2. Differential geometry in curved spacetimes

In Newtonian physics, there are two a priori different masses:

$$\bar{F} = \underset{\substack{\uparrow \\ \text{inertial mass}}}{m_i} \cdot \bar{a} \quad , \quad \bar{F} = -\underset{\substack{\uparrow \\ \text{gravitational mass}}}{m_g} \nabla\phi \leftarrow \text{gravitational potential } \phi$$

For a body falling with an acceleration $\bar{a}$ in a gravitational field these two different ways of writing the gravitational force $\bar{F}$ acting on the body yield:

$$\bar{a} = -\frac{m_g}{m_i} \nabla\phi$$

If $m_g \neq m_i$ we would see different objects falling at at different acceleration in the same field. This appears not to be the case in the nature and observations tell us that $m_i = m_g$. This is promoted to a conjecture in GR.

Weak equivalence principle: All freely falling objects have the same acceleration $-\nabla\phi$ in a gravitational field

$$\bar{a} = -\nabla\phi \Rightarrow m_i = m_g$$

This says that gravity is uniform, it acts on the same way on all massive bodies.

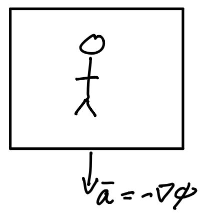
This differs from e.g. electromagnetism:

$$\bar{a} = \frac{q}{m} \bar{E}$$

$\nwarrow$ particles with a different charge q have different trajectories

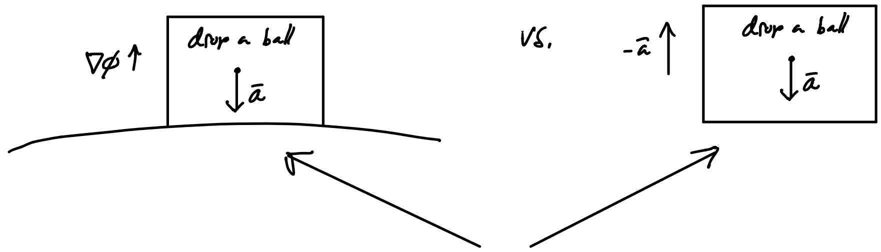
The weak equivalence principle implies that uniform gravitational field cannot be distinguished from uniform acceleration.

Consider the famous example of a physicist in a freely falling elevator:



Observer feels no acceleration $\Rightarrow$ feels herself massless!

a lab at rest in grav. field — an accelerated lab + no gravity



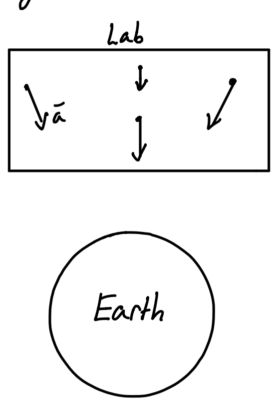
Motion of freely falling (no other forces than gravity) objects exactly the same in both cases. Uniform $\nabla\Phi$ indistinguishable from constant acceleration.

Promote this to a broader conjecture:

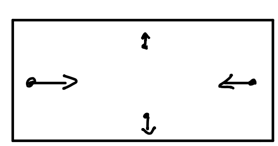
Strong equivalence principle: All physics reduces to Special Relativity in small enough regions.

Here small enough means locally, i.e. in the limit $\Delta x \to 0$.

In larger regions we start to see inhomogeneities of the gravitational field:

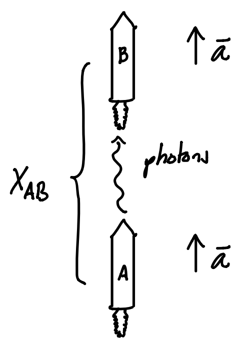


see tidal forces within the box



In SR the inertial frames have a special role, laws of physics manifestly invariant under inertial transformations. In the presence of gravity we also have special frames. Frames of freely falling observers are locally inertial and physics $\to$ SR. However, these frames cannot be extended over larger spacetime region. Hence global Lorentz transformations will no longer be a symmetry but instead they will be replaced by local, space and time dependent, transformations.

The strong equivalence principle predicts gravitational redshift:



Both A, B move with the same constant acceleration, their distance X_{AB} remain constant. A sends a light ray to B.

No gravity and $v \ll c$, the photon travel time is $\Delta t = X_{AB}$

As B receives the signal, she moves with a velocity $\Delta v = a\Delta t$ with respect to the instantaneous rest frame of A at the sending time t_0. This gives rise to the ordinary Doppler effect, e.g. from (1.35)

$$\lambda_{obs} = \frac{-2\pi}{k^\mu u_\mu}$$

in A's inst. rest. frame at t_0

$$u_B^\mu = (\gamma, \gamma\bar{v}) = (1, \bar{v}) = (1, a\Delta t\, \hat{v})$$ unit vector

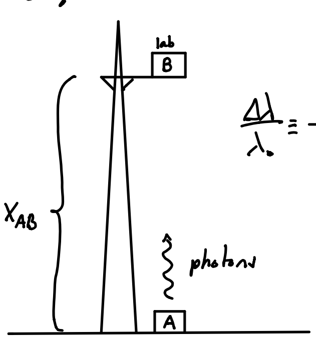
$$k^\mu_{signal} = \frac{2\pi}{\lambda_0}(1, \hat{v})$$

$$\lambda_{obs} = \frac{-2\pi}{\frac{2\pi}{\lambda_0}(-1+\Delta t a)} = \frac{\lambda_0}{1-a\Delta t} = \lambda_0(1+a\Delta t) \quad , \Delta v = a\Delta t \ll 1$$

$$\frac{\lambda_{obs}-\lambda_0}{\lambda_0} = a\Delta t = a X_{AB} \quad \text{Doppler shift}$$

Strong equivalence principle: should see the same redshift in a grav. field $\nabla\phi = \bar{a}_g$

acc of a ref. frame wrt. freely falling body

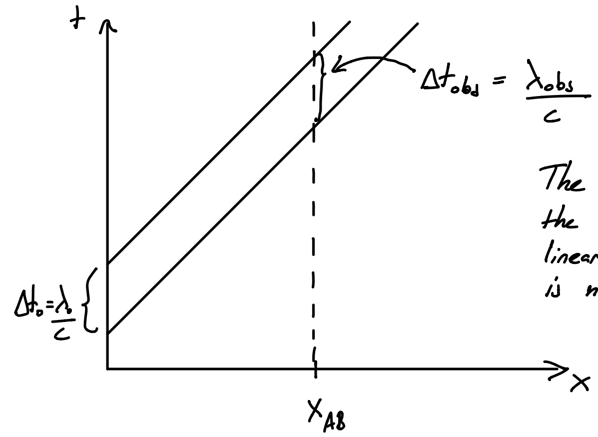


$$\frac{\Delta\lambda}{\lambda_0} \equiv \frac{\lambda_{obs}-\lambda_0}{\lambda_0} = a_g X_{AB} = a_g \frac{X_{AB}}{c^2}$$

$$= \frac{a_g}{m/s^2} X_{AB} \cdot 10^{-17}$$

tiny effect for "normal" gravitational fields/distances, observable in cosmic scales.

