2. Basics of GR and cosmology

2.1. Metric, expanding space-time

Usually we assume that gravity is described by the Einsteins general relativity theory (GR). GR is based on the concept of metric space times. The basic concept is the metric

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu} \qquad (2.1)$$

invariant line element (distance) — metric tensor — infinitesimal displacements in (contravariant) coordinates

(Repeated indices are always summed over)

Metric is actually a generalization of the Pythagoran theorem, according to which in the Euclidian 3-space:

$$ds^2 = dx^2 + dy^2 + dz^2 \equiv \eta_{ij}\, dx^i dx^j$$

where

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.2)$$

In different coordinates the metric components can be also dimensionful quantities.

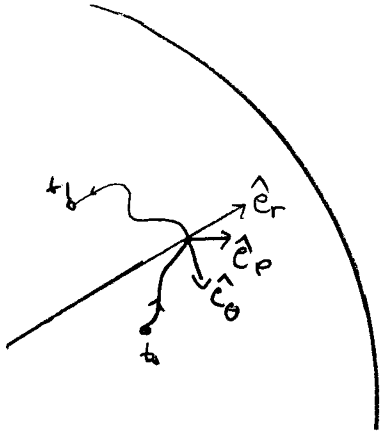
For example in flat 2-dim. space

$$ds^2 = dx^2 + dy^2 = d\rho^2 + \rho^2 d\varphi^2 \Rightarrow g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}.$$

Similarly on a 2-dim curved space (surface of a sphere)

$$ds^2 = R^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} \qquad (2.3)$$



Metric gives the length of an arbitrary curve in this space:

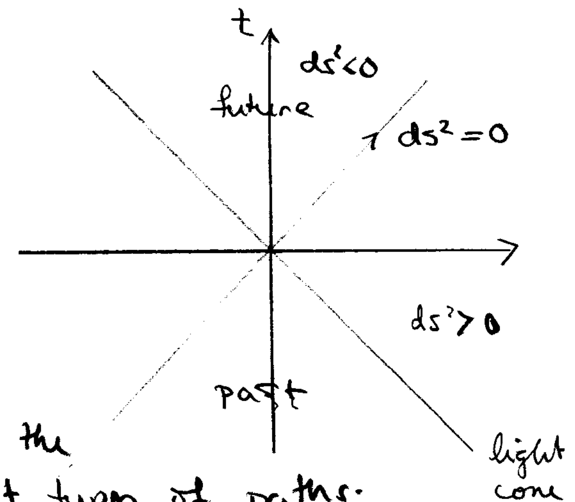
$$l(s) = \int ds = \int \sqrt{g_{ij} dx^i dx^j}$$

$$= \int_{t_0}^{t_1} \sqrt{R^2 d\theta^2 + R^2 \sin\theta^2 d\varphi^2} = R \int_{t_0}^{t_1} dt \sqrt{\dot\theta^2 + \sin^2\theta \, \dot\varphi^2}$$

A metric of particular importance in the <u>Minkowski metric</u> of the special relativity. $(c \equiv 1)$

$$ds^2 = -dt^2 + d\vec{x}^2$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \qquad (2.4)$$



Difference in signs gives rise to the peculiar geometry with 3 different types of paths:

$$ds^2 < 0 \quad \text{time-like}$$
$$ds^2 = 0 \quad \text{light-like}$$
$$ds^2 > 0 \quad \text{space-like}$$

Light-like paths define the <u>light-cone</u>. We can get information only from the past light cone, and pass information to the future light cone. ($c = 1 =$ constant). Minkowski space is <u>flat</u>.

Thes properties of the Minkowski space are passed as such to the <u>local coordinate systems</u> of <u>curved</u> spaces.

Massive particles follow time-like paths $ds^2 > 0$. The <u>proper time</u> τ, defined through

$$ds^2 \equiv -c^2 d\tau^2$$

corresponds to the time shown by a clock moving along the particle. The <u>4-motion</u> of a particle

coordinate velocity

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{d\tau}{dt}(1; \vec{v}) \qquad (2.5)$$

coordinate time

Where $x^\mu \equiv (t, \vec{x})$ are the contravariant components of a vector. The length of a space-like curve is the proper time $\ell(s) \equiv \int\sqrt{-ds^2} = \int d\tau = \tau$ ($c \equiv 1$).

<u>Homogenous and isotropic</u>, <u>flat</u>, nonstatic spacetime is a generalization of the Minkowski space:

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$
$$= -dt^2 + a(t)^2 \Big(dr^2 + \underbrace{r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}_{= r^2 d\Omega^2} \Big) \qquad (2.6)$$

In the cartesian system then:

$$g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2). \qquad (2.7)$$

More generally Homog. & isotropic ST can be curved. The most general line element turns out to be the FRW-metric:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + r^2 \sin^2\theta \, d\varphi^2) \right] \qquad (2.8)$$

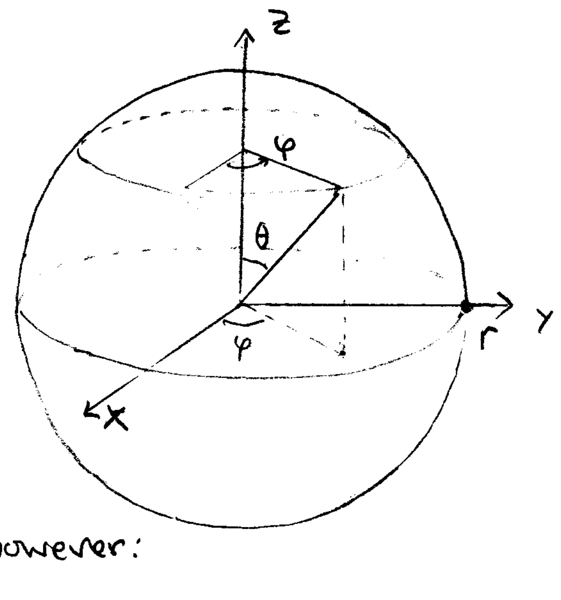
where $k=0$ gives back the flat space and $k>0$ ($k<0$) gives the closed (open) space with a positive (negative) curvature.

The metric (2.8) is written in funny coordinates, which are even singular for the case $k>0$, when $kr^2 \to 1$. Let us try to understand this using a simpler example.

2-sphere

We saw the metric for this space in eq. (2.3). We can also consider the 2-sphere **embedded** in Euclidean 3-space, as a set of points (x, y, z) with

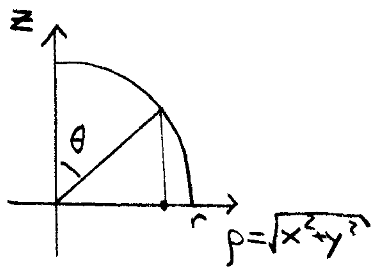
$$x^2 + y^2 + z^2 = r^2$$



z-coordinate can be eliminated, however:

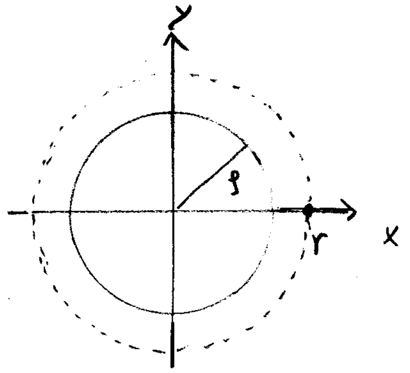
$$z^2 = r^2 - \rho^2 \qquad (\rho^2 \equiv x^2 + y^2)$$

$$\Rightarrow \quad dz^2 = \left(\frac{\rho}{z}\right)^2 d\rho^2 = \frac{\rho^2}{r^2 - \rho^2} d\rho^2$$



Let us further define

$$\begin{cases} x = \rho \sin\varphi \\ y = \rho \cos\varphi \end{cases}$$



and we get

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= d\rho^2 + \rho^2 d\varphi^2 + \frac{\rho^2}{r^2-\rho^2}\, d\rho^2 = \frac{d\rho^2}{1-\frac{\rho^2}{r^2}} + \rho^2 d\varphi^2 \qquad (2.9)$$

Defining the positive curvature

$$k \equiv 1/r^2 \qquad (2.10)$$

the $d\rho^2$-part of (2.9) reduces to the form seen in (2.8).

Note in particular that the singularity in (2.9) at $\rho = r$ is just an apparent one. It just signals the fact that the coordinate system (ρ, φ) does not uniquely cover the entire manifold. (Beyond the point $\rho = r$ ρ starts to shrink again, and one cannot distinguish (from ρ & φ alone) whether one is continuing towards south or did one turn back.) Defining

$$\rho = r \sin\theta$$

one gab back to the form (2.3) which covers the entire space.

Similarly, if we define in (2.8)

$$\begin{cases} r \equiv R \sin\chi \\ k \equiv 1/R^2 \end{cases} \qquad (2.11)$$

We get the equation (2,8) to the form ($k=+1$)

$$ds^2 = -dt^2 + a(t)^2 R^2 \overbrace{\left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right]}^{\text{unit 3-sphere}} \qquad (2,12)$$

which is a complete, nonsingular metric for a 3-sphere (in the M-space).

Note that one can always rescale $aR \equiv \tilde{a}$, such that $k \to \pm 1$ (assuming $k \neq 0$) as was implicitly assumed in (2.8).

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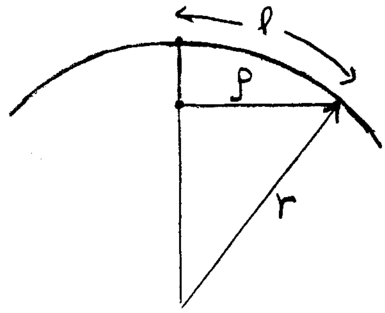
Physically, the coordinate ρ in the metric (2,9) is the Euclidian distance in the embedding space, corresponding to the actual physical distance l on the curved surface. ρ is still measurable through closed curves.
Consider a circumference of a circle drawn on the sphere.
Clearly

$$s = 2\pi\rho < 2\pi l \quad !$$



Based on this we can derive a local measure of the curvature (Ex.)

$$k \equiv \frac{3}{\pi} \lim_{l \to 0} \frac{2\pi l - s}{l^3} = \frac{1}{r^2} \qquad (2,13)$$

Similarly, the coordinate r, in the case of metric (2,8) corresponds to the surface area of a sphere with a physical radius d:

$$A = \underset{\int r^2 d\Omega}{\underset{\shortparallel}{4\pi r^2}} \begin{matrix} > \\ = \\ < \end{matrix} 4\pi d^2 \qquad \begin{pmatrix} k>0 \\ k=0 \\ k<0 \end{pmatrix}$$

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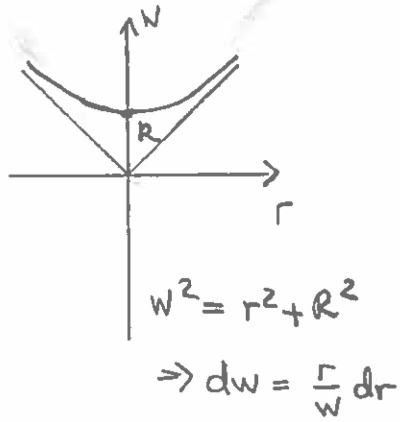
Negative curvature needs embedding onto a Minkowski space:
Now set

$$x^2+y^2+z^2-w^2 \equiv -R^2$$



and $\quad ds^2 \equiv dx^2+dy^2+dz^2-dw^2$

$$= dr^2 + r^2 d\Omega - \frac{r^2}{r^2+R^2} dr^2$$

$$= \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega \,.$$

This is of course just (2,5) with the curvature $k = -1/R^2$.

In fact the Friedman-Robertson-Walker metric (2,1) is the most general metric for homogeneous and isotropic space-time, locally consistent with the Minkowski space. It is completely parametrized by the two variables

$$a(t) \qquad \text{and} \qquad k$$

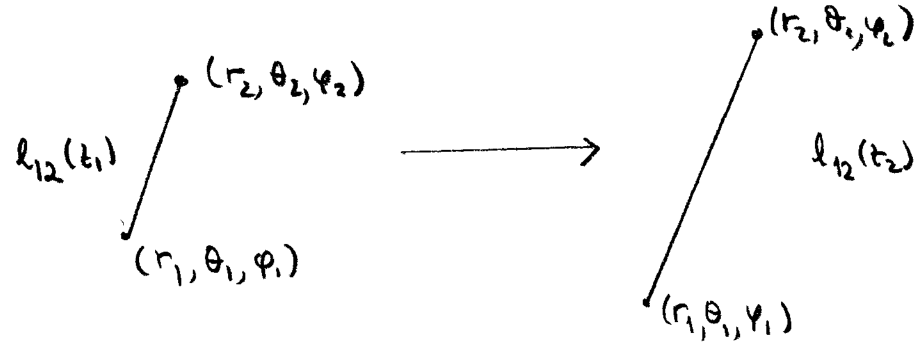
scale factor $\qquad\qquad$ geometry

Both these quantities are set by the Einstein equations, (which are of course general equations for $g_{\mu\nu}$).

We can study several conceptual quantities relevant for expanding space-times without use of E-equations.

Co-moving coordinates

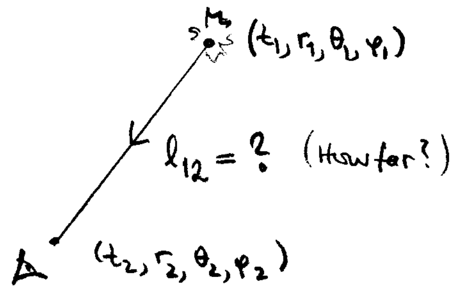
The coordinates (t, r, θ, φ) are called co-moving coordinates. Notation follows from the fact that stationary particles retain their co-moving coordinates even though the spacetime expands. Their physical distances scale however:



$$\ell_{12}(t_2) = \frac{a(t_2)}{a(t_1)}\, \ell_{12}(t_1)$$

Different distances

Because of the expansion, curvature and finite speed of light, the concept of distance is somewhat involved in cosmology.



