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Finite Temperature Field theory

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Abstract: These lectures cover some basic topics in the finite temperature field theory. We start from quantum statistical physics for boson and fermion systems. Then we quantize the free bosonic, fermionic and gauge-field theories and derive their basic thermal properties using path integral methods and imaginary time formalism. Phenomena studied include Bose condensation and black body radiation. We move to interacting field theories again starting from Bosonic systems. We study renormalization in finite temperature and compute the pressure up to one-loop and introduce resummation techniques to overcome infrared singularities. After this we study the effective action and in particular the effective potential and its applications in first order phase transitions, including evaluation of the transition strength, thermodynamical quantitites and bubble nucleation rate and growth. We then introduce the real-time formulation of the finite temperature field theory. To be continued.

Contents

Most of this course is concerned with *equilibrium* thermal field theory. More generally thermal field theory can be understood to encompass also the out-of-equilibrium systems, although the latter case perhaps is better refererred as quantum transport theory. The difference between the two is that in equilibrium systems the quantum density operator of the system is known, so that one can compute all quantum correlation functions as expectation values of this operator at a finite amount of labour.
-quantum statistics
-loop corrections
-infrared singularities and resummations

Our notations and convensions are summarized in the appendix A Relevant literature: [1–3]

1 Quantum statistical physics

Thermal field theories are essentially quantum statistical systems. The key object is the quantum density operator $\hat{\rho}$, whose form depends on the nature of the system: whether it is isolated (microcanonical ensemble), closed (canonical ensemble) or open (grand canonical ensemble). Quantum density operator allows computing the partitition function and all correlation functions. In this section we review some basic results on quanum statistical mechanics. These results will be re-derived in subsequent sections in the relativistic quantum field thery language and then extended to interacting field theories.

Keywords: *Quantum density operator, partition function, harmonic oscillator, path integral, Matsubara frequencies.*

In relativistic theory, where particle number is not conserved and system may be in thermal exchange with the surroundings, it is natural to use *grand canonical ensemble.* If the system is described by a Hamiltonian operator $\hat{H}$ and a set of conserved quantum nubers, corresponding to number operators $\hat{N}_i$, then the quantum statistical density operator for the system is given by

$$\hat{\rho} = e^{-\beta(\hat{H}-\sum_i \mu_i \hat{N}_i)}. \tag{1.1}$$

In the canoncial ensemble, with no particle exchange, or with no conserved particle numbers, one can use canonical density operator $\hat{\rho} = e^{-\beta\hat{H}}$. At any rate, using $\hat{\rho}$, we can compute the statistical expectation value of any quantum operator as

$$\langle A \rangle = \frac{\mathrm{Tr}[\hat{\rho}\hat{A}]}{\mathrm{Tr}[\hat{\rho}]}. \tag{1.2}$$

Normalization factor in this expectation value corresponds to the *grand canonical partition function*

$$\boxed{\mathcal{Z}(V,T;\{\mu_i\}) = \mathrm{Tr}[\hat{\rho}].} \tag{1.3}$$

Partition function is the central quantity quantum statistics. It allows us to write down the central thermodynamical quantities. In grand-canonical ensemble partition functin is related to grand potential Ω:

$$\Omega(V,T;\{\mu_i\}) = -T\log\mathcal{Z}(V,T;\{\mu_i\}). \tag{1.4}$$

$\Omega = -PV$ is a function of extensive variable V and intensive variables T and μ:

$$\mathrm{d}\Omega = -P\mathrm{d}V - S\mathrm{d}T - \sum_i N_i \mathrm{d}\mu_i. \tag{1.5}$$

We can then write (everywhere in these notes log refers to the natural logarithm):

$$P = -\Big(\frac{\partial\Omega}{\partial V}\Big)_{\mu,T} = T\frac{\partial\log\mathcal{Z}}{\partial V} \tag{1.6}$$

$$N = -\Big(\frac{\partial\mu_i}{\partial\mu}\Big)_{T,V} = T\frac{\partial\log\mathcal{Z}}{\partial\mu_i} \tag{1.7}$$

$$S = -\Big(\frac{\partial\Omega}{\partial T}\Big)_{V,\mu} = \log\mathcal{Z} + T\frac{\partial\log\mathcal{Z}}{\partial T}. \tag{1.8}$$

$$
\begin{array}{lll}
U(V,S;N) = TS - PV + \mu N & \to & H(P,S;N) = U + PV \\
\downarrow & & \downarrow \\
F(V,T;N) = U - TS & \to & G(P,T;N) = H - TS = F + PV = \mu N \\
\downarrow & & \downarrow \\
\Omega(V,T;\mu) = F - \mu N = -PV & \to & 0
\end{array}
$$

Table 1: Reminder: change of thermodynamical potentials under Legendre transformations. The transform to the left is always done by adding PV. Going down in first step one adds $-ST$ and in second $-\mu N$.

Backward compatibility of thermal systems Grand potential can be related to other thermodynamical potentials and the internal energy $U(V,S,N)$ via a series of Legendre transformations (see table 1). The internal energy U is relevant for the isolated systems or a microcanonical ensemble[1], and it obeys the Gibbs relation: $\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N$. Canonical ensemble is relevant for systems which are thermally connected to their surroundings, but are closed *wrt* particle exchanges. Such systems are described by Helmholtz free energy $F = U - TS$ (see table 1. Finally, if system is both in thermal contact and open for particle exchanges, it is described by a grand canonical ensemble and the relevant thermodynamical potential is the grand potential.

In a grand canonical system energy and particle number fluctuate and are replaced by expectation values. For example:

$$
U \to \langle E \rangle = \frac{1}{\mathcal{Z}} \mathrm{Tr}[\hat{H} e^{-\beta(\hat{H}-\mu\hat{N})}]. \tag{1.9}
$$

Now use $\mathrm{d}\Omega = -S\mathrm{d}T - P\mathrm{d}V - N\mathrm{d}\mu$ so that $S = -(\partial\Omega/\partial T)_{V,\mu}$. From and the fundamental relation $\Omega = -T\log\mathcal{Z}$ (here $\mu = 0$ so that $\Omega = F$), so that we get

$$
S = -\frac{\partial\Omega}{\partial T} = \log\mathcal{Z} + \frac{T}{\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial T} = -\frac{\Omega}{T} + \frac{\langle E\rangle - \mu\langle N\rangle}{T} \quad \Rightarrow \quad \Omega = \langle E\rangle - TS - \mu\langle N\rangle. \tag{1.10}
$$

The fixed internal energy U has been replaced by $\langle E\rangle$ and fixed particle number N by $\langle N\rangle$.