Consider the spacetime diagram of the process:



$$\Delta t_{obs} = \frac{\lambda_{obs}}{c} = \frac{\lambda_0 + \Delta\lambda}{c} > \Delta t_0$$

The clock of B appears to tick slower than the clock of A, t-axis cannot have a linear scale which means that the spacetime is not Minkowski

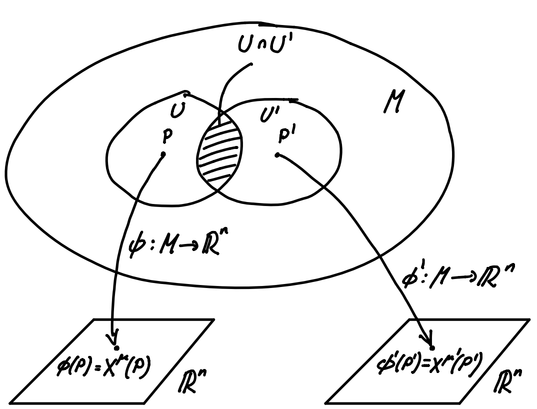
$\Longrightarrow$ Gravity affects the geometry of the spacetime. The spacetime is no longer Minkowski but it becomes curved.

↓ 22.1.

2.1 Manifolds

Curved spacetimes are represented by manifolds (suom. = monisto) equipped with a metric.

An n-dim. manifold is essentially a set M which can be locally mapped to $\mathbb{R}^n$. A single mapping $M \longrightarrow \mathbb{R}^n$ may not extend over the full manifold but the entire set M can be covered by smoothly patching together different maps*.



A chart (U, ϕ)

$U \subset M$ a subset of M

$\phi: U \to \mathbb{R}^n$ one-to-one map which gives the coordinates of $p \in U$

$\phi(p) = X^\mu(p)$

$X^\mu(p)$: coordinates of p in the chart (U, ϕ)

*C.f. strong equivalence principle: physics locally $\Leftrightarrow$ SR (Minkowski = $\mathbb{R}^n$ topologically) but SR does not hold globally.

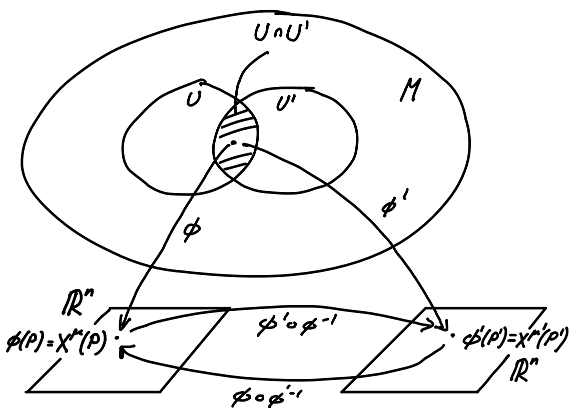
An atlas $\{(U_\alpha, \phi_\alpha)\}$ is a collection of all charts s.t.:

1) the union of U_α equals M $\quad \bigcup_\alpha U = M$ (i.e. the atlas covers M and no extra points)
2) transition from one chart to another is smooth:

$$P \in U_\alpha \cap U_{\alpha'} \neq \emptyset \quad \text{(the intersection is not an empty set)}$$

$$\phi'(P) = \phi'(\phi^{-1}(X^\mu(P))) = X^{\mu'}(X^\mu(P))$$
$$\phi(P) = \phi(\phi'^{-1}(X^{\mu'}(P))) = X^\mu(X^{\mu'}(P))$$

C^∞ differentiable, all order derivatives finite.



Both $\phi' \circ \phi^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ and $\phi \circ \phi'^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ C^∞ functions in $U \cap U'$.

We can now define a C^∞ manifold M as a set of points equipped with a maximal atlas (all possible charts). In GR we also define a metric for the spacetime. (See e.g. [M. Nakahara: Geometry, topology and physics] for more details.)

In practice, the smoothness of transitions btw. different crd functions means that the Jacobians of

$$X^\mu \to X^{\mu'}: \quad \frac{\partial X^{\mu'}}{\partial X^\nu} \qquad \left(\text{For global Lorentz } \frac{\partial X^{\mu'}}{\partial X^\nu} = \Lambda^{\mu'}{}_\nu\right)$$

$$X^{\mu'} \to X^\mu: \quad \frac{\partial X^\mu}{\partial X^{\nu'}}$$

have no singularities. The transitions are also invertible (one-to-one maps)

$$\det\left[\frac{\partial X^{\mu'}}{\partial X^\nu}\right] \neq 0, \quad \det\left[\frac{\partial X^\mu}{\partial X^{\nu'}}\right] \neq 0 \qquad \frac{\partial X^{\mu'}}{\partial X^\sigma}\frac{\partial X^\sigma}{\partial X^{\nu'}} = \delta^{\mu'}{}_{\nu'}$$

In particular, the smoothness guarantees that the chain rule always holds:

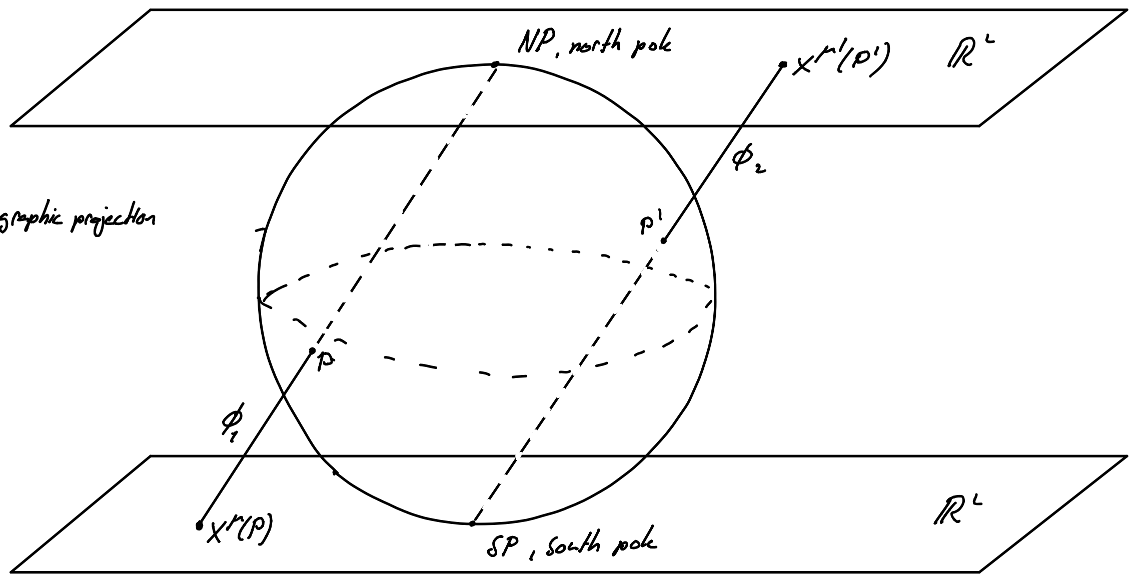
$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\nu'}}$$