Consider two galaxies at co-moving coordinates $r_1 \equiv 0$ and r_2.
Their co-moving distance depends on geometry.

$$\ell_{cm}(r_2) = \int_0^{r_2} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \arcsin r_2 \; ; & k=1 \\ r_2 \; ; & k=0 \\ \operatorname{arsinh} r_2 \; ; & k=-1 \end{cases} \qquad (2.14)$$

where we chose a straight path with $d\theta = d\varphi = 0$. The "physical" distance between galaxies along spatial hypersurfaces (dt = const) then are

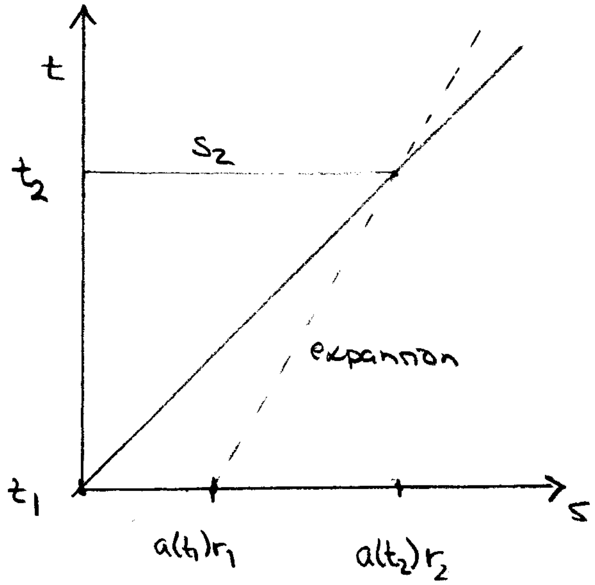
$$s_1 = a(t_1)\, \ell_{cm}(r_0) \quad \neq \quad s_2 \equiv a(t_2)\, \ell_{cm}(r_2)$$

Obviously r_2 is but a convenient parameter. We can define the co-moving distance in terms of physical observables (eventually redshifts & matter & radiation densities) by use of paths travelled by light. Indeed for light: ($d\theta = d\varphi \equiv 0$ still)

$$ds^2 = -dt^2 + a(t)^2 \frac{dr}{\sqrt{1-kr^2}} = 0$$

$$\Rightarrow l_{cm}(r_2) = \int_{t_1}^{t_2} \frac{dt}{a(t)}$$

$$\Rightarrow \boxed{s_i = a(t_i) \int_{t_1}^{t_2} \frac{dt}{a(t)}} \qquad (2.15)$$



Of course, if $a(t)$ (the expansion history of the universe) is known, eq. (2.15) answers all questions. The real question however is, how can we specify $a(t)$ from observations of some standard candles? Then the definition of distance becomes even more involved.

Redshift

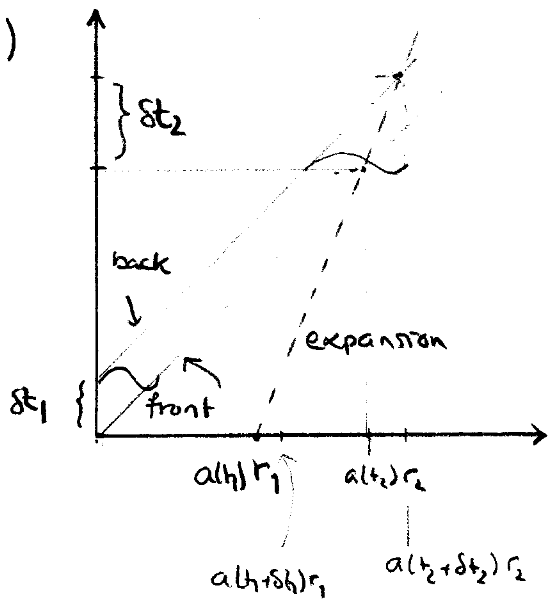
Using the definition (2.15) we can easily understand how the cosmological redshift arises.

Consider some light emitter situated at $r = r_1$. Assume that the wave length of the emitted light is λ_{em}. To emit one wavelength then takes the time $\delta t_1 = \lambda_{em}/c$. Assuming that the source is stationary, both ends travel the same co-moving distance between the emission and absorption:

$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{t_1+\delta t_1}^{t_2+\delta t_2} \frac{dt}{a(t)}$$

$$\Rightarrow \quad 0 = \int_{t_1+\delta t_1}^{t_1} \frac{dt}{a(t)} + \int_{t_2}^{t_2+\delta t_2} \frac{dt}{a(t)} \simeq \frac{\delta t_2}{a(t_2)} - \frac{\delta t_1}{a(t_1)}$$

$$\Rightarrow \quad \frac{\delta t_2}{\delta t_1} = \frac{a(t_2)}{a(t_1)} \qquad (2.16)$$



This spread in times is called <u>cosmological</u> time-dilatation.

Because $\lambda_i = c\,\delta t_i$, we immediately get the redshift:

$$\underline{\underline{1+z}} \equiv 1 + \frac{\lambda_{obs}-\lambda_{em}}{\lambda_{em}} = \frac{\lambda_{obs}}{\lambda_{em}}$$

$$= \frac{\delta t_2}{\delta t_1} = \underline{\underline{\frac{a(t_{em})}{a(t_{obs})}}} \qquad ;\left(\begin{array}{l} t_{em} = t_1 \\ t_{obs} = t_2 \end{array} \right)$$

(2.18)

In addition to redshift, expansion also makes all massive particles slow down and eventually stop. (Ex.)

Expanding Newtonian space

(Simple way to include dynamics
$\Rightarrow$ can compute $a(t)$)

Let us consider an infinite, homogenous and isotropic universe. Consider the position of a spherical shell at co-moving distance r from the observer:

$$\underbrace{R(t)}_{\text{phys. distance}} = \underbrace{a(t)}_{\text{scale factor}}\, r \qquad (2.19)$$

(This follows from the cosmological principle) From (2.19) it follows that

$$v(t) = \dot{R} = \dot{a} r = \frac{\dot{a}}{a} R = H R$$

Now assume that universe is filled by dust (particles) with

$$n = \frac{N}{V} = \frac{N}{R^3} \sim a^{-3}$$

and

$$\rho = m n \sim a^{-3}$$

$$\Rightarrow \quad \partial_t \rho = \partial_t \rho_0 \left(\frac{a_0}{a(t)}\right)^3 = -3H\rho$$

This is in fact just the continuity equation:

$$\vec{j} = \rho \vec{v} = H\rho \vec{R} \Rightarrow \nabla_R \cdot \vec{j} = H\rho\, \nabla_R \cdot \vec{R} = 3H\rho$$

$$\Rightarrow \quad \partial_t \rho + \nabla \cdot \vec{j} = 0 .$$

Now assume that R is sufficiently small: $|\vec{V}| = H|\vec{R}| \ll 1$, so that we can apply the Newtonian mechanics. Then, for a test body situated at $R(t)$-radius shell:

$$\mu \ddot{R} = -\frac{G\mu M}{R^2} = -\frac{4\pi}{3} G\mu \left[\frac{M}{\frac{4\pi}{3}\rho R^3}\right] \rho R$$

$$= -\frac{4\pi}{3} G\mu\rho R$$

Using $R(t) = a(t)\, r$ this implies

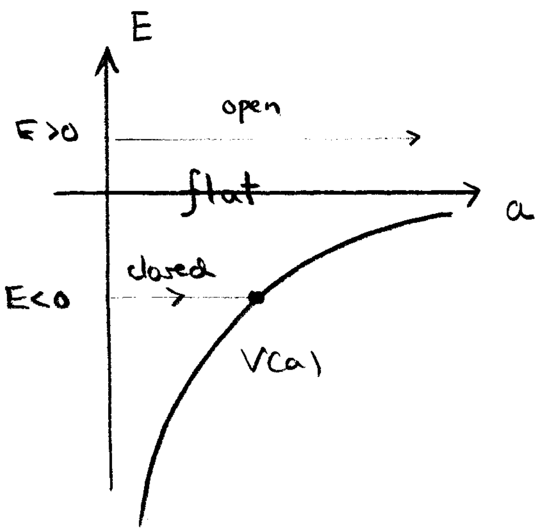
$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\rho} \qquad (NI)$$

This <u>acceleration equation</u> in fact generalizes to GR if only $\rho \sim a^{-3}$ holds.

Multiplying NI by $a\dot{a}$ and integrating we get

$$\int \overbrace{\dot{a}\ddot{a}\, dt}^{=\frac{1}{2}(\dot{a})^2} = -\frac{4\pi}{3} G\rho_0 a_0^3 \int \frac{\overbrace{\dot{a}dt}^{da}}{a^2} + \overset{E}{\overset{\equiv}{\text{const}}}$$

$$\Leftrightarrow \quad \frac{1}{2}\dot{a}^2 \underbrace{- \frac{4\pi}{3} G\rho_0 \frac{a_0^3}{a}}_{\equiv V(a)} = E$$



$(T+V=E)$. The equation for the scale factor is thus exactly analogous to the equation of a projectile in gravity field.

Dividing this equation by a^2 we get it to a form

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{2E}{a^2} = \frac{8\pi}{3} G \rho_0 \left(\frac{a_0}{a}\right)^3$$

$$\Leftrightarrow \quad \boxed{H^2 - \frac{2E}{a^2} = \frac{8\pi G}{3}\rho} \qquad (N\,II)$$

Where $-k \equiv 2E$ describes <u>curvature</u>, as shown by the fig. in the previous page:

$$E < 0 \Rightarrow \exists \; a_{max} = \frac{3E}{4\pi G \rho_0 a_0^3} \quad ; \text{ closed } (k=+1)$$

$$E > 0 \Rightarrow \text{ a grows without a bound } ; \text{ open } (k=-1)$$

$$E = 0 \Rightarrow \quad -\text{''}- \quad ; \text{ flat } (k=0)$$

(Note that we can always choose units such that $k = 2E = \pm 1$, or 0)

The flat solution $E = 0$ defines the <u>critical density</u>

$$H^2 = \frac{8\pi G}{3}\rho_{cr} \Leftrightarrow \underline{\underline{\rho_{cr} = \frac{3H^2}{8\pi G}}} \qquad (N\,III)$$

Note that $\rho_{cr} = \rho_{cr}(t)$!

Moreover, from NII one finds

$$2E = a^2 H^2 - \frac{8\pi G}{3} a^3 \rho = \frac{8\pi}{3} G \rho_{cr} a^2 \left(1 - \frac{\rho}{\rho_{cr}}\right)$$

$$= \underline{\frac{8\pi G}{3} a(t)^2 \rho_{cr}(t) \left(1 - \Omega(t)\right)}$$

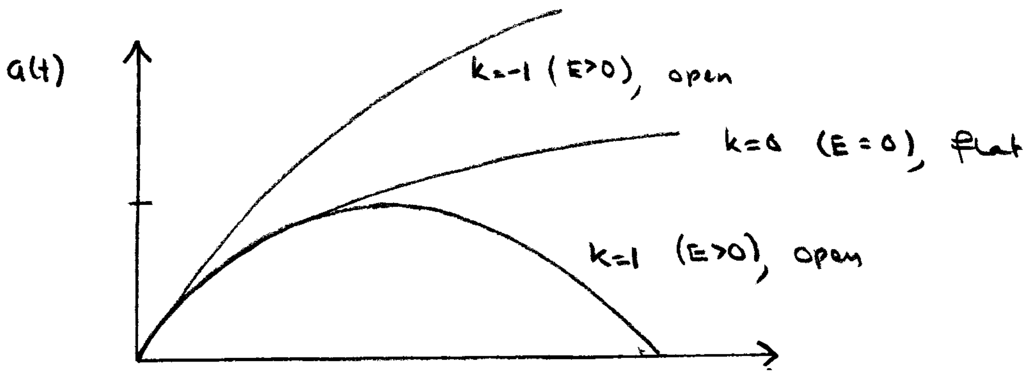
where the quantity

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{cr}(t)} \qquad (N\,IV)$$

is a cosmological parameter that tells the ratio of the matter density to the critical density as a function of time. Note that since $E = \text{const}$, we have

$$\text{sgn}(E) = \text{sgn}\,(1 - \Omega(t)),$$

i.e. $1-\Omega(t)$ - term cannot change its sign in the course of evolution,
$\Rightarrow$ Measuring $\Omega(t_0)$ at any given time t_0, one can decide what is the geometry of the universe. (One needs $H(t_0)$ and $\rho(t_0)$.)



Let us still introduce the deceleration parameter

$$q \equiv -\frac{\ddot{a}}{aH^2} \qquad (N\,V)$$

If $q>0$ ($q<0$), the expansion is decelerating (accelerating).

In the Newtonian dust universe

$$q = -\frac{\ddot{a}}{a}\frac{1}{H^2} = +\frac{1}{2}\,\frac{8\pi G}{3}\rho\,\frac{1}{H^2} = \frac{1}{2}\frac{\rho}{\rho_{cr}} > 0$$

i.e. the dust universe decelerates always (but not necessarily enough to stop (closed case). The flat border-line case corresponds to $\rho = \rho_{cr} \Rightarrow q = 1/2$.)