Gibbs enropy Let us note that if we use normalized density operator $\tilde{\rho} \equiv \hat{\rho}/\mathrm{Tr}[\hat{\rho}]$, the Gibbs entropy is also consistent with our system:

$$
\begin{aligned}
S &\equiv -\mathrm{Tr}[\frac{\hat{\rho}}{\mathcal{Z}}\log\frac{\hat{\rho}}{\mathcal{Z}}] = -\frac{1}{\mathcal{Z}}\mathrm{Tr}[\hat{\rho}(\log\hat{\rho} - \log\mathcal{Z})] \\
&= \frac{1}{T\mathcal{Z}}\mathrm{Tr}[\hat{H}e^{-\beta(\hat{H}-\mu\hat{N})}] - \log\mathcal{Z} = \frac{1}{T}(\langle E\rangle - \mu\langle N\rangle - F).
\end{aligned} \tag{1.11}
$$

We clearly reproduce the standard thermodynamical relation with Gibbs entropy.

[1] Actually, it would be more consistent to identify entropy as the thermodynamical potential for microcanonical system instead of the internal energy. Indeed, in microcanonical system entropy is related to number of microscopic states W similarly that Helmoholz free energy and grand potential are related to partition functions: $S = \log W$ and $-\beta F = \log\mathcal{Z}_c$ or $-\beta\Omega = \log\mathcal{Z}_{gc}$.

1.1 Harmonic oscillator in thermal bath

Let us start by studying a simple harmonic oscillator (SHO) in a heat bath. This is useful, because the leading quantum statistcial properties of more complicated systems can be computed summing the contribution from an infinite number of SHOs arising from second quantization of the field. The Hamiltonian operator of a simple harmonic oscillator is

$$\hat{H}_{\rm SHO} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 \tag{1.12}$$

where the momentum operator $\hat{p}$ and the position operator $\hat{q}$ obey the commuation relation

$$[\hat{p},\hat{q}] = i \quad \text{and} \quad [\hat{p},\hat{p}] = [\hat{q},\hat{q}] = 0. \tag{1.13}$$

(we are using $\hbar \equiv 1$). We can express $\hat{p}$ and $\hat{q}$ in terms of the raising- and lowering operators $\hat{a}^\dagger$ and $\hat{a}$:

$$\hat{q} \equiv \quad \frac{1}{\sqrt{2m\omega}}\Big(\hat{a} + \hat{a}^\dagger\Big), \qquad \hat{p} \equiv -i\sqrt{\frac{m\omega}{2}}\Big(\hat{a} - \hat{a}^\dagger\Big) \tag{1.14}$$

As a result of (1.13) $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commuation relations

$$[\hat{a},\hat{a}^\dagger] = 1 \quad \text{and} \quad [\hat{a},\hat{a}] = [\hat{a}^\dagger,\hat{a}^\dagger] = 0. \tag{1.15}$$

Because these are *commutation* relations, we understand that we are studying a *bosonic* system here. Using the raising and lowering operators we can write the SHO-Hamiltonian in the form:

$$\hat{H}_{\rm SHO} = \omega^2\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big) = \omega^2\Big(\hat{N} + \frac{1}{2}\Big), \tag{1.16}$$

where we also introduced the *number operator* $\hat{N} \equiv a^\dagger\hat{a}$.

Now consider a system of SHO:s in a thermal bath of temperature T. Let us also imagine that the number of oscillators is not fixed, so that we are studying a grand canonical ensemble of SHO:s. The partition function for this system can now be easily computed using a complete set of eigenfunctions $|n\rangle$ of the number operator (or the Hamiltonian) which satisfy: $\hat{N}|n\rangle = n|n\rangle$. One finds:

$$\begin{aligned} Z_{\rm SHO} &= {\rm Tr}[e^{-\beta(\hat{H}_{\rm SHO}-\mu\hat{N}_{\rm SHO})}] = {\rm Tr}[e^{-\beta(\omega-\mu)\hat{N}_{\rm SHO}-\frac{1}{2}\beta\omega}] && (1.17)\\ &= e^{-\frac{1}{2}\beta\omega}\sum_{n=0}^{\infty}\langle n|e^{-\beta(\omega-\mu)\hat{N}_{\rm SHO}}|\, n\rangle = e^{-\frac{1}{2}\beta\omega}\sum_{n=0}^{\infty}e^{-\beta(\omega-\mu)n} \\ &= \frac{e^{-\frac{1}{2}\beta\omega}}{1-e^{-\beta(\omega-\mu)}} = \frac{e^{\frac{1}{2}\beta\mu}}{2\sinh(\frac{1}{2}\beta(\omega-\mu))}. && (1.18) \end{aligned}$$

To be precise, the properly normalized bosonic number-operator eigenstate is a Fock state[2]: $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$, where $|0\rangle$ is the vacuum state of the system. We can now connect

[2] A Fock state is any state that can be created from the vacuum using a finite number of creation operators.

the particle number, defined as the expectation value of the particle number operator $\hat{N}$, with the chemical potential:

$$N_{\text{SHO}}(\mu) \equiv \langle \hat{N}_{\text{SHO}} \rangle = \frac{\text{Tr}[\hat{\rho}_{\text{SHO}} \hat{N}_{\text{SHO}}]}{\text{Tr}[\hat{\rho}_{\text{SHO}}]} = T \frac{\partial \log Z_{\text{SHO}}}{\partial \mu} = \frac{1}{e^{\beta(\omega-\mu)} - 1}. \tag{1.19}$$