note that the Jacobian in general depends on x^{μ}, i.e. the matrix elements are not constants.

Example

2-dim sphere S^2 needs at least two charts (the topology of S^2 different from $\mathbb{R}^2$)

stereographic projection



$U_1 = S^2 \setminus NP$ tangent of north pole never goes through the sphere

$U_2 = S^2 \setminus SP$ tangent of south pole never goes through the sphere

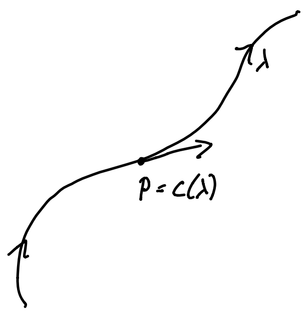
2.2 Vectors, duals and tensors

Definition of tensorial quantities on general C^∞ manifolds (physically especially in curved spacetimes) is very similar to our previous discussion of Minkowski space.

(just replace $\Lambda^{\mu'}{}_{\nu} \to \frac{\partial x^{\mu'}}{\partial x^{\nu}}$)

Vectors

As before, we define vectors as directional derivatives along curves



$c: \mathbb{R} \to M$ curve
$f: M \to \mathbb{R}$ function

(2.1) $$V = \frac{d x^\mu(c(\lambda))}{d\lambda} \frac{\partial}{\partial x^\mu(c(\lambda))} \equiv \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$$

components $V^\mu = \frac{dx^\mu}{d\lambda}$

basis vector $e_\mu = \frac{\partial}{\partial x^\mu}$

could also choose other than the crd basis $e_\mu = \partial_\mu$ but in this course we will use the crd basis

A vector acting on function gives the derivative of the function along the curve:

(2.2) $$V[f] = \frac{dx^\mu(c(\lambda))}{d\lambda} \frac{\partial f(c(\lambda))}{\partial x^\mu(c(\lambda))} \equiv \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} = \frac{df}{d\lambda}$$

Transformation properties under $x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} x^\nu$:

$$V = V^{\mu'}\partial_{\mu'} = V^\mu \partial_\mu = V^\mu \frac{\partial x^{\nu'}}{\partial x^\mu}\partial_{\nu'}$$

$$\Rightarrow \quad V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} V^\nu \qquad (2.3)$$

This is just like in Minkowski where $\frac{\partial x^{\mu'}}{\partial x^\nu} = \Lambda^{\mu'}{}_\nu$ under global Lorentz transf.

However, in (2.3) $\frac{\partial x^{\mu'}}{\partial x^\nu}$ is the Jacobian of any crd transformation which in general will be a non-trivial function of t and x^i.

Dual vectors

As before: $\omega : T_p \to \mathbb{R}$ a linear map from vectors to real numbers.

(2.4) $$\omega = \omega_\mu dx^\mu$$

component

dual basis vector $\theta^\mu = dx^\mu$ in the crd basis

(Note that $dx^\mu = d(x^\mu) = \frac{\partial x^\mu}{\partial x^\nu}\theta^\nu = \delta^\mu{}_\nu \theta^\nu$ is just a gradient of the crd function x^μ)

(2.5) $$\omega[V] = \omega_\mu dx^\mu[V^\nu \partial_\nu] = \omega_\mu V^\nu \underbrace{dx^\mu[\partial_\nu]}_{=\delta^\mu{}_\nu} = \omega_\mu V^\mu \quad , \; V \in T_p$$

Transformation properties

chain rule for the differential

$$\omega = \omega_{\mu'} dx^{\mu'} = \omega_\mu dx^\mu = \omega_\mu dx^{\nu'} \frac{\partial x^\mu}{\partial x^{\nu'}}$$

$$\Rightarrow \quad \omega_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} \omega_\nu \qquad (2.6)$$

General tensors

The definition is again exactly the same as before. Consider an (m,n) tensor T

$$T : \underbrace{T_p^* \times \ldots \times T_p^*}_{m \text{ copies}} \times \underbrace{T_p \times \ldots \times T_p}_{n \text{ copies}} \longrightarrow \mathbb{R} \quad \text{linear in all arguments.}$$

(2.7)
$$\begin{aligned}
&T[\omega^{(1)}, \ldots, \omega^{(m)}, V_{(1)}, \ldots, V^{(n)}] \\
&= T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_m} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_n} [\omega^{(1)}_{\alpha_1} dx^{\alpha_1}, \ldots, \omega^{(m)}_{\alpha_m} dx^{\alpha_m}, V^{\beta_1}_{(1)} \partial_{\beta_1}, \ldots, V^{\beta_n}_{(n)} \partial_{\beta_n}] \\
&= T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \omega^{(1)}_{\alpha_1} \ldots \omega^{(m)}_{\alpha_m} V_{(1)}^{\beta_1} \ldots V_{(n)}^{\beta_n} \underbrace{\partial_{\mu_1}[dx^{\alpha_1}]}_{=\delta_{\mu}{}^{\alpha_1}} \ldots \partial_{\mu_m}[dx^{\alpha_m}] \underbrace{dx^{\nu_1}[\partial_{\beta_1}]}_{=\delta_\beta^{\nu_1}} \ldots dx^{\nu_n}[\partial_{\beta_n}] \\
&= T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \omega^{(1)}_{\mu_1} \ldots \omega^{(m)}_{\mu_m} V_{(1)}^{\nu_1} \ldots V_{(n)}^{\nu_n}
\end{aligned}$$