Let us finally compute the time-evolution of a when $\rho = \rho_{cr}$. Here

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{cr,0} \left(\frac{a_0}{a}\right)^3$$

$$\Leftrightarrow \quad a\dot{a}^2 = \text{const} \quad \Leftrightarrow \quad \int da \sqrt{a} = \# \int dt$$

$$\Rightarrow \quad a^{3/2} = c_1 t + c_2$$

Obviously $a \geq 0$ always, so that $\exists\, t$ for which $a = 0$ $\left(t = -\frac{c_2}{c_1}\right)$. We can set $c_2/c_1 = 0$

$$\Rightarrow \quad a \propto t^{2/3}$$

$$\Leftrightarrow \quad H = \frac{\dot{a}}{a} = \frac{2}{3t}$$

We can thus express the age of the universe using the Hubble expansion rate:

$$t_0 = \frac{2}{3H_0}$$

Further still:

$$\rho(t) = \frac{3H^2}{8\pi G} = \frac{1}{6\pi G t^2} \longrightarrow \begin{cases} \infty & ; \ t \to 0 \\ 0 & ; \ t \to \infty \end{cases}$$

(flat case)

2,2 Vectors (and tensors) I

GR is based on the mathematics of differentiable manifolds, and the basic concepts on those manifolds are expressed in terms of vector and tensor fields. For a number of lectures now we review and build up the mathematical tools necessary in cosmology & GR.

A vector is a coordinate independent object, whose components depend on the basis chosen.

$$\underset{\text{(4)-vector}}{w} = \underset{\substack{\uparrow \\ \text{component}}}{w^\alpha} \hat{e}_\alpha \qquad \text{a basis vector} \tag{2.19}$$

To each coordinate basis ($(\hat{e}_\alpha)_\beta = \delta_{\alpha\beta}$) we can associate a normalized (orthonormal if $\hat{e}_\alpha$ is orthogonal) basis

$$\hat{e}_{\hat{\alpha}} \equiv \frac{1}{\sqrt{g_{\alpha\alpha}}} \hat{e}_\alpha \;^{*} \tag{2.20}$$

In a normalized basis of course $w = w^{\hat{\alpha}} \hat{e}_{\hat{\alpha}}$ with $w^{\hat{\alpha}} = \sqrt{g_{\alpha\alpha}}\, w^\alpha$

The square of a vector

$$w \cdot w \equiv g_{\alpha\beta} w^\alpha w^\beta = w^\alpha w^\beta (\hat{e}_\alpha \cdot \hat{e}_\beta) \tag{2.21}$$

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From $ds_1 = \sqrt{g_{11} dx^1 dx^1} = \sqrt{g_{11}}\, dx^1 \Rightarrow |\hat{e}_1| = \sqrt{g_{11}}$

i.e. we can conclude (we later learn that this is a bit backword thinking)

$$\hat{e}_\alpha \cdot \hat{e}_\beta = g_{\alpha\beta} \qquad (2.22)$$

so that in an orthonormal basis $\hat{e}_{\hat{\alpha}} \cdot \hat{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$.

In addition to contravariant components w^α we define the covariant components

$$w_\alpha \equiv g_{\alpha\beta} w^\beta \qquad (2.23)$$

whereby

$$w \cdot w = g_{\alpha\beta} w^\alpha w^\beta = w_\beta w^\beta \qquad (2.24)$$

In particular if $g_{\alpha\beta} = \eta_{\alpha\beta}$ we recover the familiar special relativity formulas:

$$w \cdot w = -w^{0\,2} + \vec{w}^{\,2} \qquad (2.25)$$

And defining

$$x^\mu \equiv (t, \vec{x}) \quad ; \quad x_\mu = (-t, \vec{x}) \qquad (2.26)$$

$$p^\mu \equiv (E, \vec{p}) \quad ; \quad p_\mu = (-E, \vec{p}) \qquad (2.27)$$

and in particular

$$p^2 = \eta_{\alpha\beta} p^\alpha p^\beta = -E^2 + p^2 = -m^2 \qquad (2.28)$$

↑ due to sign-convention

Equation (2.23) is an example of (raising and) lowering the indices by use of the metric. These formal operations can be defined also for tensors (more precise definition will be given below).

The <u>contravariant metric</u> is defined by the relation

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma} \qquad (2.29)$$

It then follows that:

$$g^{\alpha\beta} w_{\beta} = g^{\alpha\beta} g_{\beta\gamma} w^{\gamma} = \delta^{\alpha}{}_{\gamma} w^{\gamma} = w^{\alpha} \qquad (2.30)$$

This shows that $g^{\alpha\beta}$ can be used to <u>raise</u> the indices. This allows defining mixed tensors as well: Assume that $A^{\alpha\beta}$ is a contravariant tensor. We can then define:

$$A_{\alpha}{}^{\beta} = g_{\alpha\gamma} A^{\gamma\beta}$$
$$A_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} A^{\gamma\delta} \qquad (2.31)$$

etc. (Under what conditions $A_{\alpha}{}^{\beta} = A^{\beta}{}_{\alpha}$?)

The covariant and contravariant components have the same magnitude in orthonormal basis only. These coordinate systems are special as they will define the local Lorenz-coordinate systems in the curved spaces.

Reminder Lorenz-transformation (= coordinate transformation between inertial frames)

Lorenz transformations connect different inertial frames in the Minkowski space of special relativity. Mathematically, Lorenz-transform

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad (2.32)$$

leave the line-element

$$ds^2 = -dt^2 + d\vec{x}^2$$

invariant. This immediately leads to the property

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_\rho \Lambda^{\nu'}{}_\sigma \eta_{\mu'\nu'} \quad \text{ie} \quad \eta = \Lambda^T \eta \Lambda \qquad (2.33)$$

Possible transforms include the rotations:

$$\Lambda^\mu{}_\nu(\text{rotation around z-axes}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & s\theta & 0 \\ 0 & -s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; ; \; \begin{array}{l} s\theta = \sin\theta \\ c\theta = \cos\theta \end{array}$$

and boosts

$$\Lambda^\mu{}_\nu(\text{boost to x-direction}) = \begin{pmatrix} ch\phi & -sh\phi & 0 & 0 \\ -sh\phi & ch\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; ; \; \begin{array}{l} ch\phi = \cosh\phi \\ sh\phi = \sinh\phi \end{array}$$

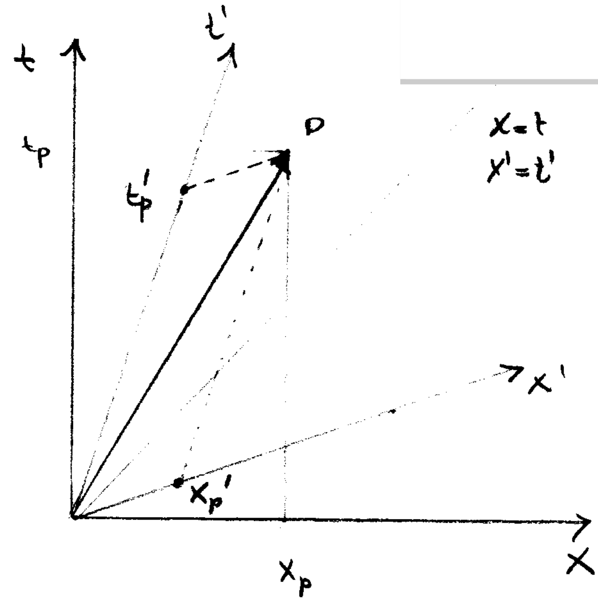
in the latter then

$$\begin{cases} t' = \operatorname{ch}\phi\, t - \operatorname{sh}\phi\, x \\ x' = -\operatorname{sh}\phi\, t + \operatorname{ch}\phi\, x \end{cases}$$

so if $x' = 0 \Rightarrow v = \frac{x}{t} = \tanh\phi \Rightarrow \operatorname{ch}\phi = \frac{1}{\sqrt{1-v^2}} \equiv \gamma$

$$\operatorname{sh}\phi = \gamma v$$

$$\Rightarrow \begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \end{cases} \qquad (c \equiv 1)$$



etc.

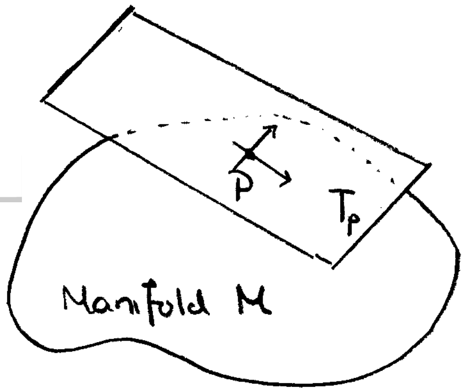
(Note that only the coordinates of a vector associated with the point P change, not the vector itself.

$$x^\mu(P) = x^\mu \hat{e}_\mu = x'^\mu \hat{e}'_\mu .$$

also these change!)

2.3 Vectors and tensors in curved spacetimes (manifolds)

To define vector (and tensor) fields in curved spacetimes, we need the concept of a tangent space



With each point P belonging to a curved manifold M, we can associate a linear vector space T_P (tangent space):

$$\forall \; U, W \in T_P \quad \text{and} \quad a, b \in \mathbb{R}$$

$$X = (a+b)(U+W) = aU + aW + bU + bW \in T_P \qquad (2.34)$$

Other operations between vectors, such as the dot-product encountered in the previous section, can be introduced as additional structures on T_P

(The set of all tangent spaces definable on a manifold M is called the tangent bundle $T(M)$.)

The unit vectors of a curved manifold thus actually live in the tangent space. (In a manifold a vector has to be defined on a point!).

$V^\mu(\lambda) = \frac{dx^\mu}{d\lambda}$

$$V(\lambda) = V^\mu(\lambda)\,\hat{e}_\mu(\lambda)$$

In each point $P \in M$ one can define the tangent vector V^μ to a curve $x^\mu(\lambda)$ going through P.

Uner a Lorenz transformation (in T_p) the components of x^μ transform as (2.32), while parametrization is unchanged. Hence

$$V^\mu = \frac{dx^\mu}{d\lambda} \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu \qquad (2.35)$$

However, because a vector is invariant, we get

$$V = V^\mu \hat{e}_\mu = V^{\nu'} \hat{e}_{\nu'} = \Lambda^{\nu'}{}_\mu V^\mu \hat{e}_{\nu'}$$

$$\Rightarrow \hat{e}_\mu = \Lambda^{\nu'}{}_\mu \hat{e}_{\nu'} \Leftrightarrow \underline{\hat{e}_{\nu'} = \Lambda^\mu{}_{\nu'} \hat{e}_\mu}^{\,*} \qquad (2.36)$$

That is, the unit vectors transform as covariant vector under L-transformation.

---

* The last equality follows from

$$\eta = \Lambda^T \eta \Lambda \Leftrightarrow \Lambda^\mu{}_{\nu'} \Lambda^{\nu'}{}_\rho = \delta^\mu{}_\rho \quad (\Lambda^{\sigma'}{}_\lambda \Lambda^\lambda{}_{\tau'} = \delta^{\sigma'}{}_{\tau'})$$

Dual vectors (1-forms)

The tangent space T_p has a dual linear vector space, called cotangent space T_p^*, which is formed by all linear transformations from T_p to R. Mathematically, if $\omega \in T_p^*$, then $\forall\, a, b \in \mathbb{R}$ and $U, W \in T_p$.

$$\omega(aU + bW) = a\,\omega(U) + b\,\omega(W) \qquad (2.37)$$

and if ω & $\eta \in T_p^*$

$$(a\omega + b\eta)(V) = a\,\omega(V) + b\eta(V) \qquad (2.38)$$

We can define a basis $\hat{\theta}^\nu$ for the dual space by requiring that

$$\hat{\theta}^\mu(e_\nu) \equiv \delta^\mu{}_\nu \qquad (2.39)$$

Just like the contravariant vectors $V \in T_p$ are often referred only via components V^μ, one often refers to contravariant vectors $\omega \in T_p^*$ by components ω_μ:

$$\omega = \omega_\mu \hat{\theta}^\mu \qquad (2.40)$$

Note that:

$$\omega(V) = \omega_\mu \hat{\theta}^\mu (V^\nu \hat{e}_\nu) = \omega_\mu V^\nu \hat{\theta}^\mu(\hat{e}_\nu)$$

$$= \omega_\mu V^\nu \delta^\mu{}_\nu = \omega_\mu V^\mu \in \mathbb{R} \qquad (2.41)$$

(The set of all dual spaces definable on a manifold M is called the cotangent bundle $T^*(M)$.)

- Note that by (2,40)

$$\omega(V) = \omega_\mu V^\mu = V^\mu \omega_\mu = V(\omega) \qquad (2,42)$$

so that $(T_p^*)^* = T^p$; hence the name <u>dual</u> space.