We can understand $N_{\text{SHO}}(\mu)$ as the occupation number in the state defined by the energy ω and chemical potential μ. It clearly satisfies $N(\mu) \in [0, \infty[$. Similarly the expected energy of the simple harmonic oscillator is:

$$E_{\text{SHO}}(\mu) \equiv \langle \hat{H}_{\text{SHO}} \rangle = \frac{\text{Tr}[\hat{\rho}_{\text{SHO}} \hat{H}_{\text{SHO}}]}{\text{Tr}[\hat{\rho}_{\text{SHO}}]} = \omega \Big(N_{\text{SHO}}(\mu) + \frac{1}{2} \Big). \tag{1.20}$$

1.1.1 Fermions

We can extend the original bosonic system to fermionic SHO, by introducing raising and lowering operators obeying anticommutation rules.

$$\{\hat{\alpha}, \hat{\alpha}^\dagger\} = 1 \quad \text{and} \quad \{\hat{a}, \hat{a}\} = \{\hat{a}^\dagger, \hat{a}^\dagger\} = 0. \tag{1.21}$$

The essential difference introducd by the anticommutation rules is that a fermionic SHO has only two different states: the vacuum $|0\rangle$ and the occupied state $|1\rangle$, such that:

$$\hat{\alpha}^\dagger |0\rangle = |1\rangle \quad \text{and} \quad \hat{\alpha}|1\rangle = |0\rangle. \tag{1.22}$$

We might call an object obeying these rules a simple fermionic oscillator (SFO). The appropriate Hamiltonian for this system is:

$$\hat{H}_{\text{SFO}} \equiv \omega^2 \Big(\hat{\alpha}^\dagger \hat{\alpha} - \frac{1}{2} \Big) = \omega^2 \Big(\hat{N}_{\text{SFO}} - \frac{1}{2} \Big), \tag{1.23}$$

This form becomes clear below when we quantize the fermionic quantum field, but for now we can take this as a definition for the fermionic non-interacting quantum system. The partition function for a thermal bath of these objects is

$$\begin{aligned} Z_{\text{SFO}} &= \text{Tr}[e^{-\beta(\hat{H}_{\text{SFO}} - \mu \hat{N}_{\text{SFO}})}] = e^{\frac{1}{2}\beta\omega} \sum_{n=0}^{1} \langle n | e^{-\beta(\omega-\mu)\hat{N}_{\text{SFO}}} | \, n \rangle \\ &= e^{\frac{1}{2}\beta\omega}(1 + e^{-\beta(\omega-\mu)n}) = 2e^{\frac{1}{2}\beta\mu} \cosh(\tfrac{1}{2}\beta(\omega - \mu)). \end{aligned} \tag{1.24}$$

Moreover,

$$N_{\text{SFO}}(\mu) \equiv \langle \hat{N}_{\text{SFO}} \rangle = \frac{\text{Tr}[\hat{\rho}_{\text{SFO}} \hat{N}_{\text{SFO}}]}{\text{Tr}[\hat{\rho}_{\text{SFO}}]} = T \frac{\partial \log Z_{\text{SFO}}}{\partial \mu} = \frac{1}{e^{\beta(\omega-\mu)} + 1}. \tag{1.25}$$

We can again understand $N_{\text{SFO}}(\mu)$ as the occupation number in the state defined by the energy ω and chemical potential μ. A striking difference to the bosonic system is that this time $N_{\text{SFO}}(\mu) \in [0, 1]$. Similarly the expected energy in the SFO heat bath is

$$E_{\text{SFO}}(\mu) \equiv \langle \hat{H}_{\text{SFO}} \rangle = \frac{\text{Tr}[\hat{\rho}_{\text{SFO}} \hat{H}_{\text{SFO}}]}{\text{Tr}[\hat{\rho}]_{\text{SFO}}} = \omega \Big(N_{\text{SFO}}(\mu) - \frac{1}{2} \Big). \tag{1.26}$$

1.2 Interacting particles in a box

In the above example the simple oscillators had only a single energy state. In reality particles come with a spectrum of states coming with some dispersion relation. We can see how this structure emerges by quantizing the field within a box. Consider a field ψ constrained in a 3D-box where each side has a length L. Boundary condition (illustrated in figure 15 implies quantization

$$\psi(x_i = 0) = \psi(x_i = L) = 0, \quad \Rightarrow \quad L = \frac{n_i \lambda_i}{2} \tag{1.27}$$

where $n_i \in \mathbb{N}$. This implies a momentum discretation

$$|p_i| = \frac{2\pi}{\lambda_i} = \frac{\pi}{L} n_i, \tag{1.28}$$

Each mode is equivalent with a SHO or SFO with energy given by $\omega = \omega(\boldsymbol{p})$, so we can construct the Hamiltonian and number operators as simple sums

$$\hat{H} = \sum_i \hat{H}_i \tag{1.29}$$

$$\hat{N} = \sum_i \hat{N}_i \tag{1.30}$$

where i runs over all lattice sites (represents 3 different sums) along different axis. The partition function then turns out to be a product of SHO and SFO partition functions:

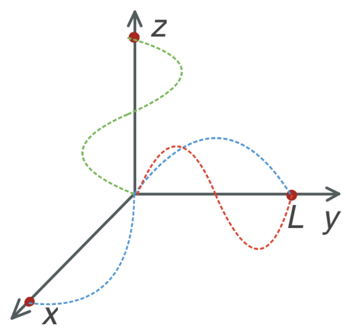

Figure 1: Particle wave solutions in a box with sides of length L.