Transformation properties:

$$T^{\mu_1'\ldots\mu_m'}{}_{\nu_1'\ldots\nu_n'}\partial_{\mu_1'}\otimes\ldots\otimes\partial_{\mu_m'}\otimes dx^{\nu_1'}\otimes\ldots dx^{\nu_n'}$$
$$= T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}\partial_{\mu_1}\otimes\ldots\otimes\partial_{\mu_m}\otimes dx^{\nu_1}\otimes\ldots dx^{\nu_n}$$
$$= T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}\frac{\partial x^{\alpha_1'}}{\partial x^{\mu_1}}\ldots\frac{\partial x^{\alpha_m'}}{\partial x^{\mu_m}}\frac{\partial x^{\beta_1}}{\partial x^{\nu_1'}}\ldots\frac{\partial x^{\beta_n}}{\partial x^{\nu_n'}}\partial_{\alpha_1'}\otimes\ldots\otimes\partial_{\alpha_m'}\otimes dx^{\beta_1'}\otimes\ldots dx^{\beta_n'}$$

$$\Rightarrow \quad T^{\mu_1'\ldots\mu_m'}{}_{\nu_1'\ldots\nu_n'} = \frac{\partial x^{\mu_1'}}{\partial x^{\alpha_1}}\ldots\frac{\partial x^{\mu_m'}}{\partial x^{\alpha_m}}\frac{\partial x^{\beta_1}}{\partial x^{\nu_1'}}\ldots\frac{\partial x^{\beta_n}}{\partial x^{\nu_n'}}T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n} \qquad (2.8)$$

So far everything has been exactly analogous to Minkowski space. An important difference concerns derivatives of tensors. In Minkowski space e.g. $\partial_\mu V^\nu$ forms a $(1,1)$ tensor (in cartesian crd's and under global Lorentz transformations). This is not true in general curved spacetime under $x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}x^\nu$

$$\partial_{\mu'}V^{\nu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^{\nu'}}{\partial x^\beta}V^\beta\right)$$
$$= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\partial_\alpha V^\beta + \underbrace{\frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\alpha \partial x^\beta}V^\beta}_{\text{non-tensorial part}} \neq \partial_\mu V^\nu$$

$\Rightarrow \partial_\mu V^\nu$ is not a tensor in general

Later we will introduce a covariant derivative ∇_μ which gives a tensorial generalisation of the partial derivative ∂_μ. But before that we will need to discuss the metric in more detail.

2.3 The metric

Mathematically a manifold does not need to have a metric. The atlas defines the topology of the manifold M and the metric is an additional structure which defines the geometry. In GR we will always be discussing manifolds equipped with the metric and the metric will be the physical object which describes gravity in the setup.

The metric is a symmetric (0,2) tensor:

$$(2.9) \qquad g = g_{\mu\nu} dx^\mu \otimes dx^\nu \qquad g_{\mu\nu} = g_{\nu\mu} \quad ,$$

which is non-singular:

$$\det g_{\mu\nu} \neq 0 \Rightarrow \exists \, g^{\mu\nu} \text{ s.t. } g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$$

The (2,0) tensor $g^{\mu\nu}$ is called the inverse metric

In General Relativity the metric plays a key role. It gives:

1) notions of past and future & causality (lightcones)
2) proper time and proper length
3) geodesics which are trajectories of freely falling particles
4) generalisation of Newtonian gravitational potential
5) local inertial frames
6) inner products

The line element

The metric gives the notion of distances in the spacetime. This is often expressed in terms of the line element:

$$(2.10) \qquad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

components of the metric

infinitesimal crd displacements

$ds^2 < 0$ timelike
$ds^2 = 0$ lightlike
$ds^2 > 0$ spacelike

The infinitesimal displacement $x^\mu \to x^\mu + dx^\mu$ is generated by the vector $dx^\mu \frac{\partial}{\partial x^\mu}$, where the components are just the displacements. Now the line element (2.10) is just the action of g on two displacement vectors:

$$g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \left(dx^\mu \frac{\partial}{\partial x^\mu}, dx^\nu \frac{\partial}{\partial x^\nu} \right) = g_{\alpha\beta}\, dx^\mu dx^\nu \delta_\mu{}^\alpha \delta_\nu{}^\beta = g_{\mu\nu}\, dx^\mu dx^\nu = ds^2$$

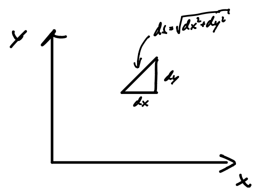
We will often call (2.10) the metric although strictly speaking the metric is the $(0,2)$ tensor $g_{\mu\nu}\, dx^\mu \otimes dx^\nu$.

Example

Euclidean 2-dim space in Cartesian coordinates (x,y)

$$g = dx \otimes dx + dy \otimes dy$$

$$ds^2 = dx^2 + dy^2$$



The same in polar coordinates (r,ϕ)

$$g = dr \otimes dr + r^2\, d\phi \otimes d\phi$$

$$ds^2 = dr^2 + r^2 d\phi^2$$

$x = r\cos\phi$
$y = r\sin\phi$

$$dx = \cos\phi\, dr - r\sin\phi\, d\phi$$

$$dy = \sin\phi\, dr + r\cos\phi\, d\phi$$

$$dx^2 + dy^2 = (\cos^2\phi + \sin^2\phi) dr^2 + r^2(\cos^2\phi + \sin^2\phi) d\phi^2 = dr^2 + r^2 d\phi^2$$

As the above example demonstrates, the same metric can look very different in different coordinates. Whether the spacetime is flat or not may therefore not be immediately obvious from the form of the metric given in some crd's. However, if the spacetime has flat geometry there exists a crd system where the components $g_{\mu\nu} = \text{constant} \quad \forall\, p \in M$. In curved spacetimes it is not possible to find coordinates which cover the entire manifold and where $g_{\mu\nu} = \text{constant}$. Later we will also define a tensorial quantity which directly measures the curvature and vanishes (in all crd systems) if the spacetime is flat.

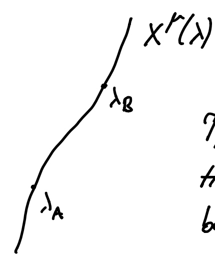
In the Euclidean space a distance between two points P and P' is the length of the straight line which connects the points. In curved spacetime we have no uniquely defined distance between two points P and P'! We can only compute lengths of different curves which connect the points P and P'. The result obviously depends on the curve chosen.