- More generally, any vector field V defined on M can be seen as a mapping

$$V : M \to T(M)$$

and then ω is a mapping

$$\omega : T(M) \to \mathbb{R}.$$

- As a little gymnastics consider a parametrized curve $x^\mu(\lambda)$ on manifold M. That is, $x^\mu(\lambda)$ is a mapping

$$x^\mu(\lambda) : I \to M \qquad (\lambda \in I \subset \mathbb{R})$$

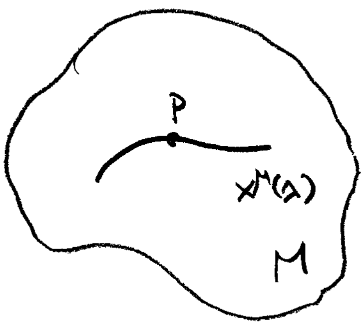
then

$$V(x^\mu) : M \to T(M)$$

$$V(x^\mu(\lambda)) : I \to T(M)$$

$$\omega(V) : T(M) \to \mathbb{R}$$

$$\omega(V(x^\mu(\lambda))) : I \to \mathbb{R}$$



Tensors

Tensors are a straightforward generalization of vectors and dual vectors to multilinear mappings. For example a (k,l)-tensor, defined on a point P is a mapping

$$T \quad : \underbrace{T_p^* \times \ldots \times T_p^*}_{k\text{-times}} \times \underbrace{T_p \times \ldots \times T_p}_{l\text{-times}} \rightarrow \mathbb{R}$$

Eg:

$$T(a\omega + b\eta, cV + dW) \equiv ac\,T(\omega,V) + ad\,T(\omega,W) + bc\,T(\eta,V) + bd\,T(\eta,W). \qquad (2.44)$$

We can define a basis for such tensor-space:

$$\hat{e}_{\mu_1} \otimes \ldots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \ldots \otimes \hat{\theta}^{\nu_l}$$

So that

$$T = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}\, \hat{e}_{\mu_1} \otimes \ldots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \ldots \otimes \hat{\theta}^{\nu_l}$$

is an invariant under coordinate transformations. One again typically refers to a tensor by its components

$$T(k,l) \sim T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}$$

Under Lorenz-transformation:

$$T^{\mu_1' \ldots \mu_k'}{}_{\nu_1' \ldots \nu_l'} = \Lambda^{\mu_1'}{}_{\mu_1} \ldots \Lambda^{\mu_k'}{}_{\mu_k} \Lambda^{\nu_1}{}_{\nu_1'} \ldots \Lambda^{\nu_l}{}_{\nu_l'} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}$$

<u>Examples of tensor quantities:</u> Metric is a (0,2)-tensor:

$$g = g_{\alpha\beta}\,\hat{\theta}^{\alpha}\otimes\hat{\theta}^{\beta} \qquad (2,45)$$

which defines the inner product on a metric manifold:

$$g(U,W) = g_{\alpha\beta}\,\hat{\theta}^{\alpha}(U)\,\hat{\theta}^{\beta}(W) = g_{\alpha\beta}\,U^{\alpha}W^{\beta} = U\cdot W \in \mathbb{R}. \qquad (2.46)$$

ie. $g: T_p\otimes T_p \rightsquigarrow \mathbb{R}$. (Now remember eq. (2,22). It can be found from (2.46) by taking $U=\hat{e}_{\alpha}$, $W=\hat{e}_{\beta}$: $\hat{e}_{\alpha}\cdot\hat{e}_{\beta} = g(\hat{e}_{\alpha},\hat{e}_{\beta}) = g_{\mu\nu}\,\hat{\theta}^{\mu}(\hat{e}_{\alpha})\hat{\theta}^{\nu}(\hat{e}_{\beta})$ $= g_{\mu\nu}\,\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} = g_{\alpha\beta}$ $\square$.)

We want the dot-product to be symmetric (as well as distances), so that

$$g(U,W) \equiv g(W,U) \iff g_{\alpha\beta} = g_{\beta\alpha} \qquad (2,47)$$

<u>Lowering of indices</u>, Consider $U\in T_p$:

$$g(U,*) \equiv g_{\alpha\beta}\,\hat{\theta}^{\alpha}(U)\,\hat{\theta}^{\beta} = g_{\alpha\beta}\,U^{\beta}\,\hat{\theta}^{\beta} \equiv U_{\beta}\hat{\theta}^{\beta} \in T_p^{*} \qquad (2,48)$$

$$(\neq U^{\beta}\hat{e}_{\beta}\ !!)$$

Thus, it is natural to associate $U_{\beta} \equiv g_{\alpha\beta}U^{\beta}$ with a covariant vector belonging to T_p^{*}. (In this sense $g(U,*): T_p \rightsquigarrow T_p^{*}$.)

<u>Inverse metric</u> is a (2,0)-tensor $g^{-1}: T_p^{*}\times T_p^{*} \rightsquigarrow \mathbb{R}$

$$g^{-1} = g^{\mu\nu}\,\hat{e}_{\mu}\otimes\hat{e}_{\nu} \qquad (2,49)$$

Obviously $g^{-1}(\omega,\eta) = g^{\alpha\beta}\omega_{\alpha}\eta_{\beta} = \omega\cdot\eta$. Moreover, $g^{-1}(\omega,*) = g^{-1}(*,\omega) =$

$$g^{\alpha\beta}\omega_\mu\,\hat{e}_\alpha(\hat{\theta}^\mu)\,\hat{e}_\beta = (g^{\alpha\beta}\omega_\alpha)\,\hat{e}_\beta \equiv \omega^\beta\hat{e}_\beta \in T_p. \quad \text{etc.}$$

This raising and lowering of indices directly generalizes to arbitrary rank tensors. Eg; starting from (2,2) tensor we can get (3,1) and (1,3) tensors:

$$g_{\mu\alpha}\,T^{\alpha\beta}{}_{\gamma\delta} = T_\mu{}^\beta{}_{\gamma\delta}$$

$$g^{\mu\gamma}\,T^{\alpha\beta}{}_{\gamma\delta} = T^{\alpha\beta\mu}{}_\delta$$

and so on.

Another important tensor is the __energy-momentum__ __tensor__ T. Eg: in (2,0)-notation

$$T(2,0) = T^{\mu\nu}\,\hat{e}_\mu\otimes\hat{e}_\nu \qquad (2,50)$$

where

$$T^{\mu\nu} = (\underset{\substack{\uparrow\\ \text{density}}}{\rho}+\underset{\substack{\uparrow\\ \text{pressure}}}{P})\,u^\mu u^\nu - P g^{\mu\nu} \qquad (2,51)$$

for an ideal fluid.

__Contraction__. Upper and lower indices can be __contracted__ to give a lower rank tensor $(k,l) \to (k-1,l-1)$: Eg:

let act $\to \hat{\theta}^\gamma(\hat{e}_\alpha) = \delta^\gamma{}_\alpha$

$$\underline{T^{\mu\nu\alpha}{}_{\beta\gamma}}\;\hat{\theta}^\beta\otimes\hat{\theta}^\gamma\otimes\hat{e}_\mu\otimes\hat{e}_\nu\otimes\hat{e}_\alpha = \underline{T^{\mu\nu\alpha}{}_{\beta\alpha}}\;\hat{\theta}^\beta\otimes\hat{e}_\mu\otimes\hat{e}_\nu \qquad (2,52)$$

You cannot do this with only lower or only upper indices!

Manifold

Above we have referred to curved spaces, as manifolds. Let us now make this definition more precise.

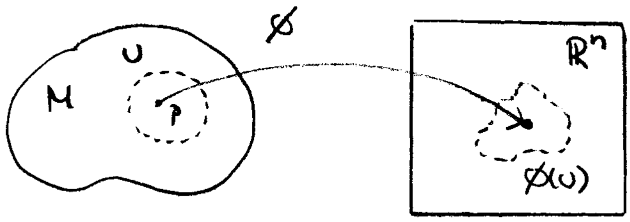
An <u>n-dimensional manifold</u> is a possibly curved, and topologically non-trivial space, which <u>locally</u> <u>looks like $\mathbb{R}^n$</u>, where

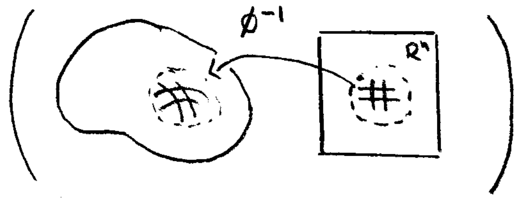
$\mathbb{R}^n$ = n-dimensional Euclidian space

Consider a point P on a manifold M. The neigbourhood of P looks locally like $\mathbb{R}^n$ if exists an open set U_p and an $\underbrace{\infty \text{ differentable}}_{C^\infty}$ mapping

$$\phi : U_p \to \mathbb{R}^n$$

such that $\phi(U_p)$ is open in $\mathbb{R}^n$.

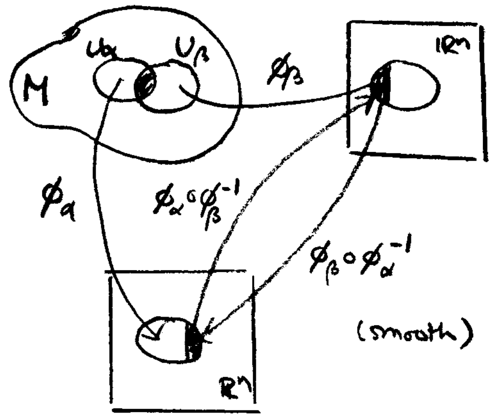




The physical meaning of this is clear. ϕ^{-1} defines a local coordinate system on M. (a <u>map</u>).

<u>Atlas</u> is a set of maps $\{(U_\alpha, \phi_\alpha)\}$

such that



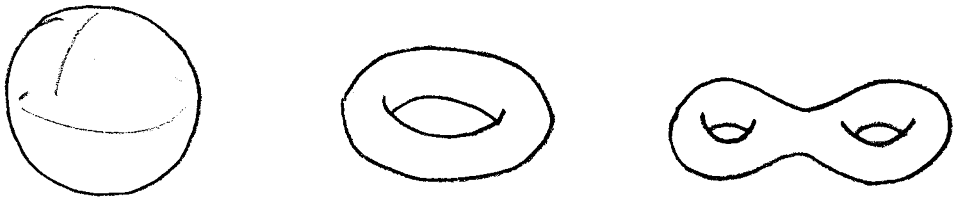
1) $\cup U_\alpha = M$ (2,53)

2) if $U_\alpha \cap U_\beta \neq \phi$ then

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

A manifold is a set M with a maximal atlas. (i.e. all of its points look locally like $\mathbb{R}^n$.)

Examples of manifolds

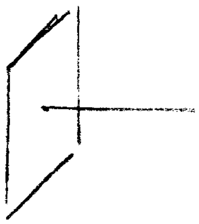


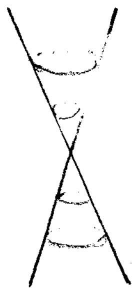
sphere (genus 0)

torus (genus 1)

genus 2

Not manifolds:





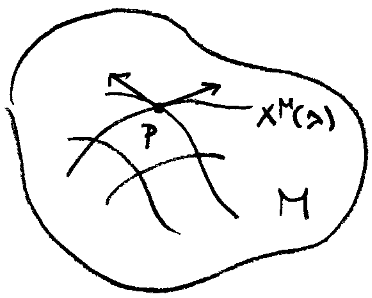
Tangent space, construction at a point.

Tangent space $\hat{=}$ space of all vectors on point P.

How to define T_p without the embedding space for M which we imagined earlier? By use of directional derivatives.

Claim. (see Carroll p. 63-66)

T_p can be identified with the space of directional derivatives along curves passing through P.



$x^\mu(\lambda) : \mathbb{R} \to M$

$f : M \to \mathbb{R} \; ; f : C^\infty$

$f(x^\mu(\lambda)) : \mathbb{R} \to \mathbb{R}$: (a function)

Let f be a C^∞-function $M \to \mathbb{R}$. Then the directional derivative $d/d\lambda$ is a mapping $f \to df/d\lambda$ defining a vector field on M.

$$\frac{d}{d\lambda} f = \frac{d}{d\lambda} f(x^\mu(\lambda)) = \frac{dx^\mu}{d\lambda} \partial_\mu f \quad ; \quad \forall \; f : C^\infty : M \to \mathbb{R} \qquad (2,54)$$

This definition obviously fulfills the Leibniz-rule $\frac{d}{d\lambda}(fg) = \frac{df}{d\lambda} g + f \frac{dg}{d\lambda}$. Moreover, operation is clearly linear, and one can show that also $a\frac{d}{d\lambda} + b\frac{d}{d\eta}$, where $\frac{d}{d\lambda}$ and $\frac{d}{d\eta}$ represent directional derivatives along curves $x^\mu(\lambda)$ and $x^\mu(\eta)$, fulfills the Leibniz rule. (i.e. it is a derivative).

$\Rightarrow$ Directional derivatives form a vector space.

Moreover, because f is arbitrary in the equation (2,54),

$$\Rightarrow \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu . \qquad (2,55)$$

$\left(= V^\mu(\lambda)\,\hat{e}_\mu = V(\lambda)\right.$ derivative along a curve)

Ie. the partial derivatives $\{\partial_\mu\}$ form a basis for T_p.

It is now easy to see that under a <u>general coordinate transformation</u>

$$\underline{\hat{e}_{\mu'}} = \partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_{\mu'} = \underline{\frac{\partial x^\mu}{\partial x^{\mu'}}\hat{e}_{\mu'}} \qquad (2,56)$$

Because vectors and tensors are invariant (this is their <u>definition</u>!), it follows that eg.

$$V^\mu\partial_\mu = V^{\mu'}\partial_{\mu'} = V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \Rightarrow V^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}}V^{\mu'}$$

$$\Leftrightarrow \underline{\underline{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}V^\mu}} \qquad (2,57)$$

Compare this with Lorenz-transf.:

$$x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu \Rightarrow \Lambda^{\mu'}{}_\mu = \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)_{\text{Lorenz.}} \qquad (2,58)$$

That is, the L-transf. properties of vectors and tensors are special cases of the general coordinate transformation (2,57).