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta \sum_i (\hat{H}_i - \mu \hat{N}_i)}] = \prod_i Z_i \tag{1.31}$$

We can now compute the pressure in the system from the grand potential

$$P = -\frac{\Omega}{V} = \frac{T}{V} \log \mathcal{Z}, \tag{1.32}$$

where

$$\begin{aligned} T \log \mathcal{Z} = T \sum_i \log Z_i \quad &\overset{L\to\infty}{\longrightarrow} \quad \left(\frac{L}{\pi}\right)^3 T \iiint_0^\infty \Big(\prod_i \mathrm{d}|p_i|\Big) \log Z_{\mathbf{p}} \\ = VT \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \log Z_{\boldsymbol{p}} \end{aligned} \tag{1.33}$$

Then, using (1.18) and (1.24) mode by mode, we get:

$$P = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \Big(\mp \frac{\omega_{\boldsymbol{p}}}{2} \mp T \log \big(1 \mp e^{-\beta(\omega_{\boldsymbol{p}} - \mu)}\big)\Big). \tag{1.34}$$

Here on the last line we combined results from (1.18) and (1.24), such that with upper signs (1.34) returns the bosonic and with lower signs the fermionic partition function for a free quantum gas in a box. For massive excitations one can simply take $\omega = \sqrt{\boldsymbol{p}^2 + m^2}$. From the partition function we can compute

$$n \equiv \frac{\langle \hat{N} \rangle}{V} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{e^{\beta(\omega_{\boldsymbol{p}}-\mu)} \pm 1} \tag{1.35}$$

$$E \equiv \frac{\langle \hat{H} \rangle}{V} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \Big(\pm \frac{\omega_{\boldsymbol{p}}}{2} + \frac{\omega_{\boldsymbol{p}}}{e^{\beta(\omega_{\boldsymbol{p}}-\mu)} \mp 1} \Big) = \pm E_0 + E_T^{\mp}, \tag{1.36}$$

where upper signs again refer to bosons and lower signs to fermions.

Antiparticles In the above considerations we only accounted for particle solutions. In field theory approach we will naturally also get the antiparticle solutions. Here we can put them in by hand using Dirac's hole theory interpretation of the antistates. If we describe a state with antiparticle by $|\bar{n}\rangle$ and intepret this state as a positive energy state ($\omega > 0$) of a missing particle $n < 0$, then:

$$\langle \bar{n}|\hat{\rho}|\bar{n}\rangle = \langle \bar{n}|e^{\beta(\hat{H}-\mu\hat{N})}|\bar{n}\rangle \equiv e^{-\beta(\omega-\mu(-1))n \mp \frac{1}{2}\beta\omega} = e^{\mp\frac{1}{2}\beta\omega} e^{-\beta(\omega+\mu)n}. \tag{1.37}$$

Then the total *thermal* energy of a system with both particles and antiparticles (we drop the vacuum energy parts here) then is

$$E_T^{\pm} = V \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(\frac{\omega_{\boldsymbol{p}}}{e^{\beta(\omega_{\boldsymbol{p}}-\mu)} \pm 1} + \frac{\omega_{\boldsymbol{p}}}{e^{\beta(\omega_{\boldsymbol{p}}+\mu)} \pm 1} \right). \tag{1.38}$$

If our particles have a charge $-e$, then antiparticles have a charge e and the total charge in the system is

$$Q_{\pm} = -eV \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(\frac{1}{e^{\beta(\omega_{\boldsymbol{p}}-\mu)} \pm 1} - \frac{1}{e^{\beta(\omega_{\boldsymbol{p}}+\mu)} \pm 1} \right). \tag{1.39}$$

1.3 Thermal integral J_T.

The expression (1.34) contains an important thermal integral function, which will turn up continuously in these lectures. Combining (1.32) and (1.34) we can write the pressures of the free bosonic and fermionic gases as[3]

$$P_{\pm} = \mp J_0 + J_T^{\pm}, \tag{1.40}$$

where upper signs refer to fermions and lower signs to bosons. The divergent *vacuum pressure* term $J_0(m)$ can be evaluated using the dimensional regularization:

$$J_0(m) = \rightarrow \frac{\mu^\epsilon}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2 + m^2)^{-1/2}} = \mu^\epsilon \Phi(m, 3-\epsilon, -\tfrac{1}{2}), \tag{1.41}$$

where the function $\Phi(m, d, \alpha)$ is defined in the appendix D. The thermal integrals $J_T^{\mp}$ then are

$$J_T^{\pm}(m,T) = \pm \frac{T^4}{2\pi^2} \int_0^\infty \mathrm{d}y\, y^2 \log\left(1 \pm e^{-\sqrt{y^2+x^2}}\right), \tag{1.42}$$

[3] We consider only the case $\mu = 0$ here, since this is the form in which we will encounter these integrals in typical cosmology and particle physics applications.

where $x \equiv m/T$. The vacuum can often be ignored be, since it does not affect thermal properties of the system. Sometimes it is relevant however, (for example the effective potential) and in this case it needs to be renormalized. The thermal part on the other hand is finite and we can rewrite it using a partial integration to give a more familiar form for the thermal pressure:

$$P_T^{\pm} = J_T^{\pm} = \frac{T^4}{2\pi^2}\int_0^\infty \mathrm{d}y\, y^2 \frac{y^2}{3\sqrt{y^2+x^2}}\frac{1}{e^{\sqrt{y^2+x^2}} \pm 1}. \tag{1.43}$$

We list several mathematical properties of integrals J_0 and $J_T^{\mp}$ in the appendix D.

1.4 Path integral for SHO

We return to study the bosonic oscillator, but compute its partition function using path integral methods. To this order we note that the *transition amplitude*

$$K(q',t';q,t) \equiv \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle, \tag{1.44}$$

where

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{1.45}$$

can be expressed as a path integral

$$K(q',t';q,t) \equiv \int [\mathcal{D}q]_{\substack{q(t_i)=q \\ q(t_i)=q'}} e^{iS[q]}, \tag{1.46}$$

where the classical action is given by

$$S_M[q] = \int_{t_i}^{t_f} \mathrm{d}t\Big(\frac{1}{2}m\dot{q}^2 - V(q)\Big), \tag{1.47}$$

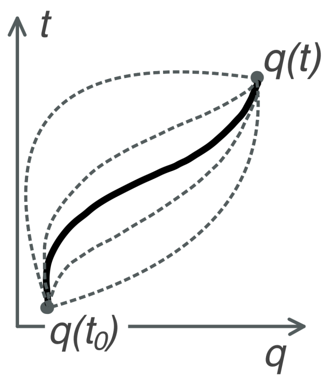

Figure 2: Examples of possible paths that quantum system can take (dashed). The thick solid line is the classical path.

where $q(t)$ and $\dot{q} = \partial q/\partial t$ now are ordinary c-numbers. The index M refers to the fact that the time t is the usual Minkowski space time. In fact, for the path integral to be well defined, the time has to be deformed to move along a complex contour which tilts down as the real time increases. This tilt is connected to the Feynman prescription which ensures that the ordered propagator has the correct boundary conditon.