Proper time (for $ds^2 < 0$)

Using the metric, or line element, we can compute lengths of different curves on the manifold. For timelike curves we define the proper time as a straightforward generalisation of the Minkowski case:

$$(2.11) \qquad d\tau \equiv \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$$

$$\tau_{AB} = \int_{\lambda_A}^{\lambda_B} d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

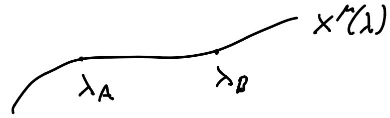


τ_{AB} gives the physical time elapsed along $x^\mu(\lambda)$ between λ_A and λ_B.

Proper length (for $ds^2 > 0$)

For spacelike curves $x^\mu(\lambda)$, the physical length between λ_A and λ_B is given by

$$s_{AB} = \int_{\lambda_A}^{\lambda_B} d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \qquad (2.12)$$



Lightlike curves ($ds^2 = 0$)

As in Minkowski, light and any other massless particles move along null curves

$$ds^2 = 0 \quad \text{for massless particles}$$

These curves have zero length.

Raising and lowering of indices with the metric

As in the Minkowski space we define:

$$\omega^\mu = g^{\mu\nu}\omega_\nu \quad , \text{ maps the dual } \omega_\mu \text{ to a vector } \omega^\mu$$

$$V_\mu = g_{\mu\nu} V^\nu \quad , \text{ maps the vector } V^\mu \text{ to a dual}$$

And similarly for tensors of any rank, e.g. $A^\mu{}_{\nu\rho} = g^{\mu\sigma} A_{\sigma\nu\rho}$.
Given that we will always discuss metric spaces, why don't we just map all duals to vectors and avoid defining the duals altogether? The point is that the metric is a dynamical degree of freedom in GR and we need to solve for its equations of motion before we know it. Therefore we also need to be careful in defining vectors and duals separately.

Inner product

$$u \cdot v \equiv g(u,v) \underset{\substack{\uparrow \\ \text{metric is symmetric}}}{=} g(v,u) = g_{\mu\nu} u^\mu v^\nu = u_\mu v^\mu$$

Using the inner product we define the norm (or its square) as before:

$$u \cdot u = g_{\mu\nu} u^\mu u^\nu = u_\mu u^\mu$$

$u_\mu u^\mu < 0$ timelike vector, tangent to a curve $ds^2 < 0$

$u_\mu u^\mu = 0$ lightlike vector, — " — $ds^2 = 0$

$u_\mu u^\mu > 0$ spacelike vector, — " — $ds^2 > 0$

Example

An expanding homogeneous and isotropic spacetime is described by the Robertson–Walker (RW) metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \underbrace{a^2(t)}_{\text{scale factor}}(dx^2 + dy^2 + dz^2)$$

The coordinates (t, x, y, z) are the so called comoving coordinates where the symmetries of the spacetime is manifest.

In the comoving coordinates:

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & a^2(t) & & \\ & & a^2(t) & \\ 0 & & & a^2(t) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & a^{-2}(t) & & \\ & & a^{-2}(t) & \\ 0 & & & a^{-2}(t) \end{pmatrix}$$

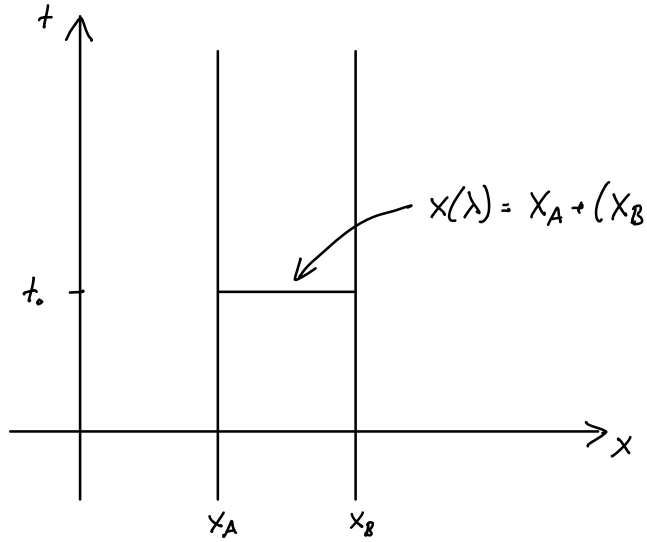
Freely falling observers are at rest in the comoving frame $x^i = \text{const.}$, hence also called comoving observers. Consider two comoving observers

A: $x^\mu_A = (t, x_A, 0, 0)$ ← rotate the crd's s.t. $y_A = y_B = z_A = z_B = 0$

B: $x^\mu_B = (t, x_B, 0, 0)$

The comoving coordinates x_A, x_B are constants and hence $x_B - x_A = \text{const.}$ However, the physical distance between A and B grows as the spacetime expands, i.e. $a(t)$ evolves.

At any constant time event, the 3d surface $t = \text{const.}$ has the geometry of $\mathbb{R}^3$ and we have a unique concept of a straight line connecting A and B : $x(\lambda) = x_A + (x_B - x_A)\lambda$



What is the physical distance along this curve from A to B? This is the proper length of the curve

$$d_{AB}(t) = \int_0^1 d\lambda \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = \int_0^1 d\lambda \sqrt{a^2(t)(x_B - x_A)^2} = a(t)\,|x_B - x_A|$$

$\Rightarrow$ The physical distance of two comoving observers changes due to the expansion of spacetime.

The RW metric describes our observable universe on large scales $d(t_0) \gtrsim 100$ Mpc (t_0 = today) where the universe is approximately homogeneous and isotropic. For most of the history of the universe, the scale factor $a(t)$ is described by a power law:

$a(t) = t^p$, $0 < p < 1$, $a(t) \to 0$ as $t \to 0$ singularity at $t=0$, the manifold ends at $t=0$ ("Big bang")

Consider the causal structure determined by the light cone. Ligh travels along null curves $ds^2=0$:

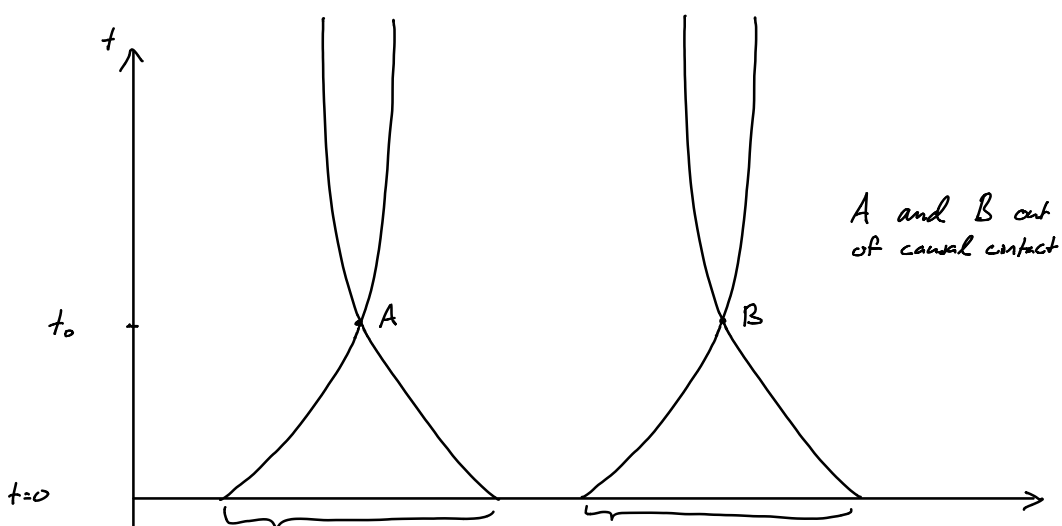
$$dt^2 = \underbrace{a^2(t)}_{=t^{2p}}\, dx^2 \qquad \text{(again rotate crd's s.t. } dy=dz=0)$$

$$\int_{x_0}^{x} dx = \pm \int_0^t \frac{dt}{t^p}$$ light ray sent at $t=0$ from $x=x_0$ along the x-axis

$$x - x_0 = \pm \frac{1}{1-p}\, t^{1-p}\Big|_0^t$$

$$x - x_0 = \pm \frac{1}{1-p}\, t^{1-p} \quad \Rightarrow \quad t = (1-p)^{\frac{1}{1-p}} \left(\mp(x-x_0)\right)^{\frac{1}{1-p}}$$

By the time t, light has travelled the coordinate distance $\Delta x(t) = \frac{1}{1-p} t^{1-p}$. Points with a greater separation $|x_A - x_B| > \Delta x$ cannot have exchanged any information by the time t.



horizon of A, i.e. points $|x_A - x| < \frac{1}{1-p} t^{1-p}$ which can have exchanged information with A

horizon of B

2.4 Local inertial frame

As already mentioned, in curved spacetime it is not possible to find coordinates where $g_{\mu\nu} = \text{const.}$ for all $p \in M$. However, it is always possible to choose coordinates $x^{\hat\mu}$ such that:

$$(2.14) \qquad g_{\hat\mu\hat\nu}(P) = \eta_{\hat\mu\hat\nu} \quad , \quad \partial_{\hat\sigma} g_{\hat\mu\hat\nu}(P) = 0 \qquad \text{at any single point } P$$

These coordinates are called local inertial coordinates or local Lorentz frame.
In general:

$$\partial_{\hat\sigma}\partial_{\hat\beta}\, g_{\hat\mu\hat\nu}(P) \neq 0 \Rightarrow g_{\hat\mu\hat\nu}(P') \neq \eta_{\hat\mu\hat\nu} \quad \text{for } P' \neq P$$

meaning that as soon as we deviate from P we see deviation from the Minkowski form of the metric. This is all just saying that by going to the local inertial frame we can locally (at a point) remove all effects of curvature in accordance to the equivalence principles.

Let us now show that the local Lorentz frame where (2.14) holds indeed exists.
Choose any $P_0 \in M$ and perform a constant shift $x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu$, $a^\mu = \text{const.}$ to set $\tilde{x}^\mu(P_0) = 0$:

$$\tilde{x}^\mu(P_0) = x^\mu(P_0) + a^\mu = 0 \Rightarrow a^\mu = -x^\mu(P_0)$$
$$\tilde{x}^\mu(P) = x^\mu(P) - x^\mu(P_0)$$

The metric components are unaffected by this:

$$g_{\tilde\mu\tilde\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta} = g_{\mu\nu}$$

Then perform another (non-trivial) change of coordinates: $\tilde{x}^\mu \to \hat{x}^\mu(\tilde{x}^\mu)$ s.t. $\hat{x}(P_0) = 0$ and expand $\tilde{x}^\mu(\hat{x}^\mu)$ around P_0:

$$\tilde{x}^\mu(P) = \tilde{x}^\mu(P_0) + \frac{\partial \tilde{x}^\mu}{\partial \hat{x}^\alpha}\bigg|_{P_0} \underbrace{\left(\hat{x}^\alpha(P) - \hat{x}^\alpha(P_0)\right)}_{=\hat{x}^\alpha(P)\equiv \hat{x}^\alpha} + \frac{1}{2}\frac{\partial^2\tilde{x}^\mu}{\partial\hat{x}^\alpha\partial\hat{x}^\beta}\bigg|_{P_0}\hat{x}^\alpha\hat{x}^\beta + \frac{1}{6} \frac{\partial^3 \tilde{x}^\mu}{\partial\hat{x}^\alpha\partial\hat{x}^\beta\partial\hat{x}^\gamma}\bigg|_{P_0}\hat{x}^\alpha\hat{x}^\beta\hat{x}^\gamma + \dots$$

$4\cdot 4 = 16$ dof

symm. $\alpha\beta$

$4\cdot\left(4 + \frac{4\cdot 3}{2!}\right) = 40$ dof

symm. $\alpha\beta\gamma$

$4\cdot\left(\underset{xxx}{4} + \underset{xx\cdot y}{4\cdot 3} + \underset{xyz}{\frac{4\cdot 3\cdot 2}{3!}}\right) = 80$ dof

The metric components in the new coordinates $\hat{x}^\mu$ are just functions of the coordinates and can also be expanded around P_0:

$$g_{\hat\mu\hat\nu}(P) = g_{\hat\mu\hat\nu}(P_0) + \partial_{\hat\sigma} g_{\hat\mu\hat\nu}\Big|_{P_0} \hat{x}^\sigma + \frac{1}{2}\partial_{\hat\sigma}\partial_{\hat\rho}\, g_{\hat\mu\hat\nu}\Big|_{P_0} \hat{x}^\sigma\hat{x}^\rho + \dots$$

$$= \frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu} g_{\alpha\beta} + \left(\frac{\partial^2 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu} g_{\alpha\beta} + (\mu\leftrightarrow\nu) + \frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial x^\gamma}{\partial\hat{x}^\sigma}\partial_\gamma g_{\alpha\beta}\right)\hat{x}^\sigma$$

$$+\frac{1}{2}\Bigg[\left(\frac{\partial^3 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\rho\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu} g_{\alpha\beta} + \frac{\partial^2 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\mu}\frac{\partial^2 x^\beta}{\partial\hat{x}^\rho\partial\hat{x}^\nu} g_{\alpha\beta} + (\mu\leftrightarrow\nu)\right) + \frac{\partial^2 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\partial_{\hat\rho} g_{\alpha\beta}$$

$$+\left(\frac{\partial^2 x^\alpha}{\partial\hat{x}^\mu\partial\hat{x}^\rho}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial x^\gamma}{\partial\hat{x}^\sigma}\partial_\gamma g_{\alpha\beta} + (\mu\leftrightarrow\nu)\right) + \frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial^2 x^\gamma}{\partial\hat{x}^\sigma\partial\hat{x}^\rho}\partial_\gamma g_{\alpha\beta}$$

$$+\frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial x^\gamma}{\partial\hat{x}^\sigma}\frac{\partial x^\delta}{\partial\hat{x}^\rho}\partial_\gamma\partial_\delta g_{\alpha\beta}\Bigg]\hat{x}^\sigma\hat{x}^\rho + \dots$$