The cotangent space T_p^* is then spanned by gradients df:

$$\underbrace{df}_{\in T_p^*}\Big(\underbrace{\frac{d}{d\lambda}}_{\in T_p}\Big) = \frac{df}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu f \quad \in \mathbb{R} \qquad (2,59)$$

Now take $f = x^\mu$ and $d/d\lambda \to \partial_\mu$:

$$dx^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu \quad \Big(= \hat{\theta}^\mu(\hat{e}_\nu) \Big) \qquad (2,60)$$

Moreover $\quad dx^{\mu'} = \dfrac{dx^{\mu'}}{dx^\mu} dx^\mu$ and therefore

$$\omega = \omega_{\mu'} dx^{\mu'} = \omega_{\mu'} \frac{dx^{\mu'}}{dx^\mu} dx^\mu \equiv \omega_\mu dx^\mu \Rightarrow \underline{\underline{\omega_\mu = \omega_{\mu'} \frac{dx^{\mu'}}{dx^\mu}}} \qquad (2,61)$$

In general:

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_k}\, \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \dots dx^{\nu_k}$$

$$\Rightarrow \boxed{T^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_k'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{dx^{\mu_k'}}{dx^{\mu_k}} \frac{dx^{\nu_1}}{dx^{\nu_1'}} \cdots \frac{dx^{\nu_e}}{dx^{\nu_e'}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_e}} \qquad (2,62)$$

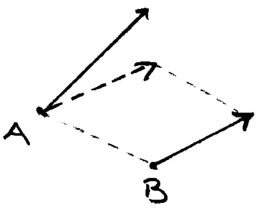
*Note that one can write

$$T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_e} = T\big(dx^{\mu_1}, \dots, dx^{\mu_k}, \partial_{\nu_1}, \dots, \partial_{\nu_e}\big). \qquad (2,63)$$

Curvature

Parallel transport and covariant derivative ; Connection

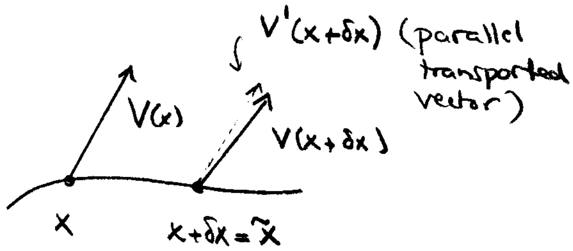
In Euclidean space comparing vectors is easy. One simply moves them to the same point.



This is not so easy in curved space where movement changes the vector

$$V'^{\mu}(\tilde{x}) = V^{\mu}(x) + \delta V^{\mu}(\tilde{x})$$

(2,64)



However, because

$$\delta V^{\mu} \to 0 \qquad \begin{cases} V^{\mu} \to 0 \\ \delta x^{\mu} \to 0 \end{cases},$$

$$\Rightarrow \quad \delta V^{\mu} = -\Gamma^{\mu}{}_{\rho\sigma} V^{\rho} \delta x^{\sigma} \qquad (2,65)$$

connection coefficient ≠ 0 on a curved manifold

Covariant derivative tells the rate of change of a vector (field) with respect to a parallel transported vector;

$$\nabla_{\mu} V^{\nu} \equiv \lim_{\delta x \to 0} \frac{V^{\nu}(x+\delta x) - V'^{\nu}(x+\delta x)}{\delta x^{\mu}}$$

$$= \lim_{\delta x \to 0} \frac{V^{\nu}(x+\delta x) - V^{\nu}(x) + \Gamma^{\nu}{}_{\rho\sigma} V^{\rho} \delta x^{\sigma}}{\delta x^{\mu}}$$

$$= \partial_{\mu} V^{\nu} + \Gamma^{\nu}{}_{\rho\mu} V^{\rho} \qquad (2,66)$$

Requiring that $\nabla_\mu V^\nu$ transforms as a tensor implies that

$$\Gamma^{\mu'}{}_{\nu'\sigma'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\sigma}}{\partial x^{\sigma'}} \Gamma^{\mu}{}_{\nu\sigma} + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x^{\nu'} \partial x^{\sigma'}} , \qquad (2,67)$$

That is the connection $\Gamma^{\mu}{}_{\nu\sigma}$ is <u>not</u> a tensor.

<u>Obs.</u> Defining a connection does not need metric. That is, parallel transport is a topological, not a metric concept. (conversely however; given $g \Rightarrow \Gamma_g$)

Vector V^μ is <u>parallel transported</u> along a curve $x^\mu(\lambda)$ when:

$$\frac{D}{d\lambda} V^\mu \equiv \underbrace{\frac{dx^\nu}{d\lambda}}_{=u^\nu} \nabla_\nu V^\mu = \frac{dx^\nu}{d\lambda}\left(\partial_\nu V^\mu + \Gamma^{\mu}{}_{\rho\nu} V^\rho\right)$$

$$= \frac{dV^\mu}{d\lambda} + \Gamma^{\mu}{}_{\rho\nu} V^\rho \frac{dx^\nu}{d\lambda} = 0 \qquad (2,68)$$

(change is only due to curvature)

This equation defines the <u>geodesic</u> as a <u>generalization of the Euclidian concept of straight line</u>:

> A geodesic is a curve which parallel transports its own tangent $V^\mu = \frac{dx^\mu}{d\lambda}$

$$\Rightarrow \frac{d^2x^\mu}{d\lambda^2} + \Gamma^{\mu}{}_{\rho\nu} \frac{dx^\rho}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \qquad (2,69)$$

Clarifying comment:

The change of a vector under a parallel transport $u^\lambda \nabla_\lambda V^\lambda \equiv 0$, results only from the rotation of the tangent space itself. The vector does not rotate at all in the co-moving local frame.

Other covariant derivatives.

- Contravariant vectors.

Because $\omega(V) = \omega_\lambda V^\lambda$ is a scalar (invariant), we have

$$\nabla_\mu (\omega_\lambda V^\lambda) = \partial_\mu (\omega_\lambda V^\lambda) \qquad ; \quad \begin{aligned} \nabla_\mu \omega_\lambda &\equiv \partial_\mu \omega_\lambda + (\Gamma\omega)_{\mu\lambda} \\ \nabla_\mu V^\lambda &= \partial_\mu V^\lambda + \Gamma^\lambda{}_{\rho\mu} V^\rho \end{aligned}$$

$$\Rightarrow \quad (\Gamma\omega)_{\mu\lambda} V^\lambda + \omega_\lambda \Gamma^\lambda{}_{\rho\mu} V^\rho = 0$$

$$\Leftrightarrow \quad \left[(\Gamma\omega)_{\mu\lambda} + \omega_\sigma \Gamma^\sigma{}_{\lambda\mu} \right] V^\lambda = 0$$

Because this must hold for arbitrary V^λ, we get

$$\underline{\nabla_\mu \omega_\lambda = \partial_\mu \omega_\lambda - \Gamma^\sigma{}_{\lambda\mu} \omega_\sigma} \qquad (2,70)$$

Similarly one can compute the covariant derivatives for more general tensors: Eg:

$$\nabla_\mu g(U,V) = \nabla_\mu (g_{\alpha\beta} U^\alpha V^\beta) = \partial_\mu (g_{\alpha\beta} U^\alpha V^\beta)$$

$$\Rightarrow \quad (\Gamma g)_{\mu\alpha\beta}\, U^\alpha V^\beta + g_{\alpha\beta}\Big(\underbrace{(\Gamma U)^\alpha{}_\mu}_{=\Gamma^\alpha{}_{\rho\mu}U^\rho}\, V^\beta + U^\alpha \underbrace{(\Gamma V)^\beta{}_\mu}_{\Gamma^\rho{}_{\rho\mu}U^\alpha V^\rho}\Big) = 0$$

$$\begin{matrix} \alpha \to \sigma & \quad & \beta \to \sigma \\ \rho \to \alpha & \quad & \rho \to \beta \end{matrix}$$

$$\Leftrightarrow \quad \left[(\Gamma g)_{\mu\alpha\beta} + \Gamma^\sigma{}_{\alpha\mu}\, g_{\sigma\beta} + \Gamma^\sigma{}_{\beta\mu}\, g_{\alpha\sigma} \right] U^\alpha V^\beta = 0 \quad \forall U^\alpha, V^\beta$$

$$\Rightarrow \quad \underline{\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \Gamma^\sigma{}_{\alpha\mu}\, g_{\sigma\beta} - \Gamma^\sigma{}_{\beta\mu}\, g_{\alpha\sigma}} \qquad (2,71)$$

And so on. The general rule for an arbitrary tensor is

$$\begin{aligned} \nabla_\mu T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} &= \partial_\mu T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \\ &+ \Gamma^{\mu_1}{}_{\sigma\mu}\, T^{\sigma \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_2}{}_{\sigma\mu}\, T^{\mu_1 \sigma \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \dots \\ &- \Gamma^\sigma{}_{\nu_1\mu}\, T^{\mu_1 \dots \mu_k}{}_{\sigma \nu_2 \dots \nu_l} - \Gamma^\sigma{}_{\nu_2\mu}\, T^{\mu_1 \dots \mu_k}{}_{\nu_1 \sigma \dots \nu_l} + \dots \end{aligned}$$

(2,72)

That is, every contravariant index creates a $+\Gamma$-term and each covariant one a $-\Gamma$-term.

—— xx —— xx ——

An alternative view to covariant derivative and connection:

Consider two vectors U and $V \in T_p$. Then also the gradient of V in the direction U is a vector $\nabla_u V \in T_p$. Now:

$$\begin{aligned} (\nabla_u V)^\sigma &= \hat{\theta}^\sigma(\nabla_u V) = \hat{\theta}^\sigma(u^\lambda \nabla_\lambda (V^\alpha \hat{e}_\alpha)) \\ &= u^\lambda \hat{\theta}^\sigma\Big(\underbrace{(\nabla_\lambda V^\alpha)}_{=\partial_\lambda V^\alpha}\hat{e}_\alpha + V^\alpha(\nabla_\lambda \hat{e}_\alpha)\Big) \end{aligned}$$

$=\partial_\lambda V^\alpha$ because V^α is just a scalar function here!

$$= u^\lambda \left(\partial_\lambda V^\alpha \hat{\theta}^\sigma(\hat{e}_\alpha) + V^\alpha \hat{\theta}^\sigma(\nabla_\lambda \hat{e}_\alpha) \right)$$

$$= u^\lambda \left(\partial_\lambda V^\sigma + V^\alpha \hat{\theta}^\sigma(\nabla_\lambda \hat{e}_\alpha) \right) \equiv u^\lambda \left(\partial_\lambda V^\sigma + \Gamma^\sigma{}_{\alpha\lambda} V^\alpha \right)$$

$$= u^\lambda \nabla_\lambda V^\sigma$$

(our initial notation)

That is, we really mean that

$$\nabla_\lambda V^\sigma \equiv (\nabla_\lambda V)^\sigma$$

and

$$\Gamma^\sigma{}_{\alpha\lambda} = \overset{dx^\sigma(\nabla_\lambda \partial_\alpha)}{\hat{\theta}^\sigma(\nabla_\lambda \hat{e}_\alpha)} \qquad (2,73)$$

(= the σ-component of the rate of change of vector $\hat{e}_\alpha$ in a movement along the direction $\hat{e}_\lambda$.)

Ex:

You can also similarly show (looking at the form $(\nabla_\mu \omega)$) that

$$\Gamma^\sigma{}_{\alpha\lambda} = -\left(\nabla_\lambda \hat{\theta}^\sigma\right)(\hat{e}_\alpha). \qquad (2,74)$$

With these results at hand equation (2,72) follows finally! (Ex).

Metric geodesic $\hat{=}$ the path of shortest distance between two points.

Let us now add the metric structure $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ to our manifold.

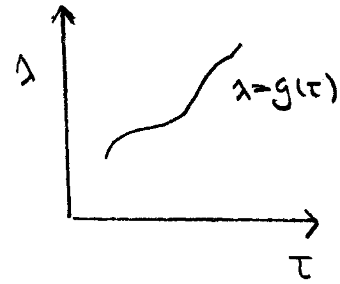
The distance between two points P_1 & P_2 along a path s is

$$l = \int_{P_1}^{P_2} ds \qquad (2,75)$$

The metric geodesic is the curve s, which minimizes l. Now observe:

$$l = \int_{P_1}^{P_2} d\lambda \, \frac{ds}{d\lambda} = \int_{P_1}^{P_2} d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \qquad (2.76)$$

is <u>reparametrization invariant</u>. That is, one can make an arbitrary change of variables $\lambda = g(\tau)$, where g is a monotonic function, without changing l. We can simplify our problem then enormously by using $\lambda \to \tau$, where τ is the <u>proper time</u> along the curve. Then:



$$ds(\lambda) = c\,d\tau(\lambda)$$

$$\Rightarrow \int_{P_1}^{P_2} ds = \frac{1}{c}\int_{P_1}^{P_2} \frac{ds^2}{d\tau} = \frac{1}{c}\int_{P_1}^{P_2} d\lambda \, \frac{ds^2}{d\lambda d\tau}$$

$$\overset{\lambda\to\tau}{=} \frac{1}{c}\int_{P_1}^{P_2} d\tau \, g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \equiv \int_{P_1}^{P_2} d\tau \, L(\tau) \qquad (2.77)$$

Note that after the step $\lambda \to \tau$ the expression has lost its reparametrization invariance! It is only invariant under <u>affine transformation</u>

$$\tau \to \tau' = a\tau + b \qquad (2.78)$$

which corresponds to arbitrary unit and the initial value of the (proper) time. Now, we get from (2.77) (Euler-Lagrange)

$$0 = \frac{\delta l}{\delta x^\rho} \overset{P_1 \& P_2 \text{ fixed}}{\Longrightarrow} \frac{\delta L}{\delta x^\rho} - \frac{d}{d\tau}\frac{\delta L}{\delta \dot{x}^\rho} = 0 \quad ; \quad \dot{x}^\rho \equiv \frac{dx^\rho}{d\tau}$$

- $\dfrac{\delta L}{\delta x^\rho} = g_{\mu\nu,\rho}\,\dot{x}^\mu \dot{x}^\nu$
- $\dfrac{\delta L}{\delta \dot{x}^\rho} = g_{\mu\nu}\left(\dot{x}^\mu \delta^\nu_\rho + \delta^\mu_\rho \dot{x}^\nu\right) \overset{g_{\mu\nu} = g_{\nu\mu}}{=} 2 g_{\mu\rho}\dot{x}^\mu$

$$\Rightarrow \frac{d}{d\tau}\frac{\delta L}{\delta \dot{x}^\rho} = 2 g_{\mu\rho}\ddot{x}^\mu + 2\dot{x}^\mu \dot{x}^\nu g_{\mu\rho,\nu} \qquad (g_{\mu\nu,\rho} \equiv \partial_\rho g_{\mu\nu})$$

$$= 2 g_{\mu\rho}\frac{d^2x^\mu}{d\tau^2} + (g_{\mu\rho,\nu} + g_{\nu\rho,\mu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$\Rightarrow g_{\mu\rho}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \quad \Big| \cdot g^{\rho\alpha};$$

$$\Leftrightarrow \frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}g^{\rho\alpha}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \quad ; \quad \begin{matrix}\alpha\to\mu\\ \mu\to\sigma\end{matrix}$$

$$\Leftrightarrow \boxed{\frac{d^2x^\alpha}{d\tau^2} + \Gamma_{ch}{}^{\alpha}{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0} \qquad (2,79)$$

where we defined the <u>Christoffel-connection</u>

$$\boxed{\Gamma_{ch}{}^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})} \qquad (2,80)$$

This is a metric-induced connection on a manifold. Particles obeying Einsteins theory will follow these geodesics, even when the manifold might have some other connection defining the "straight" paths!