However, here we are (currently) not interested in time-dependent quantities. Instead, we want to compute *expectation values* of the known thermal equilibrium density operator $\hat{\rho} = e^{-\beta\hat{H}}$. To this end first note that a transition matrix element (1.46) over imaginay time interval can be written as a path integral over the Euclidean time

$$K(q',-i\tau';q,-i\tau) = \langle q'|e^{-\hat{H}(\tau'-\tau)}|q\ \rangle = \int [\mathcal{D}q]_{\substack{q(\tau_i)=q \\ q(\tau_f)=q'}} e^{-S_E[q]}. \tag{1.48}$$

where the Euclidean action is

$$S_E[q] = \int_{\tau_i}^{\tau_f} \mathrm{d}t\Big(\frac{1}{2}m\dot{q}^2 + V(q)\Big). \tag{1.49}$$

In this expression $\dot{q} \equiv \partial q/\partial\tau$. So far we have formally computed the partition function over a complete set of eigenstates of the number operator:

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta\hat{H}}] = \sum_n e^{-\beta E_n}, \tag{1.50}$$

but we can equally well perform the trace over the eigenstates of the position operator $\hat{q}$:

$$\mathcal{Z} = \int \mathrm{d}q \langle q|e^{-\beta\hat{H}}|q\rangle = \int \mathrm{d}q K(q,-i\beta;q,0)$$
$$= \int [\mathcal{D}q]_{q(t_i)=q(t_f)=q}\, e^{-S_E[q]}\,, \tag{1.51}$$

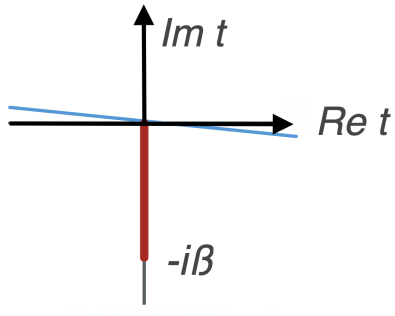

Figure 3: Complex time paths. Blue: the tilted real-time path. Red: the imaginary time contour.

where on first equality we used (1.48) with $\tau = 0$ and $\tau' = \beta$. That is, because we are computing a statistical *expectation value* over the thermal density operator, the result emerges as a path integral weighted by Euclidean action over configurations that are periodic in imaginary time with period $\beta = 1/T$. More detailed calculation connecting path integral to transition amplitude is given in appendix B.

1.5 Thermal generating functional and propagator

We now generalize the partition function into a generating function by introducing a source $j(q)$:

$$\boxed{\mathcal{Z}(\beta,j) = \int [\mathcal{D}q]_\beta\, e^{-S_E[q]+\int_0^\beta j(\tau)q(\tau)\mathrm{d}\tau}.} \tag{1.52}$$

where the index β is a shorthand reminder that the integral is taken over configurations that are periodic in τ with a period β. We can now compute the complex time-ordered propagator for our theory[4]

$$\begin{aligned}\frac{1}{\mathcal{Z}}\frac{\delta\mathcal{Z}}{\delta j(\tau_1)\delta j(\tau_2)} &= \frac{1}{\mathcal{Z}}\int [\mathcal{D}q]_\beta\, q(\tau_1)q(\tau_2)\, e^{-S_E[q]} \\ &= \mathrm{Tr}\big[e^{-\beta\hat{H}}\mathcal{T}\big(\hat{q}(\tau_1)\hat{q}(\tau_2)\big)\big] \\ &= \langle\mathcal{T}\big(\hat{q}(\tau_1)\hat{q}(\tau_2)\big)\rangle_\beta,\end{aligned} \tag{1.53}$$

where $\tau_1, \tau_1 \in [0,\beta]$ and $\mathcal{T}$ refers to time ordering in τ. Because of the periodicity it suffices to define only the propagator

$$\Delta(\tau) = \langle\mathcal{T}\big(\hat{q}(\tau)\hat{q}(0)\big)\rangle_\beta, \tag{1.54}$$

[4]To keep the notation simple, we use label our states and operators with a real valued time argument: $\hat{q}(\tau)|q\rangle \equiv q(\tau)|q\rangle$. This is consistent when we also assume a real frequency variable $p_0 = \omega_n$, since indeed $-i\tau(ip_0) = \tau p_0$. The imaginary time concept was used merely to provide a connection to the presumably more familiar real time path integral for the transition amplitude. In fact our representation of the statistical expectation value as a path integral is better defined than the one for the transition amplitude; it has a well defined (Wiener) measure from outset and it convergers to unique answer, whereas the path integral for transition amplitude indeed requires an extension to a complex time to be well defined.

Indeed, using $\hat{q}(\tau) = e^{\hat{H}\tau}\hat{q}(0)e^{-\hat{H}\tau}$ one can show that for any $\tau_1, \tau_1 \in [0, \beta]$

$$\langle \mathcal{T}\big(\hat{q}(\tau_1)\hat{q}(\tau_2)\big)\rangle_\beta = \langle \mathcal{T}\big(\hat{q}(\tau_1 - \tau_2)\hat{q}(0)\big)\rangle_\beta. \tag{1.55}$$

Moreover periodicity implies that

$$\begin{aligned}\Delta(\tau) &= \langle \mathcal{T}\big(\hat{q}(\tau)\hat{q}(0)\big)\rangle_\beta = \langle \mathcal{T}\big(\hat{q}(\tau)\hat{q}(\beta)\big)\rangle_\beta \\ &= \langle \mathcal{T}\big(\hat{q}(\tau - \beta)\hat{q}(0)\big)\rangle_\beta \\ &= \Delta(\tau - \beta).\end{aligned} \tag{1.56}$$

This can obviously be extended to points outside the periodicity interval $\Delta(\tau) = \Delta(\tau + n\beta)$ for any $n \in \mathbb{Z}$. The periodicity property is called the Kubo-Martin-Schwinger (KMS) condition.

