\sigma_{\nu\lambda} \Gamma^\lambda_{\mu\alpha} V^\alpha \\
&= \left(\partial_\mu \Gamma^\sigma_{\nu\lambda} + \Gamma^\sigma_{\mu\alpha} \Gamma^\alpha_{\nu\lambda} - \partial_\nu \Gamma^\sigma_{\mu\lambda} - \Gamma^\sigma_{\nu\alpha} \Gamma^\alpha_{\mu\lambda} \right) V^\lambda \\
&\equiv R^\sigma{}_{\lambda\mu\nu} V^\lambda
\end{aligned}
$$

This definition turns out to be equivalent point 1 above (Exercise).

We defined : $\underbrace{[\nabla_\mu, \nabla_\nu]}_{(1,2)\text{ tensor}} V^\sigma = R^\sigma{}_{\lambda\mu\nu} V^\lambda$ (1,0) tensor

Hence, $R^\sigma{}_{\mu\nu\lambda}$ is a (1,3) tensor. It is called the Riemann tensor and its components can be read off from above:

$$
\boxed{R^\sigma{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\lambda} + \Gamma^\sigma_{\mu\alpha} \Gamma^\alpha_{\nu\lambda} - \partial_\nu \Gamma^\sigma_{\mu\lambda} - \Gamma^\sigma_{\nu\alpha} \Gamma^\alpha_{\mu\lambda}} \tag{3.15}
$$

In a flat spacetime $\exists$ crd's where $g_{\mu\nu} = \text{const}\ \forall p \in M \Rightarrow \Gamma = 0 \Rightarrow R^\sigma{}_{\mu\nu\lambda} = 0$

This goes also in the other direction: $R^\sigma{}_{\mu\nu\lambda} = 0 \Rightarrow \exists$ crd's where $g_{\mu\nu} = \text{const}\ \forall p \in M$ (Exercise)

Therefore, we get the important result:

$R^\sigma{}_{\mu\nu\lambda} = 0 \Leftrightarrow$ spacetime is flat

The Riemann tensor measures deviations from flat spacetime.

Given the metric in any crd's we can now immediately compute the Christoffels using (3.4) and components of the Riemann tensor using (3.15). If any of the components $R^{\sigma}{}_{\mu\nu\lambda} \neq 0$ the spacetime is curved.

Symmetries of the Riemann tensor

In a 4-d space the Riemann tensor $R^{\sigma}{}_{\mu\nu\lambda}$ has $4^4 = 256$ components. Not all of them are independent, however, since the Riemann tensor has several symmetries.

From the very definition $[\nabla_\mu, \nabla_\nu] V^\sigma = R^{\sigma}{}_{\lambda\mu\nu} V^\lambda$ and from (3.15) we see that

$R^{\sigma}{}_{\lambda\mu\nu} = - R^{\sigma}{}_{\lambda\nu\mu}$ antisymmetric under exchange of last two indices

There are also other symmetries which are not immediately transparent. These can be seen e.g. by going to the local inertial frame (symmetry or antisymmetry of a tensor wrt. indices is a crd invariant thing).

$$R_{\hat\sigma\hat\lambda\hat\mu\hat\nu}\Big|_{P_0} = g_{\hat\sigma\hat\alpha} R^{\hat\alpha}{}_{\hat\lambda\hat\mu\hat\nu}\Big|_{P_0} \qquad \text{at } P_0 \text{ where } \Gamma^{\hat\alpha}_{\hat\mu\hat\nu}(P_0) = 0$$

$$= g_{\hat\sigma\hat\alpha}\left(\partial_{\hat\mu}\Gamma^{\hat\alpha}_{\hat\nu\hat\lambda} - \partial_{\hat\nu}\Gamma^{\hat\alpha}_{\hat\mu\hat\lambda}\right)\Big|_{P_0}$$

$$= \frac{1}{2} g_{\hat\sigma\hat\alpha}\,\partial_{\hat\mu}\left(g^{\hat\alpha\hat\beta}\left(\partial_{\hat\nu} g_{\hat\lambda\hat\beta} + \partial_{\hat\lambda} g_{\hat\beta\hat\nu} - \partial_{\hat\beta} g_{\hat\lambda\hat\nu}\right) - (\hat\mu \leftrightarrow \hat\nu)\right)\Big|_{P_0} \qquad (\partial_{\hat\mu} g_{\hat\alpha\hat\beta}(P_0) = 0)$$