Christoffel connection parallel transports the metric:

$$\nabla_\rho g_{\mu\nu} = 0 \qquad (2,81)$$

when $\Gamma = \Gamma_{ch}$. Eq. (2,81) can also be taken as a definition of the metric (Christoffel) connection: you can derive (2,80) by assuming (2,81) holds.

Exercise: Show that

$$\Gamma^{\mu}{}_{\mu\sigma} = \frac{1}{\sqrt{-g}}\,\partial_\sigma \sqrt{-g} \quad ; \quad g \equiv \det(g_{\mu\nu})$$

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,V^\mu\right) \qquad (2,82)$$

Note that the geodesic equation can be rewritten with help of the 4-velocity $u^\mu = dx^\mu/d\tau$ as:

$$\frac{du^\mu}{d\tau} + \Gamma^{\mu}{}_{\nu\sigma}u^\nu u^\sigma = 0$$

But $\dfrac{d}{d\tau} = \dfrac{dx^\sigma}{d\tau}\partial_\sigma = u^\sigma \partial_\sigma$, whereby

$$u^\sigma\left(\partial_\sigma u^\mu + \Gamma^{\mu}{}_{\nu\sigma}u^\nu\right) = \underline{u^\sigma \nabla_\sigma u^\mu = 0} \qquad (2,83)$$

In terms of 4-momentum $p^\mu = m u^\mu$ this says that

$$p^\sigma \nabla_\sigma p^\mu = 0 \qquad (2,84)$$

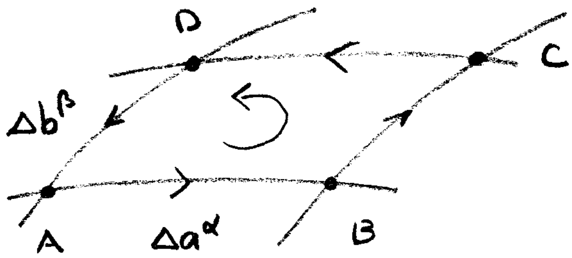
That is, freely falling particles are going where their momentum is pointing to. Sensible!

For light $m=0$ of course, but even then one can choose λ such that $p^\mu = dx^\mu/d\lambda$ and the physical content of eq. (2.84) remains intact. (see Carroll, p.110).

Curvature tensor

Connection coefficients tell us everything about curvature. Eg. does a vector change when parallel transported over a closed loop, or do parallel geodesic remain parallel at all times etc.

Let us look more closely at the first of these issues. Consider parallel transporting a vector over an infinitesimal loop



After going around the loop $V^\mu \to V^\mu + \delta V^\mu$ at point A. Because V^μ is a vector so is δV^μ, and hence $V \rightsquigarrow \delta V$ is a linear mapping, i.e. a tensor with at least one upper and lower indices. However, the result must depend also on vectors Δa^α and Δb^β, so that

$$\delta V^\mu \equiv -R^\mu{}_{\nu\alpha\beta} V^\nu \Delta a^\alpha \Delta b^\beta \qquad (2.85)$$

(+ if ↻ ; arbitrary definition)

Now, we clearly need to have $\delta V^\mu \to -\delta V^\mu$ if we reverse the order of traversing the loop, i.e. if $\Delta a \leftrightarrow \Delta b$, and hence

$$R^\mu{}_{\nu\alpha\beta} = -R^\mu{}_{\nu\beta\alpha} \qquad (2.86)$$

(Then also $\delta V \to 0$ for $\Delta a = \Delta b$!)

The 4-index object $R^{\mu}{}_{\nu\alpha\beta}$ is the <u>Riemann curvature tensor</u>:

$$R : T_p \otimes T_p \otimes T_p \to T_p$$

$$R = R^{\mu}{}_{\nu\alpha\beta}\, \hat{e}_{\mu} \otimes \hat{\theta}^{\nu} \otimes \hat{\theta}^{\alpha} \otimes \hat{\theta}^{\beta}.$$

Let us now express $R^{\mu}{}_{\nu\alpha\beta}$ using connection.

$$\delta V^{\mu} = \delta V^{\mu}_{AB} + \delta V^{\mu}_{BC} - \delta V^{\mu}_{AD} - \delta V^{\mu}_{DC}$$

$$= -\Gamma^{\mu}{}_{\nu\alpha}(A)\, V^{\nu} \Delta a^{\alpha} - \Gamma^{\mu}{}_{\nu\beta}(A+\Delta a)\, V^{\nu}(A+\Delta a)\, \Delta b^{\beta}$$
$$+ \Gamma^{\mu}{}_{\nu\beta}(A)\, V^{\nu} \Delta b^{\beta} + \Gamma^{\mu}{}_{\nu\alpha}(A+\Delta b)\, V^{\nu}(A+\Delta b)\, \Delta a^{\alpha}$$

Use: $\Gamma^{\mu}{}_{\nu\delta}(A+\Delta n)\, V^{\nu}(A+\Delta n)$

$$= \Gamma^{\mu}{}_{\nu\delta}(A) V^{\nu}(A) + \partial_{\sigma}\left(\Gamma^{\mu}{}_{\nu\delta}(A) V^{\nu}(A)\right) \Delta n^{\sigma} + \mathcal{O}(\Delta n^2)$$

$$\Rightarrow \delta V^{\mu} = \left[\partial_{\beta}(\Gamma^{\mu}{}_{\nu\alpha} V^{\nu}) - \partial_{\alpha}(\Gamma^{\mu}{}_{\nu\beta} V^{\nu}) \right] \Delta a^{\alpha} \Delta b^{\beta}$$

$$\begin{pmatrix} \nu \to \rho \\ \rho \to \nu \end{pmatrix}$$

$$= \Big[\partial_{\beta}\Gamma^{\mu}{}_{\nu\alpha} V^{\nu} + \Gamma^{\mu}{}_{\nu\alpha} \underbrace{\partial_{\beta} V^{\nu}}_{= -\Gamma^{\nu}{}_{\rho\beta} V^{\rho}} - (\alpha \leftrightarrow \beta) \Big] \Delta a^{\alpha} \Delta b^{\beta}$$

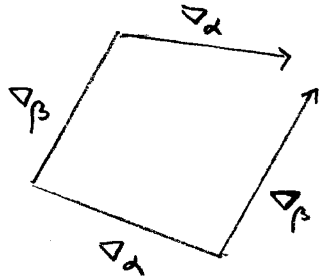
$$= -\left(\partial_{\alpha}\Gamma^{\mu}{}_{\nu\beta} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta} - (\alpha \leftrightarrow b) \right) V^{\nu} \Delta a^{\alpha} \Delta b^{\beta}$$

$$= -R^{\mu}{}_{\nu\alpha\beta} V^{\nu} \Delta a^{\alpha} \Delta b^{\beta}$$

That is

$$\boxed{R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta} - \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\nu\alpha}} \quad (2,87)$$

Note that $R^{\mu}{}_{\nu\alpha\beta} = R^{\mu}{}_{\nu\alpha\beta}(\Gamma)$, There is no reference to a metric here.



Ex. Show that

$$[\nabla_\alpha, \nabla_\beta] V^\mu = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu$$

$$= R^{\mu}{}_{\nu\alpha\beta} V^\nu - \underbrace{2\Gamma^{\lambda}{}_{[\alpha\beta]}}_{\equiv T^{\lambda}{}_{\alpha\beta}} \nabla_\lambda V^\mu \quad (2,88)$$

$\equiv T^{\lambda}{}_{\alpha\beta}$; torsion tensor.

(=0 for metric connection)

Space has torsion if parallel transports do not commute. $R^{\mu}{}_{\nu\alpha\beta}$ describes all spaces, and it coincides with the commutator of the parallel transports in torsionless spaces.

Notation above $T^\lambda = 2\Gamma^{\lambda}{}_{[\alpha\beta]} \equiv \Gamma^{\lambda}{}_{\alpha\beta} - \Gamma^{\lambda}{}_{\beta\alpha}$ (2,89)

3,7.

Riemann tensor has many symmetry properties (see Caroll or MSW). In the end only 20 of the total of $4^4 = 256$ components are independent. ($\frac{1}{12}d^2(d^2-1)$ in dimension d).

From the Riemann tensor we can define other tensors: For example the contraction

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \qquad (2.90)$$

Defines the <u>Ricci tensor</u>, which you might have seen in the Einstein equation. If the connection defining $R^{\lambda}{}_{\mu\alpha\nu}$ is Christoffel, then (2.90) is the only independent contraction.

It is easy to verify that for Ch-connection $R_{\mu\nu} = R_{\nu\mu}$, $\Rightarrow$ <u>$R_{\mu\nu}$ has 10 components.</u> (For now, until said otherwise we take $\Gamma \equiv \Gamma_{ch.}$)

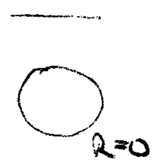
From $R_{\mu\nu}$ we can define the <u>curvature scalar</u>

$$R \equiv g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu} \qquad (2.91)$$

For the metric (ch-) connection R is the <u>only</u> invariant scalar function that can be constructed from the metric (containing two derivatives)

Some specific values of R:

- Euclidian n-sphere: $R = \dfrac{n(n-1)}{r^2} = \begin{cases} 0 \; ; \; n=1 \\ \frac{2}{r^2} \; ; \; n=2 \\ \text{etc} \end{cases}$



- Schwarzhild exterior solution: $R = 0$?

- FRW: $R = 6\left(\dfrac{\ddot{a}}{a} + \left(\dfrac{\dot{a}}{a}\right)^2 + \dfrac{\kappa}{a^2} \right)$

$\left(R \sim \pm \frac{1}{a^2} \text{ (or 0) if } \dot{a} = \ddot{a} = 0 \right)$

Above all, the curvature scalar defines the <u>Einstein-Hilbert action</u>

$$S \equiv \underbrace{\frac{1}{16\pi G_N}}_{\text{Newtons constant}} \int \underbrace{d^4x \sqrt{-g}}_{\text{inv. volume}} \underbrace{R}_{\text{Ricci scalar}} + \Big(\underbrace{S_m[\psi, g]}_{\substack{\text{Matter-action} \\ \text{(MSM + extensions)}}} \Big)$$

$$= S_{EH}[g] + S_m[\psi, g] \qquad (2.92)$$

When S is varied with respect the metric, one directly gets the famous Einstein equations:

$$16\pi G_N \frac{\delta S}{\delta g^{\mu\nu}} = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \quad \Big(\equiv \sqrt{-g}\, \underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} \Big)$$

$$\frac{1}{2} \frac{\delta S_m}{\delta g^{\mu\nu}} = -\sqrt{-g}\, T_{\mu\nu}$$

$$\frac{\delta S}{\delta g} = 0 \quad \Rightarrow \quad \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}} \qquad (2.93)$$

Extra: Derive (2,93) from (2,92). First put $S_m \equiv 0$.

