Finite temperature field theory (FTFT) Excercise 3.

Return by 12.00, Monday 5.10.2021,
(electronically to olli.a.koskivaara@student.jyu.fi or in paper to a box outside Fys.1.)

1. Show that in the low temperature limit $m/T \gg 1$ the thermal integrals

$$J_T^{\mp}(m,T) \equiv \mp T \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \log(1 \mp e^{-\beta\omega_{\boldsymbol{p}}}) \to J_T^{\mp} = T n(T),$$

where $\omega_{\boldsymbol{p}}^2 = \boldsymbol{p}^2 + m^2$ and n is the number density.

2. Show carefully the following identitites on fermionic integrals

$$\begin{aligned} \int \prod_{i=1}^{N} \mathrm{d}\theta_i^* \mathrm{d}\theta e^{-\theta_i^* A_{ij}\theta_j} &= \det A \\ \int \prod_{i=1}^{N} \mathrm{d}\theta_i^* \mathrm{d}\theta_k \theta_l^* e^{-\theta_i^* A_{ij}\theta_j} &= \det A (A^{-1})_{kl}. \end{aligned}$$

3. Show that in the high-temperature limit

$$J_T^{-}(m,T) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{2(4\pi)^2}\left[\log(\frac{m e^{\gamma_E}}{4\pi T}) - \frac{3}{4}\right] + \frac{m^6 \zeta(3)}{3(4\pi)^4 T^2} + \mathcal{O}\Big(\frac{m^8}{T^4}\Big)$$

for bosonic integral. Note that the third term arises purely from the zero mode. Similarly for Fermions show that

$$J_T^{+}(m,T) = \frac{7}{8}\frac{\pi^2 T^4}{90} - \frac{m^2 T^2}{48} - \frac{m^4}{2(4\pi)^2}\left[\log(\frac{m e^{\gamma_E}}{\pi T}) - \frac{3}{4}\right] + \frac{7 m^6 \zeta(3)}{3(4\pi)^4 T^2} + \mathcal{O}\Big(\frac{m^8}{T^4}\Big).$$

The two first 2-3 terms in the expansions are easy, but the logarithmic corrections are much harder. The difficulty comes from the fact that J's are not analytic around $m = 0$. Look at section 2.3 of the book by Laine and Vuorinen or the paper by Doland and Jackw, which you find temporarily from the course homepage.

Olli: please take into account the difficulty of the excercise when giving points. Here and also in previous excercises.