$$= \frac{1}{2} \underbrace{g_{\hat\sigma\hat\alpha} g^{\hat\alpha\hat\beta}}_{=\delta^{\hat\beta}_{\hat\sigma}}\left(\partial_{\hat\mu}\partial_{\hat\nu} g_{\hat\lambda\hat\beta} + \partial_{\hat\mu}\partial_{\hat\lambda} g_{\hat\beta\hat\nu} - \partial_{\hat\mu}\partial_{\hat\beta} g_{\hat\lambda\hat\nu} - (\hat\mu \leftrightarrow \hat\nu)\right)\Big|_{P_0}$$

cancels against $-\partial_{\hat\nu}\partial_{\hat\mu} g_{\hat\sigma\hat\beta}$

$$= \frac{1}{2}\left(\partial_{\hat\mu}\partial_{\hat\lambda} g_{\hat\sigma\hat\nu} - \partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\lambda\hat\nu} - \partial_{\hat\nu}\partial_{\hat\lambda} g_{\hat\sigma\hat\mu} + \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\lambda\hat\mu}\right)\Big|_{P_0}$$

While this result holds only at P_0 where $g_{\hat\mu\hat\nu}(P_0) = \eta_{\hat\mu\hat\nu}$, $\partial_{\hat\mu} g_{\hat\alpha\hat\beta}(P_0) = 0$, the symmetries of the result hold for any $P \in M$ and in any frame.

We thus find that the Riemann tensor has the following symmetries:

$$(3.16)\quad \begin{cases} R_{\sigma\lambda\mu\nu} = -R_{\sigma\lambda\nu\mu} \\ R_{\sigma\lambda\mu\nu} = -R_{\lambda\sigma\mu\nu} \\ R_{\sigma\lambda\mu\nu} = R_{\mu\nu\sigma\lambda} \end{cases}$$

$$\left(\underset{\sigma}{\partial_{\hat\mu}}\underset{\nu}{\partial_{\hat\lambda}} \underset{\mu\lambda}{g_{\hat\sigma\hat\nu}} - \underset{\sigma}{\partial_{\hat\mu}}\underset{\mu}{\partial_{\hat\sigma}} \underset{\nu\lambda}{g_{\hat\lambda\hat\nu}} - \underset{\lambda}{\partial_{\hat\nu}}\underset{\nu}{\partial_{\hat\lambda}} \underset{\mu\sigma}{g_{\hat\sigma\hat\mu}} + \underset{\lambda}{\partial_{\hat\nu}}\underset{\mu}{\partial_{\hat\sigma}} \underset{\nu\sigma}{g_{\hat\lambda\hat\mu}}\right)_{\sigma\to\mu,\ \lambda\to\nu}$$
$$= \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\lambda\hat\mu} - \partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\lambda\hat\nu} - \partial_{\hat\nu}\partial_{\hat\lambda} g_{\hat\sigma\hat\mu} + \partial_{\hat\mu}\partial_{\hat\lambda} g_{\hat\sigma\hat\nu}$$

How many degrees of freedom are left among the $4^4 = 256$ components?

$$R_{\sigma\lambda\mu\nu} \qquad = \qquad R_{\underbrace{\sigma\lambda}_{6}\underbrace{\mu\nu}_{6}}$$

antisymm. wrt. $\sigma\lambda$: $\dfrac{4\times 4 - 4}{2} = 6$ dof

antisymm. wrt. $\mu\nu$: $\dfrac{4\times 4 - 4}{2} = 6$ dof

symm. under $\mu\nu \leftrightarrow \sigma\lambda$, the same number of dof's as for a symm. 6×6 matrix:

$$\underbrace{\frac{6\cdot 6 - 6}{2}}_{1/2\text{ of off-diagonals}} + \underbrace{6}_{\text{diagonals}} = \underline{\underline{21}}$$

$\Rightarrow$ (3.16) leaves 21 independent components in $R_{\sigma\lambda\mu\nu}$

There is one more symmetry that follows from the expression of $R_{\hat\sigma\hat\lambda\hat\mu\hat\nu}(P_0)$ above:

$$(3.17)\qquad R_{\sigma\lambda\mu\nu} + R_{\sigma\mu\nu\lambda} + R_{\sigma\nu\lambda\mu} = 0 \qquad \text{cyclic permutation of last 3 indices}$$

This leaves altogether $21 - 1 = 20$ independent components in $R^{\sigma}{}_{\lambda\mu\nu}$. These are precisely the 20 second derivatives $\partial_{\hat\rho}\partial_{\hat\nu} g_{\hat\alpha\hat\beta}$ which we could not set to zero in going to the local Lorentz frame. The Riemann curvature tensor is a tensorial quantity that contains these 20 dof's which measure deviation from a flat spacetime.

Bianchi identity

The Bianchi identity is a derivative constraint equation identically satisfied by the Riemann tensor.