This yields:

$$g_{\hat\mu\hat\nu}(P_0) = \frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu} g_{\alpha\beta} \qquad \text{10 eqs}$$

$$\partial_{\hat\sigma} g_{\hat\mu\hat\nu}\Big|_{P_0} = \frac{\partial^2 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu} g_{\alpha\beta} + (\mu\leftrightarrow\nu) + \frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial x^\gamma}{\partial\hat{x}^\sigma}\partial_\gamma g_{\alpha\beta} \qquad \text{40 eqs}$$

$$\partial_{\hat\sigma}\partial_{\hat\rho}\, g_{\hat\mu\hat\nu}\Big|_{P_0} = \Bigg(\frac{\partial^3 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\rho\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu} g_{\alpha\beta} + \frac{\partial^2 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\mu}\frac{\partial^2 x^\beta}{\partial\hat{x}^\rho\partial\hat{x}^\nu} g_{\alpha\beta} + \frac{\partial^2 x^\alpha}{\partial\hat{x}^\mu\partial\hat{x}^\rho}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial x^\gamma}{\partial\hat{x}^\sigma}\partial_\gamma g_{\alpha\beta} + (\mu\leftrightarrow\nu)$$

$$+\frac{\partial^2 x^\alpha}{\partial\hat{x}^\sigma\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\partial_{\hat\rho} g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial^2 x^\gamma}{\partial\hat{x}^\sigma\partial\hat{x}^\rho}\partial_\gamma g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\frac{\partial x^\gamma}{\partial\hat{x}^\sigma}\frac{\partial x^\delta}{\partial\hat{x}^\rho}\partial_\gamma\partial_\delta g_{\alpha\beta}\Bigg) \qquad \text{100 eqs}$$

Now showing that we can set (2.14) amounts to comparing the available dof to number of eqs.

1) $g_{\hat\mu\hat\nu}(P_0) = \underbrace{\frac{\partial x^\alpha}{\partial \hat x^\mu}\frac{\partial x^\beta}{\partial \hat x^\nu}}_{16\text{ dof}} g_{\alpha\beta} = \eta_{\hat\mu\hat\nu}$ 10 eqs $\Rightarrow$ 16-10=6 dof left in $\frac{\partial x}{\partial \hat x}$

The remaining 6 parameters in $\frac{\partial x}{\partial \hat x}$ are the 6 parameters of the Lorentz group which leave $g_{\mu\nu}$ invariant.

2) $$\partial_{\hat\sigma} g_{\hat\mu\hat\nu}\Big|_{P_0} = \underbrace{\frac{\partial^2 x^\alpha}{\partial \hat x^\sigma \partial \hat x^\mu}}_{\substack{40\text{ dof after imposing 1) and}\\ \text{removing the 6 dof of Lorentz group}}}\frac{\partial x^\beta}{\partial \hat x^\nu} g_{\alpha\beta} + (\mu\leftrightarrow\nu) + \frac{\partial x^\alpha}{\partial \hat x^\mu}\frac{\partial x^\beta}{\partial \hat x^\nu}\frac{\partial x^\gamma}{\partial \hat x^\sigma}\partial_\gamma g_{\alpha\beta} = 0$$ 40 eqs

So we have 40 eqs and 40 dof which is just enough to set $\partial_{\hat\sigma} g_{\hat\mu\hat\nu}\Big|_{P_0} = 0$

3) $$\partial_{\hat\sigma}\partial_{\hat\rho} g_{\hat\mu\hat\nu}\Big|_{P_0} = \Big(\underbrace{\frac{\partial^3 x^\alpha}{\partial \hat x^\sigma \partial \hat x^\rho \partial \hat x^\mu}}_{80\text{ dof after 1) and 2)}}\frac{\partial x^\beta}{\partial \hat x^\nu} g_{\alpha\beta} + \frac{\partial^2 x^\alpha}{\partial \hat x^\sigma \partial \hat x^\mu}\frac{\partial^2 x^\beta}{\partial \hat x^\rho \partial \hat x^\nu} g_{\alpha\beta} + \frac{\partial^2 x^\alpha}{\partial \hat x^\mu \partial \hat x^\rho}\frac{\partial x^\beta}{\partial \hat x^\nu}\frac{\partial x^\gamma}{\partial \hat x^\sigma}\partial_\gamma g_{\alpha\beta} + (\mu\leftrightarrow\nu)$$
$$+ \frac{\partial^2 x^\alpha}{\partial \hat x^\sigma \partial \hat x^\mu}\frac{\partial x^\beta}{\partial \hat x^\nu}\partial_{\hat\rho} g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial \hat x^\mu}\frac{\partial x^\beta}{\partial \hat x^\nu}\frac{\partial^2 x^\gamma}{\partial \hat x^\sigma \partial \hat x^\rho}\partial_\gamma g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial \hat x^\mu}\frac{\partial x^\beta}{\partial \hat x^\nu}\frac{\partial x^\gamma}{\partial \hat x^\sigma}\frac{\partial x^\delta}{\partial \hat x^\rho}\partial_\gamma\partial_\delta g_{\alpha\beta}\Big)$$ 100 eqs

After 1) and 2) we are left with 20 components of $\partial_{\hat\sigma}\partial_{\hat\rho} g_{\hat\mu\hat\nu}$ which in general cannot be set to zero. These 20 components encode information of the curvature of spacetime and the same 20 dof will appear in the Riemann curvature tensor which we define later.

The local Lorentz frame is the rest frame of a freely falling observer where physics is locally (at the point where $g_{\hat\mu\hat\nu} = \eta_{\hat\mu\hat\nu}$) described by SR. Therefore it is often convenient to compute things in the local Lorentz frame using SR results and then recast the results into covariant crd independendent form.