Propagator for a simple harmonic oscillator Let us now assume the harmonic potential that we studied above with usual quantum statistical methods:

$$V_0(q) = \frac{1}{2}m\omega^2 q^2. \tag{1.57}$$

We now absorb the mass m into q redefining $mq^2 \to q^2$ (that is we are measuring q in units of mass). The generating function is a Gaussian functional integral that can in this case be computed in a closed form:

$$\begin{aligned}\mathcal{Z}[\beta, j] &= \int [\mathcal{D}q]_\beta \, e^{-\int_0^\beta \mathrm{d}\tau [\frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 - jq]} \\ &= \int [\mathcal{D}q]_\beta \, e^{-\int_0^\beta \mathrm{d}\tau [\frac{1}{2}q(-\partial_\tau^2 + \omega^2)q - jq]}.\end{aligned} \tag{1.58}$$

Here we left out the surface term in the action $\sim \int_0 \beta \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau} q\dot{q} = |_0^\beta q\dot{q}$, which in fact does not vanish based on general arguments: all we can say without a detailed calculation is that $q(0) = q(\beta)$, but we cannot argue that $\dot{q}(0) = \dot{q}(\beta)$. However, performing the calculation exactly shows that dropping this term is still legitimate (excercise 4).

Now denote by $\Delta_0(\tau, \tau')$ the Greens function which is the inverse of the differential operator appearing in the integral in (1.58), *e.g.* is a solution to the equation

$$\big(-\partial_\tau^2 + \omega^2\big)\Delta_0(\tau, \tau') = \delta(\tau - \tau'). \tag{1.59}$$

Given this Greens function, we can easily perform the Gaussian integral in (1.58) to get

$$\mathcal{Z}[\beta, j] = \mathcal{Z}(\beta) e^{\frac{1}{2}\int_0^\beta \mathrm{d}\tau \mathrm{d}\tau' j(\tau)\Delta_0(\tau,\tau')j(\tau')}, \tag{1.60}$$

where $\mathcal{Z}(\beta)$ is the partition function, *e.g.* the generating function with zero external sources. Using the solution (1.60) in (1.53) we can see that $\Delta_0(\tau, \tau') = \Delta_0(\tau - \tau')$ is the propagator for the simple harmonic oscillator. We can work out the explicit form of the propagator going to the Fourier space:

$$\Delta_0(\tau) = T\sum_n \Delta_0(\omega_n) e^{-i\omega_n \tau}. \tag{1.61}$$

$$\Delta_0(\omega_n) = \int \mathrm{d}\tau \Delta_0(\tau) e^{i\omega_n \tau} \tag{1.62}$$

Using the periodicity condition (1.56) on equation (1.61) imposes the condition $e^{-i\beta\omega_n} = 1$, which implies:

$$\omega_n = 2\pi nT. \tag{1.63}$$

These energy levels are called bosonic *Matsubara frequences*. Inserting (1.61) back into (1.59), we easily find the propagator in the Fourier space:

$$\boxed{\Delta_0(\omega_n, \omega) = \frac{1}{\omega_n^2 + \omega^2}}. \tag{1.64}$$

1.6 Matsubara sums and the partition function

Finding the propagator $\Delta_0(\tau, \omega)$ is a good excercise in performing sums over Matsubara frequencies. The common trick is to express the Matsubara sum as a complex integral with a suitable function such that the original Matsubara sum emerges as a sum over the residues encompassed by the contour.

We shall use the contour shown in figure 4, as follows:

$$\begin{aligned}\Delta_0(\tau, \omega) &= T \sum_{n=-\infty}^{\infty} \Delta_0(\omega_n, \omega) e^{-i\omega_n \tau} \\ &= T \sum_{n=-\infty}^{\infty} \frac{1}{\omega^2 + \omega_n^2} e^{-i\omega_n \tau} \\ &= \frac{T}{2\pi i} \oint_{\mathcal{C}_1} \frac{e^{-\tau z}}{\omega^2 - z^2} \frac{\beta e^{\beta z}}{e^{\beta z} - 1} \end{aligned} \tag{1.65}$$

At this point one simply reinterprets the contous $\mathcal{C}_1$ as encompassing the poles of the complex function $\Delta_0(z, \omega)$, which allows us to compute it as a sum of its two residues only. In this interpretation the contour $\mathcal{C}_1$ runs *clockwise* so there is an additional minus sign. We find:

$$\Delta_0(\tau, \omega) = \frac{1}{2\omega} \left(\frac{e^{\omega(\beta-\tau)}}{e^{\beta\omega} - 1} + \frac{e^{-\omega(\beta-\tau)}}{e^{-\beta\omega} - 1} \right) \tag{1.66}$$

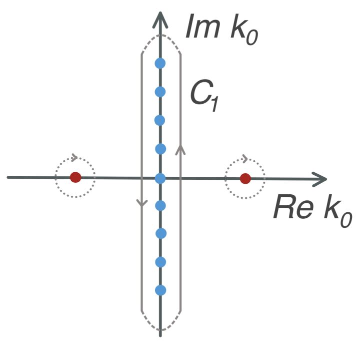

Figure 4: Complex integration contour $\mathcal{C}_1$ that encompasses the Matsubara frequencies as positive sign residues or the poles at $\pm\omega$ as negative sign residues.

where $\tau \in [0, \beta]$. Defining the Bose-Einstein distribution function (this is just our old N_{SHO} with zero chemical potential):

$$n_{\text{BE}}(\omega) = \frac{1}{e^{\beta\omega} - 1}, \tag{1.67}$$

and finally noting that clearly $\Delta_0(\tau, \omega) = \Delta_0(-\tau, \omega)$, we can write (1.66) as

$$\Delta_0(\tau, \omega) = \frac{1}{2\omega} \Big((1 + n_{\text{BE}}(\omega)) e^{-\omega|\tau|} + n_{\text{BE}}(\omega) e^{\omega|\tau|} \Big). \tag{1.68}$$

One can also easily check that (1.68) satisfies the KMS-condition (1.56). This result can also be obtained directly from (1.59) (see excercise **1.4**). Note that the absolute value of τ in exponentials in (1.68) give rise to the delta-function in (1.59).