Go to the local Lorentz frame again consider the point P_0 where $g_{\hat\mu\hat\nu}(P_0)=0$, $\partial_{\hat\mu} g_{\hat\alpha\hat\rho}(P_0)=0$

$\Rightarrow \Gamma^{\hat\mu}_{\hat\alpha\hat\beta}(P_0)=0$ and :

$$\nabla_{\hat\alpha} R^{\hat\beta}{}_{\hat\gamma\hat\mu\hat\nu}\Big|_{P_0} = \partial_{\hat\alpha} R^{\hat\beta}{}_{\hat\gamma\hat\mu\hat\nu}\Big|_{P_0}$$

$$= \partial_{\hat\alpha}\left(\partial_{\hat\mu}\Gamma^{\hat\beta}_{\hat\nu\hat\gamma} - \partial_{\hat\nu}\Gamma^{\hat\beta}_{\hat\mu\hat\gamma} + \underbrace{\Gamma\Gamma}_{0} - \underbrace{\Gamma\Gamma}_{0}\right)\Big|_{P_0} \quad \text{as } \Gamma(P_0)=0$$

$$= \left(\partial_{\hat\alpha}\partial_{\hat\mu}\Gamma^{\hat\beta}_{\hat\nu\hat\gamma} - \partial_{\hat\alpha}\partial_{\hat\nu}\Gamma^{\hat\beta}_{\hat\mu\hat\gamma}\right)\Big|_{P_0}$$

$$\left(\nabla_{\hat\alpha} R^{\hat\beta}{}_{\hat\gamma\hat\mu\hat\nu} + \nabla_{\hat\mu} R^{\hat\beta}{}_{\hat\gamma\hat\nu\hat\alpha} + \nabla_{\hat\nu} R^{\hat\beta}{}_{\hat\gamma\hat\alpha\hat\mu}\right)_{P_0} = \Big(\cancel{\partial_{\hat\alpha}\partial_{\hat\mu}\Gamma^{\hat\beta}_{\hat\nu\hat\gamma}} - \cancel{\partial_{\hat\alpha}\partial_{\hat\nu}\Gamma^{\hat\beta}_{\hat\mu\hat\gamma}}$$
$$+ \cancel{\partial_{\hat\mu}\partial_{\hat\nu}\Gamma^{\hat\beta}_{\hat\alpha\hat\gamma}} - \cancel{\partial_{\hat\mu}\partial_{\hat\alpha}\Gamma^{\hat\beta}_{\hat\nu\hat\gamma}}$$
$$+ \cancel{\partial_{\hat\nu}\partial_{\hat\alpha}\Gamma^{\hat\beta}_{\hat\mu\hat\gamma}} - \cancel{\partial_{\hat\nu}\partial_{\hat\mu}\Gamma^{\hat\beta}_{\hat\alpha\hat\gamma}}\Big)\Big|_{P_0}$$

$$=0$$

Since we can do this for any point $P\in M$, we find the tensorial Bianchi identity:

(3.18) $$\nabla_\alpha R^\beta{}_{\gamma\mu\nu} + \nabla_\mu R^\beta{}_{\gamma\nu\alpha} + \nabla_\nu R^\beta{}_{\gamma\alpha\mu} = 0$$ cyclic permutation over the 1st and last two indices

This turns out to be a useful result. The Bianchi identity is essentially the Jacobi identity for covariant derivatives (Exercise):

$$[\nabla_\alpha,[\nabla_\mu,\nabla_\nu]] + [\nabla_\mu,[\nabla_\nu,\nabla_\alpha]] + [\nabla_\nu,[\nabla_\alpha,\nabla_\mu]] = 0$$

Recall $R^\beta{}_{\lambda\mu\nu}u^\lambda = [\nabla_\mu,\nabla_\nu]u^\beta$

Decomposition of the Riemann tensor

Recall that a generic tensor can be uniquely decomposed into symmetric and antisymmetric parts and the symmetric part further into trace and trace-free parts.

Consider e.g. a $(0,2)$ tensor $A_{\mu\nu}$:

$$A_{\mu\nu} = A_{(\mu\nu)} + A_{[\mu\nu]} \quad , \quad A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \quad \text{symm.}$$
$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \quad \text{antisymm.}$$

$$A \equiv g^{\mu\nu} A_{\mu\nu} \quad \text{trace}$$
$$A^{[\mu}{}_{\mu]} = 0 , \quad A^{(\mu}{}_{\mu)} = A$$
$$A_{(\mu\nu)} = A_{(\mu\nu)} - \frac{1}{\underset{\text{dim}(M)}{n}} A g_{\mu\nu} + \frac{1}{n} A g_{\mu\nu} \equiv \underset{\text{trace-free part}}{\hat{A}_{\mu\nu}} + \frac{1}{n} A g_{\mu\nu}$$

$$\Rightarrow A = \hat{A}_{\mu\nu} + \frac{1}{n} A g_{\mu\nu} + A_{[\mu\nu]}$$

Crd independent decomposition into trace, trace-free and antisymm. parts.