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, R$$

$$\Rightarrow \quad \delta S_{EH} = \frac{1}{16\pi G_N} \int d^4x \left[\sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g}\, R_{\mu\nu} \delta g^{\mu\nu} + R\, \delta\sqrt{-g} \right]$$

$$\equiv \delta S_1 + \delta S_2 + \delta S_3 \equiv 0$$

Now: $\delta\sqrt{-g} = \delta\sqrt{-\det(g)} = \delta\, i\, e^{\frac{1}{2}\mathrm{Tr}\ln g_{\mu\nu}}$

$$= \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$$

$$\Rightarrow \quad \delta S_{EH} = \delta S_1 + \frac{1}{16\pi G_N}\int d^4x \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\delta g^{\mu\nu} = 0$$

So if $\delta S_1 = 0$, we would get the lhs of the Einstein equation with $T_{\mu\nu} \equiv 0$. Now:

move to normal coordinates where $\Gamma = 0$ but $\partial\Gamma$ is not.

$$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} = \delta(\partial_\lambda \Gamma^\lambda{}_{\nu\mu}) - \delta(\partial_\mu \Gamma^\lambda{}_{\nu\lambda})$$

$$= \partial_\lambda(\delta\Gamma^\lambda{}_{\nu\mu}) - \partial_\mu(\delta\Gamma^\lambda{}_{\nu\lambda})$$

$$= \nabla_\lambda(\delta\Gamma^\lambda{}_{\nu\mu}) - \nabla_\mu(\delta\Gamma^\lambda{}_{\nu\lambda})$$

in general coordinates, because $\delta\Gamma^\lambda{}_{\nu\mu}$ is a tensor. (covariant generalization)

($\nabla_\mu V^\mu = \nabla_{\hat\mu} V^{\hat\mu} = \partial_{\hat\mu} V^{\hat\mu}$ in normal coord. for a vector)

Note that although $\Gamma^{\alpha}{}_{\mu\nu}$ is not a tensor $\delta\Gamma^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\alpha}{}_{\mu\nu} - \hat{\Gamma}^{\alpha}{}_{\mu\nu}$ is. Indeed

$$\underbrace{\nabla_\mu V^\lambda - \hat{\nabla}_\mu V^\lambda}_{\text{Tensor}} = \partial_\mu V^\lambda + \Gamma^{\lambda}{}_{\rho\mu} V^\rho - \partial_\mu V^\lambda - \hat{\Gamma}^{\lambda}{}_{\rho\mu} V^\rho$$

$$= (\Gamma^{\lambda}{}_{\rho\mu} - \hat{\Gamma}^{\lambda}{}_{\rho\mu}) V^\rho = \underbrace{\delta\Gamma^{\lambda}{}_{\rho\mu}}_{\text{tensor}} \underbrace{V^\rho}_{\text{vector}}$$

Thus we have:

$$\delta S_1 = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(\nabla_\lambda(\delta\Gamma^{\lambda}{}_{\mu\nu}) - \nabla_\mu(\overbrace{\delta\Gamma^{\lambda}{}_{\nu\lambda}}^{\equiv V_\nu}) \right)$$

$$= \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(\nabla_\lambda(\underbrace{g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\mu\nu}}_{\equiv U^\lambda}) - (\underbrace{\nabla_\lambda g^{\mu\nu}}_{=0}) \delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla^\nu V_\nu \right)$$

(• Eg: $0 = \nabla_\lambda \delta^{\mu}{}_{\alpha} = \nabla_\lambda(g^{\mu\nu} g_{\nu\alpha}) = (\nabla_\lambda g^{\mu\nu}) g_{\nu\alpha} + g^{\mu\nu}(\overbrace{\nabla_\lambda g_{\nu\alpha}}^{=0})$

$\Rightarrow \nabla_\lambda g^{\mu\nu} = 0.$)

$$= \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(\nabla_\lambda U^\lambda - \overbrace{\nabla^\nu V_\nu}^{=\nabla_\nu V^\nu} \right) \quad ; \quad U^\lambda \equiv g^{\mu\nu}\Gamma^{\lambda}{}_{\mu\nu}$$

$$V_\mu \equiv \delta\Gamma^{\lambda}{}_{\mu\lambda}$$

$$= \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(\frac{1}{\sqrt{-g}} \partial_\lambda(\overbrace{\sqrt{-g}\, U^\lambda}^{\equiv \tilde{U}^\lambda}) - \frac{1}{\sqrt{-g}} \partial_\mu(\overbrace{\sqrt{-g}\, V^\mu}^{\equiv \tilde{V}^\mu}) \right)$$

$$= \frac{1}{16\pi G_N} \int d^4x\, \partial_\lambda(\tilde{U}^\lambda - \tilde{V}^\lambda) \underset{\text{Stokes th}}{=} \frac{1}{16\pi G_N} \int_{\delta\Sigma} d^3x\, n \cdot (\tilde{U} - \tilde{V}) = 0$$

<u>Assume.</u> $\delta g, \partial\delta g \to 0$ as $x \to \infty$.

Here we used the result (2,82): $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,V^\mu)$, and eventually assumed that the surface term vanishes (i.e. $\delta g \to 0$ and $\partial\,\delta g \to 0$ on the boundary)

this one is not really that easy...

So indeed $\delta S_1 = 0$ and we get

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{1}{16\pi G_N}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$

If we now define *

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \qquad (2,94)$$

We get

$$0 = \frac{16\pi G_N}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{16\pi G_N}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} + \frac{16\pi G_N}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$$

$$\Leftrightarrow \quad \underline{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}}$$

* Eg: $\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m\phi^2$; $\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)$

$$\Rightarrow \frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{\delta}{\delta g^{\mu\nu}}\int d^4x\,\sqrt{-g}\,\mathcal{L}_\phi = \sqrt{-g}\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} - \frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\mathcal{L}_\phi$$

$$= -\frac{\sqrt{-g}}{2}\left(g_{\mu\nu}\mathcal{L}_\phi - (\partial_\mu\phi)(\partial_\nu\phi)\right)$$

Einstein equation; bottom up approach.

Starting point. Observe that gravitation couples universally to all matter and energy. (EEP)

Einstein equivalence principle

(I) Newton

$$\left.\begin{aligned} \vec{F}_I &= \overset{\text{inertial mass}}{m_I}\,\vec{a} \\ \vec{F}_G &= -\underset{\text{gravitational "charge"}}{m_G}\nabla\Phi \end{aligned}\right\} \Rightarrow \vec{a} = -\underbrace{\frac{m_G}{m_I}}_{=1}\nabla\Phi$$

$=1$ for all kinds of matter. (Eötvös)

$\Rightarrow$ WEP

Weak equivalence principle

(II) special th. of relativity: $E=mc^2$

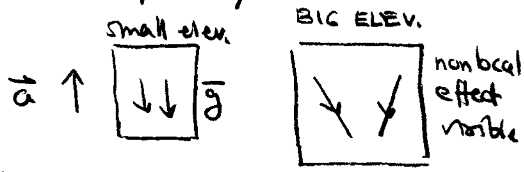
+ observation: (eg) $m_{^4He} = 2m_p + 2m_n + \underbrace{E_{binding}}_{<0}$

<0. interaction energy

yet ^{4}He and protons and neutrons fall the same way $\Rightarrow$ gravitation couples to interaction energy the same way as to any matter. $\Rightarrow$ EEP.

Einsteins elevators.

Because of the EEP, gravitational force cannot be differentiated from acceleration by <u>local</u> experiments. That is $\exists$ special inertial frames where gravity is eliminated completely.



$\Rightarrow$ one can think that gravity is <u>a property of the space</u>, not of matter.

This then leads to the idea of geodesic motion of freely falling bodies.

Because gravity disappears in freely falling inertial frames it makes sense to require that all laws of nature are reduced to special relativity in these frames. In particular $\exists$ coordinate frame where

$$g_{\mu\nu} \rightarrow g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$$

and

$$\partial_{\hat{\alpha}} g_{\hat{\alpha}\hat{\beta}} = 0 \qquad (\Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} = 0)$$

It then follows that the curvature can be seen only at order $\partial_{\hat{\alpha}}\partial_{\hat{\beta}} g_{\hat{\alpha}\hat{\beta}} \neq 0$.

<u>On d.o.f's</u>:

- $g_{\mu\nu} = g_{\nu\mu}$ is a tensor

$$\eta_{\hat{\mu}\hat{\nu}} = \underbrace{\frac{\partial x^{\alpha}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\beta}}{\partial x^{\hat{\nu}}}}_{\text{16 arbitrary functions}} \overbrace{g_{\alpha\beta}}^{\text{10 free d.o.f}}$$

$\Rightarrow$ Can choose $x^{\hat{\mu}}$ such that $g_{\alpha\beta} \rightarrow \eta_{\hat{\mu}\hat{\nu}}$. The extra 6 d.o.f.s in the transformation are just the local Lorentz transformations between the inertial frames of special relativity!

- Similarly the 40 components of $\partial_{\hat{\alpha}} g_{\hat{\alpha}\hat{\beta}}$ can be removed by coordinate transformations, while only 80 of the 100 d.o.f's of the quantity $\partial_{\hat{\alpha}}\partial_{\hat{\beta}} g_{\hat{\alpha}\hat{\beta}}$ can be taken out. This leaves just the 20 independent d.o.f's of $R^{\lambda}{}_{\mu\nu\rho}$! (See Carroll p. 74–75.)

These considerations lead to the picture where space time is a differential manifold with a metric and metric curvature, and which locally looks like a Minkowski space.

How about the dynamics? We know that in slow movements the Newtonian gravity holds:

$$\nabla^2 \Phi = 4\pi G_N \rho \qquad (2,95)$$

The problem is that this equation is not <u>covariant</u>. (Covariance corresponds to requirement that the laws of nature look the same in all coordinate systems.)

As we have seen, already the special theory of relativity was based on covariance, i.e. requirement of invariance under arbitrary Lorenz transformations. In GR this is generalized to arbitrary coordinate transformations

$\Rightarrow$ The laws of nature must be expressed as <u>tensor equations</u>.

Now, the covariant generalization of ρ in the special th. of relativity is (for an ideal fluid)

$$T^{\hat{\mu}\hat{\nu}} = (\rho + p) u^{\hat{\mu}} u^{\hat{\nu}} + p \eta^{\hat{\mu}\hat{\nu}} \qquad (2,96)$$

density pressure

where $u^{\hat{\mu}} = \gamma(1, \vec{v})$. If the fluid velocity v is small $v \ll c$, we have $u^{\hat{\mu}} \simeq (1, \vec{0})$. If furthermore $p \ll \rho$, we get

$$T^{\hat{0}\hat{0}} = \rho \quad ; \quad T^{\hat{\mu}\hat{\nu}} \ll T^{\hat{0}\hat{0}} \quad \hat{\mu} \text{ or } \hat{\nu} \neq 0$$

So it is natural to replace the r.h.s. of eqn (2,95) by the tensor

symmetric ⇒ 10 components, in general

$$\kappa T = \kappa T^{\mu\nu} \, \hat{e}_\mu \otimes \hat{e}_\nu$$

To the l.h.s. of (2,95) we need some other tensor that can depend on metric and its (2^{nd}) derivatives. Let us call it G, i.e.

$$\boxed{G = \kappa T} \qquad \kappa \text{ is some coeff. of proportionality}$$

where $G = G^{\mu\nu} \hat{e}_\mu \otimes \hat{e}_\nu$. What could $G^{\mu\nu}$ be? In the Local freely falling frame $\partial_{\hat{\alpha}} g_{\hat{\alpha}\hat{\beta}} = 0$ and in general $\nabla_\alpha g_{\alpha\beta} = 0$, so G has to be some contraction from expression involving second derivatives $\partial_\alpha \partial_\beta g_{\mu\nu}$, which is a tensor, and which has 10 components. Guess $G^{\mu\nu} \sim R^{\mu\nu}$. But now, in the freely falling frame

$$\partial_{\hat{\mu}} T^{\hat{\mu}\hat{\nu}} = 0 \Rightarrow \underline{\nabla_\mu T^{\mu\nu} = 0} \quad (2,97)$$

Energy momentum conservation law in curved space.

⇒ we have to have $\nabla_\mu G^{\mu\nu} = 0$ as well. $R^{\mu\nu}$ does <u>not</u> fill this requirement. From the Bianchi identity (Ex.)

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

One can show that

$$\nabla_\mu G^{\mu\nu} = 0 \quad \text{for} \quad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

So, we are ready to postulate

$$\underline{G = \kappa T} \quad \Leftrightarrow \quad \underline{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}} \qquad (2,98)$$

to be defined from Newton limit

Is this equation unique? (ie. the only covariant generalization consistent with EEP and locally Minkowskian)

NO !

It is the simplest generalization of special th. of relativity, consistent with EEP to curved spaces. It is also extremely succesfull.

However, we do not have a strong arguments like we have in QFT:

$$(\text{symmetry} + \text{renormalizability}) \overset{\text{unique}}{\Rightarrow} \mathcal{L}(\psi, A_\mu \ldots).$$

(Wilsonian sense)

So why does gravity emerge at long distances is not explained. Recently it has become somewhat popular to think that gravity might actually not be a fundamental force at all, nor even geometry, strictly speaking. Maybe gravity is an entropic force, similar to the one governing the osmosis. If so, then also the space and inertia are emergent phenomena!

Generalizations

1) Add cosmological constant Λ

$$S_{EH} \to \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\,(R - 2\Lambda) \qquad (2{,}99)$$

$$\Rightarrow \quad G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \qquad (2{,}100)$$

- Note that $\Lambda g_{\mu\nu}$ is a "tensor that depends on g", so it could have been included from the beginning to the definition of $G_{\mu\nu}$. Also note that $\nabla_\alpha g_{\mu\nu} = 0$, so this term is covariant. Today we tend to associate $\Lambda g_{\mu\nu}$ with $T_{\mu\nu}$ however.

2) Generalized $f(R)$ - models

$$S_{EH} \longrightarrow \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, \underbrace{f(R)}_{\text{arbitrary function of } R} + S_m[g_{\mu\nu}, \psi] \qquad (2{,}101)$$

if $\Gamma = \Gamma_{ch}$

$$\Rightarrow \quad F R_{\mu\nu} - \frac{1}{2} f(R)\, g_{\mu\nu} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \Box F = 8\pi G_N T_{\mu\nu} \qquad (2{,}102)$$

where $F \equiv \partial f / \partial R$. Any such theory is consistent with EEP and the requirement of equivalence with special th. of relativity in local frames.