2.5 Integration on manifolds

Using the local inertial frame at any point P_0, the geometry around that point reduces to Minkowski up to linear level in $\Delta\hat{x}^\mu = \hat{x}^\mu(P) - \hat{x}^\mu(P_0)$. In particular, the infinitesimal volume element at P_0 is just:

$$d^4V = d^4\hat{x} \qquad \text{at } P = P_0$$

A transformation to other coordinates $x^\mu(\hat{x}^\mu)$ yields the usual Jacobian determinant:

$$d^4V = d^4\hat{x} = \det\left(\frac{\partial \hat{x}^\mu}{\partial x^\nu}\right) d^4x \qquad \text{at } P = P_0$$

The Jacobian determinant can be related to the determinant of the metric:

$$g_{\mu\nu} = \frac{\partial \hat{x}^\alpha}{\partial x^\mu}\frac{\partial \hat{x}^\beta}{\partial x^\nu} g_{\hat{\alpha}\hat{\beta}} = \frac{\partial \hat{x}^\alpha}{\partial x^\mu}\frac{\partial \hat{x}^\beta}{\partial x^\nu}\eta_{\alpha\beta} \qquad \text{at } P = P_0$$

$$\Rightarrow \quad \det(g_{\mu\nu}) \equiv g = \det\left(\frac{\partial \hat{x}^\alpha}{\partial x^\mu}\right)\det\left(\frac{\partial \hat{x}^\beta}{\partial x^\nu}\right)\underbrace{\det(\eta_{\alpha\beta})}_{=\det\begin{pmatrix}-1 & & & 0\\ & 1 & & \\ & & 1 & \\ 0 & & & 1\end{pmatrix} = -1} \qquad \text{at } P = P_0$$

determinant of the component matrix ($\det(g_{\mu\nu})$)

do not confuse with the tensor $g = g_{\mu\nu}dx^\mu \otimes dx^\nu$ (g)

$$g = \left(\det\left(\frac{\partial \hat{x}^\mu}{\partial x^\nu}\right)\right)^2 (-1)$$

the index names irrelevant here, this denotes the Jacobian matrix

$$\Rightarrow \quad \det\left(\frac{\partial \hat{x}^\mu}{\partial x^\nu}\right) = \sqrt{-g} = \sqrt{|g|} \qquad \text{at } P = P_0$$

Therefore we get the result:

$$d^4V = \sqrt{-g}\, d^4x \qquad \text{at } P = P_0$$

Now since this holds for any P_0 (the crd's $\tilde{x}^{\mu}(P_0)$ of course differ for differ P_0 but for any P_0 we can go to the local Lorentz coordinates) and the RHS is expressed in general coordinates, we can take this as the definition of the volume element over the entire manifold (see below for a more formal treatment)

$$(2.13) \quad d^4V = \sqrt{-g}\, d^4x \qquad \forall P \in M$$

Integration of scalar functions $f: M \to \mathbb{R}$ over curved spacetime regions is then defined by

$$(2.14) \quad \int_{\Sigma} f(x)\, d^4V = \int_{\Sigma} d^4x \sqrt{-g}\, f(x) \qquad \Sigma \subset M$$

SUPPLEMENTARY MATERIAL

More mathematically, we define the volume element as the (0,4) tensor (4-form)

$$\Omega_M \equiv \sqrt{|g|}\, \underbrace{dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3}_{\equiv dx^0 \otimes dx^1 \otimes dx^2 \otimes dx^3 - dx^1 \otimes dx^0 \otimes dx^2 \otimes dx^3 + \dots}$$

(See also: Nakahara: Chapter 7.9, Carroll: Chapter 2.10)

antisymmetrised sum over all permutations, the operator $\wedge$ defined this way is called wedge product.

Change coordinates $x^\mu \to \tilde{x}^\mu$:

$$g_{\mu\nu}' = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta} \Rightarrow g' = \left(\det \frac{\partial x^\nu}{\partial x^{\mu'}}\right)^2 g \equiv \left|\frac{\partial x^\nu}{\partial x^{\mu'}}\right|^2 g \,, \quad \text{assume } \left|\frac{\partial x^\nu}{\partial x^{\mu'}}\right| > 0$$

$$\Omega_M = \sqrt{|g'|}\, dx^{0'} \wedge dx^{1'} \wedge dx^{2'} \wedge dx^{3'}$$

$$= \left|\frac{\partial x^\nu}{\partial x^{\mu'}}\right| \sqrt{|g|}\, \underbrace{\frac{\partial x^{0'}}{\partial x^{\mu_0}} \frac{\partial x^{1'}}{\partial x^{\mu_1}} \frac{\partial x^{2'}}{\partial x^{\mu_2}} \frac{\partial x^{3'}}{\partial x^{\mu_3}}\, dx^{\mu_0} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3}}_{= \left|\frac{\partial x^{\mu'}}{\partial x^{\nu}}\right| dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3}$$

the determinant is by definition the antisymmetrised sum over all products of elements

$$= \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

So that Ω_M is invariant under crd transformations. Consider the action of Ω_M o infinitesimal displacement vectors $v = dx^\mu \partial_\mu$:

$$\Omega_M(dx^{\nu_0}\partial_{\nu_0}, dx^{\nu_1}\partial_{\nu_1}, dx^{\nu_2}\partial_{\nu_2}, dx^{\nu_3}\partial_{\nu_3})$$

$$= \sqrt{|g|} \underbrace{\sum_P}_{\text{permutation}} \mathrm{sgn}(P)\, dx^{\nu_{P_0}} dx^{\nu_{P_1}} dx^{\nu_{P_2}} dx^{\nu_{P_3}} \delta^0_{\nu_{P_0}} \delta^1_{\nu_{P_1}} \delta^2_{\nu_{P_2}} \delta^3_{\nu_{P_3}}$$

$$= \sqrt{|g|}\, dx^0 dx^1 dx^2 dx^3$$

$$= \sqrt{|g|}\, d^4x$$

Then we define the integral $\int_\Sigma$ over a volume $\Sigma \subset M$ as the map:

$$\int_\Sigma : \Omega_M \to \mathbb{R} \quad , \quad \int_\Sigma \Omega_M = \underbrace{\int_\Sigma \sqrt{|g|}\, d^4x}_{\text{usual 4-d integral of } \sqrt{|g|}}$$

The integral of a function $f: M \to \mathbb{R}$ is defined by:

$$\int_\Sigma f\, \Omega_M \equiv \underbrace{\int_\Sigma \sqrt{|g|}\, d^4x\, f}_{\text{usual 4-d integral of } \sqrt{|g|} f}$$

SUPPLEMENTARY MATERIAL