We can also decompose the Riemann tensor into trace and trace-free parts.

Symmetries (3.16) set contractions over first and last pair of indices of $R^{\sigma}{}_{\lambda\mu\nu}$ to zero

$$R^{\sigma}{}_{\sigma\mu\nu} = 0 , \quad R^{\mu}{}_{\nu\sigma}{}^{\sigma} = 0$$

The only non-zero contraction is <u>the Ricci tensor</u>:

$$(3.19) \qquad R_{\mu\nu} \equiv R^{\sigma}{}_{\mu\sigma\nu}$$
$$R_{\nu\mu} = R^{\sigma}{}_{\nu\sigma\mu} \underset{(3.17)}{=} -\cancelto{0}{R^{\sigma}{}_{\sigma\mu\nu}} - R^{\sigma}{}_{\mu\nu\sigma} \underset{(3.16)}{=} R^{\sigma}{}_{\mu\sigma\nu} = R_{\mu\nu}$$

$\Rightarrow$ <u>$R_{\mu\nu} = R_{\nu\mu}$ Ricci tensor is symmetric</u>

The trace of $R_{\mu\nu}$ is called <u>the Ricci scalar</u>:

$$(3.20) \qquad R \equiv R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

The Ricci tensor $R_{\mu\nu}$ and Ricci scalar R encode all information of traces (= contractions) of the Riemann tensor $R^{\sigma}{}_{\lambda\mu\nu}$.

$R_{\mu\nu}$ symm $\frac{(4\times4-4)}{2}+4=6+4=10$ dof

The other 10 dof of $R^{\sigma}{}_{\lambda\mu\nu}$ are encoded in the trace-free part of $R^{\sigma}{}_{\lambda\mu\nu}$ called the Weyl tensor:

$$(3.21)\qquad C_{\lambda\sigma\mu\nu}\equiv R_{\lambda\sigma\mu\nu}-\frac{2}{\underset{\underset{\dim(M)}{\uparrow}}{n-2}}\Big(\underbrace{g_{\lambda[\mu}R_{\nu]\sigma}}_{\equiv\frac{1}{2}(g_{\lambda\mu}R_{\nu\sigma}-g_{\lambda\nu}R_{\mu\sigma})}-g_{\sigma[\mu}R_{\nu]\lambda}\Big)+\frac{2}{(n-1)(n-2)}g_{\lambda[\mu}g_{\nu]\sigma}R$$

The Weyl tensor has the same symmetries as $R_{\lambda\sigma\mu\nu}$ but all contractions of C are zero. The Weyl tensor is defined for $\dim(M)\geq3$ and for $\dim(M)=3$ $C_{\lambda\sigma\mu\nu}=0$.

Contracting the Bianchi idenity (3.18) we get:

$$\nabla_\alpha \underbrace{R^{\beta}{}_{\gamma\beta\nu}}_{=R_{\gamma\nu}}+\nabla_\beta R^{\beta}{}_{\gamma\nu\alpha}+\nabla_\nu \underbrace{R^{\beta}{}_{\gamma\alpha\beta}}_{=-R_{\gamma\alpha}}=0$$

$$\nabla_\alpha R_{\gamma\nu}+\nabla_\beta R^{\beta}{}_{\gamma\nu\alpha}-\nabla_\nu R_{\gamma\alpha}=0\quad \big/\cdot g^{\alpha\gamma}$$

$$\nabla^{\gamma}R_{\gamma\nu}+\nabla_\beta \underbrace{R^{\beta}{}_{\gamma\nu}{}^{\gamma}}_{=R_{\gamma}{}^{\beta\gamma}{}_{\nu}=R^{\gamma\beta}{}_{\gamma\nu}=R^{\beta}{}_{\nu}}-\nabla_\nu R=0$$

$$\Rightarrow 2\nabla^{\mu}R_{\mu\nu}=\nabla_\nu R$$

$$(3.22)\qquad \nabla^{\mu}R_{\mu\nu}=\frac{1}{2}\nabla_\nu R$$

It is useful to define the Einstein tensor $G_{\mu\nu}$ as:

$$(3.23)\qquad G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$$

From (3.22) we then get:

$$(3.24)\qquad \nabla^{\mu}G_{\mu\nu}=0\qquad (\text{as } \nabla_\alpha g_{\mu\nu}=0)$$

The trace of the Einstein tensor is:

$$G^{\mu}{}_{\mu} = R - \frac{4}{2} R = -R$$

We will soon see that the Einstein tensor enters directly in the Einstein equations.