1. Show that the critical temperature for forming the condensate in the non-relativistic limit ($Q/V \ll m^3$, this is only an approximative condition) can be expressed as

$$T_c = \frac{2\pi}{m}\left(\frac{Q}{V}\right)^{2/3}\zeta(\tfrac{3}{2})^{-2/3}$$

and in the relativistic limit ($Q/V \gg m^3$) as

$$T_c = \left(\frac{3}{m}\frac{Q}{V}\right)^{1/2}.$$

2. Show carefully the following identitites on fermionic integrals

$$\begin{aligned}\int\prod_{i=1}^{N}\mathrm{d}\theta_i^*\mathrm{d}\theta e^{-\theta_i^* A_{ij}\theta_j} &= \det A\\ \int\prod_{i=1}^{N}\mathrm{d}\theta_i^*\mathrm{d}\theta_k\theta_l^* e^{-\theta_i^* A_{ij}\theta_j} &= \det A(A^{-1})_{kl}.\end{aligned}$$

3. Show that in the low temperature limit $m/T \gg 1$

$$J_T^{\mp}(m,T) \equiv \mp T\int\frac{\mathrm{d}^3p}{(2\pi)^3}\log(1\mp e^{-\beta\omega_{\boldsymbol{p}}}) \to J_T^{\mp} = Tn(T),$$

where $\omega_{\boldsymbol{p}}^2 = \boldsymbol{p}^2 + m^2$ and n is the number density. Moreover, show that the J_T^--function can be expanded as:

$$J_T^-(m,T) = -\frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2(\frac{nm}{T})$$

where $K_2(x)$ is the Bessel function of the second kind.

4. Compute also the related integral:

$$I^-(m,T) \equiv \sum\!\!\!\!\!\!\!\int \Delta_0(\omega_n,\omega_{\boldsymbol{p}}) = \int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega}(1+2n_B(\omega_{\boldsymbol{p}})) \equiv I_0^-(m) + I_T^-(m,T)$$

Show that the thermal part of this integral has the expression

$$I_T^-(m,T) = \frac{mT}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}K_1(\frac{nm}{T}).$$

Finally show also that $mI_T^-(m,T) = \partial_m J_T^-(m,T)$.

5. Show that in the high-temperature limit

$$J_T^-(m,T) = \frac{\pi^2 T^4}{90} - \frac{m^2T^2}{24} + \frac{m^3T}{12\pi} + \frac{m^4}{2(4\pi)^2}\left[\log(\frac{me^{\gamma_E}}{4\pi T}) - \frac{3}{4}\right] - \frac{m^6\zeta(3)}{3(4\pi)^4T^2} + \mathcal{O}\Big(\frac{m^8}{T^4}\Big)$$

for bosonic integral. Note that the third term arises purely from the zero mode. Similarly for Fermions show that

$$J_T^+(m,T) = \frac{7}{8}\frac{\pi^2T^4}{90} - \frac{m^2T^2}{48} - \frac{m^4}{2(4\pi)^2}\left[\log(\frac{me^{\gamma_E}}{\pi T}) - \frac{3}{4}\right] + \frac{7m^6\zeta(3)}{3(4\pi)^4T^2} + \mathcal{O}\Big(\frac{m^8}{T^4}\Big).$$

The two first 2-3 terms in the expansions are easy, but the logarithmic corrections are much harder. The difficulty comes from the fact that J's are not analytic around $m = 0$. Look at section 2.3 of the book by Laine and Vuorinen or the paper by Doland and Jackw, which you find temporarily from the course homepage.