Excercise 1. Consider a plasma with 4-velocity u^μ and energy momentum tensor $T^{\mu\nu}$. We can then write a relativistic generalization of the Gibbs relation as follows:

$$\mathrm{d}s^\mu = \beta u_\nu \mathrm{d}T^{\nu\mu} - \sum_a \xi_a \mathrm{d}j_a^\mu, \tag{1}$$

where $\beta \equiv 1/T$ and s^μ is the entropy flux, j_a^μ a set of conserved currents and $\xi_a \equiv \mu_a/T$ where μ_a are chemical potentials. For a perfect fluid there is only one 4-vector available, u^μ, so that $s^\mu = su^\mu$ and $j_a^\mu = n_a u^\mu$ and $T^{\mu\nu} = (\rho + P)u^\mu u^\nu - p\eta^{\mu\nu}$, where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric. Using these defininition show that (1) gives both the differential Gibbs relation: $\mathrm{d}s = \beta \mathrm{d}\rho - \sum_a \xi_a \mathrm{d}n_a$ and thermal potential relation $s = \beta(\rho + P) - \sum \xi_a n_a$.

Excercise 2. Show that the generating funcction $Z(\beta, j)$ for the simple harmonic oscillator can be expressed in the form

$$\mathcal{Z}[\beta, j] = \mathcal{Z}(\beta) e^{-\frac{1}{2}\int_0^\beta \mathrm{d}\tau \mathrm{d}\tau' j(\tau)\Delta_0(\tau,\tau')j(\tau')},$$

(peform the Gaussian integral using periodicity). Show also that $\mathcal{Z}(\beta, j)$ can be expressed in the form

$$\mathcal{Z}(\beta, j) = \mathrm{Tr}[e^{-\beta \hat{H}} \mathcal{T}(e^{\int_0^\beta \mathrm{d}\tau j(\tau)\hat{q}(\tau)})].$$

Excercise 3. Show by direct evaluation in the τ-representation, that the solution to equation

$$(-\partial_\tau^2 + \omega^2)\Delta_0(\tau) = \delta(\tau)$$

is the propagator

$$\Delta_0(\tau, \omega) = \frac{1}{2\omega}\Big((1 + n_{\mathrm{BE}}(\omega))e^{-\omega|\tau|} + n_{\mathrm{BE}}(\omega)e^{\omega|\tau|}\Big).$$

Start by making exponential ansaz and then require that result obeys the KMS-condition and you get the correctly normalized $\delta(\tau)$-distribution from the derivative terms.

Excercise 4. Show by direct discretization of the path integral that the partition function is exactly given by

$$\mathcal{Z}(\beta) = \int_\beta \mathcal{D}q e^{-\int_0^\beta \mathrm{d}\tau(\frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2)} = \frac{e^{-\frac{1}{2}\beta\omega}}{1 - e^{-\beta\omega}}.$$

Start by dividing the quantum path into a classical parth and a perturbation $q = q_{\mathrm{cl}} + h$ and show that partition function separates $\mathcal{Z} = \mathcal{Z}_{\mathrm{cl}}\mathcal{Z}_h$. Find classical part evaluating $S_{E,\mathrm{cl}}$

by using the propagator derived in excercise 3. Then show that the fluctuation part can be written as:

$$\mathcal{Z}_h = \lim_{N\to\infty}\kappa_N^{N+1}\int\prod_{i=1}^{N}\mathrm{d}h_i\exp\Big[-\sum_{ij}h_iA_{ij}h_j\Big] \tag{2}$$

where

$$A_{ij} = \frac{1}{2a_N\Delta\tau_N}\Big(\delta_{ij} - a_N(\delta_{i+1,j}+\delta_{i-1,j})\Big), \tag{3}$$

with $a_N \equiv (2+(\Delta\tau\omega)^2)^{-1}$ and $\Delta\tau = \beta/(N+1)$. Evaluate the determinant and finally show that $\kappa_N = 1/\sqrt{2\pi\Delta\tau}$, requiring that the path integral for transition amplitude obeys

$$F(h,-i\beta;h,0) = \sum_{h'}F(h,-i\beta;h',-i(\beta-\Delta\tau))F(h',-i(\beta-\Delta\tau);h,0)$$

Excercise 5. Prove the identity

$$\frac{\sinh\pi x}{\pi x} = \prod_{n=1}^{\infty}(1+\frac{x^2}{n^2}).$$

Hint: first take a logarithm of the product expresssions, and then differentiate the result with respect to x. The differentiated result is a simple sum that can be evaluated by the complex path contour trick, using the complex function $\coth(z)$.

Note: excercise 4 is probably the most demanding, maybe equivalent to in workload to all others combined.