- This theory is 4^{th} order, however!
- Different 2^{nd} order theory results if Γ is taken as a free parameter (Palatini formulation)

- Observations (in particular from satellite tracking in the solar system) show however that

$$f(R) = R - 2\Lambda + \text{something very small.}$$

So GR just works.

3) Scalar-Tensor gravities

$$S_{EH} \to \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(F(\phi) R + Z(\phi)(\nabla\phi)^2 + V(\phi) \right)$$

$$+ \; S_m\left[A(\phi) g^{\mu\nu}, \psi \right] \qquad (2.103)$$

These constructions pop out from compactification of higher dimensional gravity theories and in particular from low energy limits of string theories.

- GR obviously belong to this class
- $f(R)$-models are classically equivalent to a subclass of STG-models
- Conformal gravity theories are included
- Observations again put strong constraints (not excluded however)
- Can "explain" DE ($\Lambda \to$ dynamical DE)

In what follows we shall restrict our attention to the model with $S_{EH} \to S_{EH} + S_\Lambda$ only. (Mostly.)

Newton limit of GR

1) (Geodetic) motion in a static gravity field.

Start from

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

In a slow movement: $\frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \frac{dt}{d\tau}v^i \ll \frac{dt}{d\tau} = \frac{dx^0}{d\tau} \; (=\gamma)$
so that

$$\frac{d^2x^\mu}{d\tau^2} \approx -\Gamma^\mu{}_{00}\left(\frac{dt}{d\tau}\right)^2$$

Now:

static field

$$\Gamma^\mu{}_{00} = \frac{1}{2}g^{\mu\lambda}\left(\cancel{\partial_0 g_{\lambda 0}} + \cancel{\partial_0 g_{0\lambda}} - \partial_\lambda g_{00}\right) = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}$$

For a weak field:

$$g_{\hat\mu\hat\nu} \equiv \eta_{\hat\mu\hat\nu} + h_{\hat\mu\hat\nu} \quad ; \quad |h_{\hat\mu\hat\nu}| \ll 1. \qquad (2,104)$$

$$\Rightarrow \quad g^{\hat\mu\hat\nu} \simeq \eta^{\hat\mu\hat\nu} - h^{\hat\mu\hat\nu} \qquad (2,105)$$

$$\Rightarrow \quad \Gamma^\mu{}_{00} \approx -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} = -\frac{1}{2}\partial^\mu h_{00}$$

$$\Rightarrow \quad \frac{d^2x^\mu}{d\tau^2} \approx \frac{1}{2}(\partial^\mu h_{00})\left(\frac{dt}{d\tau}\right)^2$$

However, since $\partial^0 h_{00} = 0 \Rightarrow \frac{dt}{d\tau} = \text{const}$, ie. in slow movements the coordinate time and the proper time are the same.

$$\Rightarrow \quad \frac{d^2x^i}{dt^2} = \frac{1}{2}\partial^i h_{00} \Leftrightarrow \underbrace{\ddot{\vec{x}} = -\nabla\Phi}_{\text{Newton!}} \quad , \text{where} \quad \boxed{\begin{array}{l} h_{00} \equiv -2\Phi \\ \Rightarrow g_{00} \approx -(1+2\Phi) \end{array}} \quad (2,106)$$

2) Newtons law of gravity

We noted that in a slow movement $T_{\hat{0}\hat{0}} \approx \rho$, and $|T_{\hat{\mu}\hat{\nu}}| \ll |T_{\hat{0}\hat{0}}|$ for $\hat{\mu}$ or $\hat{\nu} \neq 0$. Now first observe that:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad \Rightarrow \quad R = -\kappa T \qquad (2.107)$$

$$\Rightarrow R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \qquad (2.108)$$

(Here $T = g^{\mu\nu} T_{\mu\nu}$.) In a slow movement then ($T \approx -T_{00}$

$$R_{00} \approx \kappa \left(1 + \frac{1}{2} g_{00}\right) T_{00} = \frac{1}{2} \kappa \rho$$

where

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = R^{i}{}_{0i0}$$

$$= \partial_i \Gamma^i{}_{00} - \underbrace{\partial_0 \Gamma^i{}_{0i}}_{0\text{ : static}} + \overbrace{\Gamma\Gamma - \Gamma\Gamma}^{\sim(\partial h)^2 \text{ : small.}}$$

$$\approx \partial_i \frac{1}{2} \left(g^{i\lambda} \left(\underbrace{\partial_0 g_{\lambda 0}}_{0} + \underbrace{\partial_0 g_{0\lambda}}_{0} - \partial_\lambda g_{00} \right)\right)$$

$$= \frac{1}{2} \partial_i \left(g^{ij} \partial_j g_{00} \right) \approx -\frac{1}{2} \partial_i \delta^{ij} \partial_j g_{00}$$

$$= -\frac{1}{2} \nabla^2 g_{00} = \nabla^2 \Phi .$$

$$\Rightarrow \underline{\nabla^2 \Phi = \frac{1}{2} \kappa \rho} \equiv \underline{4\pi G_N \rho} \quad \Rightarrow \quad \boxed{\kappa = 8\pi G_N}$$!

Correct, and the limiting procedure defines κ.

Computation of various tensors

In practice we start from the metric. For example from the FRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \right]$$

we read:

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -a^2 \frac{1}{1-kr^2} & & \\ & & -a^2 r^2 & \\ & & & -a^2 r^2 \sin^2\theta \end{pmatrix}$$

Given the metric, you compute

(I) The connection

$$\Gamma^{\lambda}{}_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma} \right)$$

or using the geodesic eqn. if you have to do it by hand.

(II) Riemann

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\rho\beta} \Gamma^{\rho}{}_{\nu\alpha} - (\alpha \leftrightarrow \beta)$$

(III) Ricci ; $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$

(IV) R-scalar $R = g^{\mu\nu} R_{\mu\nu}$

(V) E-tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

Fortunately we do not need to do these excercises by hand. It is quite easy to write a small mathematica application that does it for you. In fact one is available at the course home page.

```
Clear[coord, metric, inversemetric, affine, Riemann, Ricci, Rscalar,
 Einstein, Tmatter, Tmattermix, DcovTmatter, r, θ, ϕ, t, B, A, m]
```

```
n = 4;
coord = {t, r, θ, ϕ};
```

$$\text{metric} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a[t]^2 \big/ (1 - K * r\hat{}2) & 0 & 0 \\ 0 & 0 & a[t]^2\, r\hat{}2 & 0 \\ 0 & 0 & 0 & a[t]^2\, r\hat{}2\, \mathrm{Sin}[\theta]\hat{}2 \end{pmatrix};$$

```
inversemetric = Inverse[metric] // FullSimplify;
inversemetric // MatrixForm;
mixmetric = metric.inversemetric;
```

$$\text{Tmattermix} = \begin{pmatrix} -\rho[t] & 0 & 0 & 0 \\ 0 & P[t] & 0 & 0 \\ 0 & 0 & P[t] & 0 \\ 0 & 0 & 0 & P[t] \end{pmatrix};$$

```
Tmatter = Tmattermix.metric // FullSimplify;
```

```
affine := affine = Simplify[
     Table[
     (1 / 2) * Sum[(inversemetric[[i, s]]) *
                 (  D[metric[[s, j]], coord[[k]] ]
                  + D[metric[[s, k]], coord[[j]] ]
                  - D[metric[[j, k]], coord[[s]] ] ), {s, 1, n}],
          {i, 1, n}, {j, 1, n}, {k, 1, n}]  ]
affine;
listaffine := Table[
              If[UnsameQ[affine[[i, j, k]], 0],
               {ToString[Γ[i, j, k]], affine[[i, j, k]]}] ,
               {i, 1, n}, {j, 1, n}, {k, 1, j}]

TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2],
               TableSpacing → {2, 2} ]
```

Γ[1, 2, 2]	$\frac{a(t)\, a'(t)}{1-K r^2}$
Γ[1, 3, 3]	$r^2\, a(t)\, a'(t)$
Γ[1, 4, 4]	$r^2\, a(t) \sin^2(\theta)\, a'(t)$
Γ[2, 2, 1]	$\frac{a'(t)}{a(t)}$
Γ[2, 2, 2]	$\frac{K r}{1-K r^2}$
Γ[2, 3, 3]	$r\left(K r^2 - 1\right)$
Γ[2, 4, 4]	$r\left(K r^2 - 1\right) \sin^2(\theta)$
Γ[3, 3, 1]	$\frac{a'(t)}{a(t)}$
Γ[3, 3, 2]	$\frac{1}{r}$
Γ[3, 4, 4]	$-\cos(\theta) \sin(\theta)$
Γ[4, 4, 1]	$\frac{a'(t)}{a(t)}$
Γ[4, 4, 2]	$\frac{1}{r}$
Γ[4, 4, 3]	$\cot(\theta)$

```
Riemann := Riemann =
      Simplify[Table[
                 D[affine[[i, j, l]], coord[[k]] ]
                -D[affine[[i, j, k]], coord[[l]] ]
               + Sum[ affine[[s, j, l]] affine[[i, s, k]]
                     -affine[[s, j, k]] affine[[i, s, l]], {s, 1, n}]
                 , {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}] ];

listriemann :=
        Table[If[UnsameQ[Riemann[[i, j, k, l]], 0],
                  {ToString[R[i, j, k, l]], Riemann[[i, j, k, l]]}] ,
              {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, k - 1}]

TableForm[Partition[DeleteCases[Flatten[listriemann], Null], 2],
          TableSpacing → {2, 2}]
```

R[1, 2, 2, 1]	$\frac{a(t)\,a''(t)}{K\,r^2-1}$
R[1, 3, 3, 1]	$-r^2\,a(t)\,a''(t)$
R[1, 4, 4, 1]	$-r^2\,a(t)\sin^2(\theta)\,a''(t)$
R[2, 1, 2, 1]	$-\frac{a''(t)}{a(t)}$
R[2, 3, 3, 2]	$-r^2\left(a'(t)^2+K\right)$
R[2, 4, 4, 2]	$-r^2\sin^2(\theta)\left(a'(t)^2+K\right)$
R[3, 1, 3, 1]	$-\frac{a''(t)}{a(t)}$
R[3, 2, 3, 2]	$\frac{a'(t)^2+K}{1-K\,r^2}$
R[3, 4, 4, 3]	$-r^2\sin^2(\theta)\left(a'(t)^2+K\right)$
R[4, 1, 4, 1]	$-\frac{a''(t)}{a(t)}$
R[4, 2, 4, 2]	$\frac{a'(t)^2+K}{1-K\,r^2}$
R[4, 3, 4, 3]	$r^2\left(a'(t)^2+K\right)$

```
Ricci := Ricci = Simplify[
   Table[Sum[Riemann[[i, j, i, l]], {i, 1, n}], {j, 1, n}, {l, 1, n}] ]

Ricci // MatrixForm // FullSimplify;

listricci := Table[If[UnsameQ[Ricci[[j, l]], 0],
    {ToString[R[j, l]], Ricci[[j, l]]}] , {j, 1, n}, {l, 1, j}]
TableForm[Partition[DeleteCases[Flatten[listricci], Null], 2],
          TableSpacing → {2, 2}]
```

R[1, 1]	$-\frac{3\,a''(t)}{a(t)}$
R[2, 2]	$\frac{2\,a'(t)^2+2\,K+a(t)\,a''(t)}{1-K\,r^2}$
R[3, 3]	$r^2\left(2\left(a'(t)^2+K\right)+a(t)\,a''(t)\right)$
R[4, 4]	$r^2\sin^2(\theta)\left(2\left(a'(t)^2+K\right)+a(t)\,a''(t)\right)$

```
Rscalar =
 Simplify[Sum[inversemetric[[i, j]] * Ricci[[i, j]], {i, 1, n}, {j, 1, n}] ]
```

$$\frac{6\left(a'(t)^2+K+a(t)\,a''(t)\right)}{a(t)^2}$$

```
Einstein := Einstein = Simplify[Ricci - (1 / 2) Rscalar * metric]

listeinstein := Table[If[UnsameQ[Einstein[[j, l]], 0],
                        {ToString[G[j, l]], Einstein[[j, l]]}] ,
                      {j, 1, n}, {l, 1, j}]
TableForm[Partition[DeleteCases[Flatten[listeinstein], Null], 2],
 TableSpacing → {2, 2}]

MixEinstein = Einstein.inversemetric // FullSimplify;
listMixeinstein := Table[If[UnsameQ[MixEinstein[[j, l]], 0],
                        {ToString[Gmix[j, l]], MixEinstein[[j, l]]}] ,
                      {j, 1, n}, {l, 1, j}]
TableForm[Partition[DeleteCases[Flatten[listMixeinstein], Null], 2],
 TableSpacing → {2, 2}]
```

G[1, 1]	$\frac{3\left(a'(t)^2+K\right)}{a(t)^2}$
G[2, 2]	$\frac{a'(t)^2+K+2\,a(t)\,a''(t)}{K\,r^2-1}$
G[3, 3]	$-r^2\left(a'(t)^2+K+2\,a(t)\,a''(t)\right)$
G[4, 4]	$-r^2\sin^2(\theta)\left(a'(t)^2+K+2\,a(t)\,a''(t)\right)$

Gmix[1, 1]	$-\frac{3\left(a'(t)^2+K\right)}{a(t)^2}$
Gmix[2, 2]	$-\frac{a'(t)^2+K+2\,a(t)\,a''(t)}{a(t)^2}$
Gmix[3, 3]	$-\frac{a'(t)^2+K+2\,a(t)\,a''(t)}{a(t)^2}$
Gmix[4, 4]	$-\frac{a'(t)^2+K+2\,a(t)\,a''(t)}{a(t)^2}$