Partition function Our second example is the partition function $\mathcal{Z}(\beta)$. We first perform the calcluation in the usual field-theory fashion leaving out the overall constants, but still paying attention to not throwing out any T-dependent terms:

$$\mathcal{Z}(\beta) = \int [\mathcal{D}q]_\beta e^{-\int_0^\beta \mathrm{d}\tau \frac{1}{2} q(\tau)\Delta_0^{-1}(\tau,\omega)q(\tau)} = C(\det \tilde{\Delta}_0)^{1/2} = Ce^{\frac{1}{2}\mathrm{Tr}[\log \tilde{\Delta}_0]}, \tag{1.69}$$

where we introduced scalings $\tau \to \beta\tau$ and $q \to \beta^{1/2} q$ to get the dimensionless propagator $\tilde{\Delta}_0 \equiv \beta^{-2}\Delta_0$. Note also that the path-integral measure does not change in this scaling, since $k_N \propto \beta^{-1/2}$ (see appendix B). Taking logarithm and dropping the constant terms we get

$$\begin{aligned}\log \mathcal{Z}(\beta) &= \frac{1}{2}\mathrm{Tr}[\log \tilde{\Delta}_0] = -\frac{1}{2}\sum_n \log\left[\beta^2(\omega^2 + \omega_n^2)\right] \\ &= -\int_1^{\beta\omega} \mathrm{d}\theta \sum_n \frac{\theta}{\theta^2 + (2\pi n)^2} - \frac{1}{2}\sum_n \log(1 + (2\pi n)^2).\end{aligned} \tag{1.70}$$

The last term is again a T-independent constant which we must drop in this calculation. We then get

$$\begin{aligned}\log \mathcal{Z}(\beta) &= -\int_1^{\omega} \mathrm{d}\omega' \sum_n \frac{\omega'}{\omega'^2 + \omega_n^2} = -\int_0^{\omega} \mathrm{d}\omega' \omega' \beta\Delta_0(\tau = 0, \omega') \\ &= -\frac{1}{2}\int_0^{\beta\omega} \mathrm{d}\theta \big(1 + 2n_{\mathrm{BE}}(\theta)\big) = -\frac{\beta\omega}{2} - \log\left(1 - e^{-\beta\omega}\right).\end{aligned} \tag{1.71}$$

To get to second line we used the middle line of equation (1.65) and to get to third line we used equation (1.68). This result agrees with (1.18), suggesting that the constants we dropped should actually vanish. Because $-T\log\mathcal{Z}$ can be related to a physical quantity, the pressure (up to the vacuum contribution, which need to be removed otherwise), one would indeed expect that (1.71) emerges exactly in a more rigorous calcluation. We shall now show that this is indeed so.

Proof that (1.71) is exact including constant terms A brute force proof, where one evaluates the path integral more carefully will be done in excercise: **1.6**. However, Laine and Vuorinen [1] introduce a nice trick to get the same result studying the $\omega \to 0$ limit, whch we reproduce here. We start by writing again the Fourier-decomposition of $q(\tau)$ separating out the zero mode:

$$q(\tau) = \frac{\hat{q}_0}{\beta} + \frac{1}{\beta}\sum_{n\neq 0} \hat{q}_n e^{-i\omega_n\tau}. \tag{1.72}$$

Using $(\partial_\tau^2 + \omega^2)q(\tau) = \omega^2\hat{q}_0 + \sum_{n\neq 0}(\omega_n^2 + \omega^2)\hat{q}_n e^{-i\omega_n\tau}$ we can write the Euclidean action in the Fourier represenatation as follows:

$$S_E = \frac{1}{2\beta}\omega^2\hat{q}_0^2 + \frac{1}{2\beta}\sum_{n\neq 0}(\omega_n^2 + \omega^2)\hat{q}_n^2. \tag{1.73}$$

Performing the gaussian integrals over $\hat{q}_n$'s one then finds

$$\mathcal{Z}(\beta) = C(\beta)\frac{\sqrt{2\pi\beta}}{\omega}\prod_{n=1}^{\infty}\frac{1}{\omega_n^2+\omega^2}, \tag{1.74}$$

were the contribution $\sqrt{2\pi\beta}/\omega$ came from the zero mode. For other modes, we displayed only the ω-dependent terms and combined all other factors into the ω-independent constant $C(\beta)$. The idea is now to determine $C(\beta)$ in the limit $\omega \to 0$. The zero-mode contribution is problematic however, because it blows up when $\omega = 0$ due to the infinite integration range in $\hat{q}_0$. However, one observes that $\int_0^\beta \mathrm{d}\tau q(\tau) = \hat{q}_0$, so that $\hat{q}^0/\beta$ is the average value of $q(\tau)$. Zero mode is then regulated in systems where the average variation of $q(\tau)$ is restricted to within some range Δq. In such regulated system zero mode contributes a factor $\beta\Delta q$ to partition function, and we find at $\omega = 0$:

$$\mathcal{Z}_{\mathrm{reg}}(\beta,\omega=0) = C(\beta)\beta\Delta q\prod_{n=1}^{\infty}\frac{1}{\omega_n^2}. \tag{1.75}$$

This partitition function can be computed also directly however (we still measure everything in units m):

$$\begin{aligned}\mathcal{Z}_{\mathrm{reg}}(\beta,\omega=0) &= \int_{\Delta q}\mathrm{d}q\langle q|e^{-\frac{1}{2}\beta\dot{q}^2}|q\rangle = \int_{\Delta q}\mathrm{d}q\langle q|e^{-\frac{1}{2}\beta p^2}|q\rangle \\ &= \int_{\Delta q}\mathrm{d}q\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\langle q|e^{-\frac{1}{2}\beta\hat{p}^2}|p\rangle\langle p|q\rangle \\ &= \int_{\Delta q}\mathrm{d}q\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}e^{-\frac{1}{2}\beta p^2}|\langle p|q\rangle|^2 = \frac{\Delta q}{\sqrt{2\pi\beta}},\end{aligned} \tag{1.76}$$

where we used $|\langle p|q\rangle|^2 = 1$ before performing the gaussian integral over p. Equations (1.75) and (1.76) have the same dependence on arbitrary interval Δq, which allows solving $C(\beta)$ independently from the regulator:

$$C(\beta) = \frac{1}{\sqrt{2\pi\beta^3}}\prod_{n=1}^{\infty}\omega_n^2 \tag{1.77}$$

Inserting this result back to (1.78) now gives

$$\begin{aligned}\mathcal{Z}(\beta) &= \frac{1}{\beta\omega}\prod_{n=1}^{\infty}\frac{\omega_n^2}{\omega_n^2+\omega^2} \\ &= \frac{1}{\beta\omega}\frac{1}{\prod_{n=1}^{\infty}\left[1+(\frac{\beta\omega}{2\pi n})^2\right]} = \frac{1}{2\sinh\frac{1}{2}\beta\omega}.\end{aligned} \tag{1.78}$$

where at last one used the identity $\frac{\sinh\pi x}{\pi x} = \prod_{n=1}^{\infty}(1+\frac{x^2}{n^2})$. This is a very important result, since all bosonic free theory partition functions return to this same form. In what follows we can then write down the partition function after reducing $\log\mathcal{Z}(\beta)$ into the form on the first line of (1.70).

Excercises to section 1

1.1 Show that (1.35) and (1.36) also follow from the thermodynamical relations

$$n \equiv -\frac{1}{V}\Big(\frac{\partial \Omega}{\partial \mu}\Big)_{V,T}$$
$$E \equiv \Omega + ST + \mu N = \Omega - T\Big(\frac{\partial \Omega}{\partial T}\Big)_{V,\mu} - \mu\Big(\frac{\partial \Omega}{\partial \mu}\Big)_{V,T}.$$

1.2 Consider a plasma with 4-velocity u^μ and energy momentum tensor $T^{\mu\nu}$. We can then write a relativistic generalization of the Gibbs relation as follows:

$$\mathrm{d}s^\mu = \beta u_\nu \mathrm{d}T^{\nu\mu} - \sum_a \xi_a \mathrm{d}j_a^\mu, \tag{1.79}$$

where $\beta \equiv 1/T$ and s^μ is the entropy flux, j_a^μ a set of conserved currents and $\xi_a \equiv \mu_a/T$ where μ_a are chemical potentials. For a perfect fluid there is only one 4-vector available, u^μ, so that $s^\mu = su^\mu$ and $j_a^\mu = n_a u^\mu$ and $T^{\mu\nu} = (\rho + P)u^\mu u^\nu - p\eta^{\mu\nu}$, where $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric. Using these definintion show that (1.79) gives *both* the differential Gibbs relation: $\mathrm{d}s = \beta \mathrm{d}\rho - \sum_a \xi_a \mathrm{d}n_a$ and thermal potential relation $s = \beta(\rho + P) - \sum \xi_a n_a$.

1.3 Show that the generating funcction $Z(\beta, j)$ for the simple harmonic oscillator can be expressed in the form (1.60) (peform the Gaussian integral using periodicity). Show also that $\mathcal{Z}(\beta, j)$ can be expressed in the form

$$\mathcal{Z}(\beta, j) = \mathrm{Tr}\big[e^{-\beta \hat{H}}\mathcal{T}\big(e^{\int_0^\beta \mathrm{d}\tau j(\tau)\hat{q}(\tau)}\big)\big].$$

1.4 Show by direct evaluation in the τ-representation, that the solution to equation

$$(-\partial_\tau^2 + \omega^2)\Delta_0(\tau) = \delta(\tau)$$

is the propagator (1.68). Start by making exponential ansaz and then require that result obeys the KMS-condition and you get the correctly normalized $\delta(\tau)$-distribution from the derivative terms.

1.5 Using canonical commuation rules and the expression for the Hamiltonian: $\hat{H} = \hat{p}\dot{\hat{q}} - \hat{L}$, show that $\hat{q}(\tau) = e^{\hat{H}\tau}\hat{q}(0)e^{-\hat{H}\tau}$.

1.6 Show by direct discretization of the path integral that the partition function is exactly given by

$$\mathcal{Z}(\beta) = \int_\beta \mathcal{D}q e^{-\int_0^\beta \mathrm{d}\tau(\frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2)} = \frac{e^{-\frac{1}{2}\beta\omega}}{1 - e^{-\beta\omega}}.$$

Start by dividing the quantum path into a classical parth and a perturbation $q = q_{\mathrm{cl}} + h$ and show that partition function separates $\mathcal{Z} = \mathcal{Z}_{\mathrm{cl}}\mathcal{Z}_h$. Find classical part evaluating $S_{E,\mathrm{cl}}$ by use of (1.68). Then show that the fluctuation part can be written as:

$$\mathcal{Z}_h = \lim_{N\to\infty}\kappa_N^{N+1}\int\prod_{i=1}^{N}\mathrm{d}h_i\exp\Big[-\sum_{ij}h_iA_{ij}h_j\Big] \tag{1.80}$$

where

$$A_{ij} = \frac{1}{2a_N\Delta\tau_N}\Big(\delta_{ij} - a_N(\delta_{i+1,j}+\delta_{i-1,j})\Big), \tag{1.81}$$

with $a_N \equiv (2+(\Delta\tau\omega)^2)^{-1}$ and $\Delta\tau_N = \beta/(N+1)$. Evaluate the determinant and finally show that $\kappa_N = 1/\sqrt{2\pi\Delta\tau_N}$, requiring that the path integral for transition amplitude obeys

$$F(h,-i\beta;h,0) = \sum_{h'}F(h,-i\beta;h',-i(\beta-\Delta\tau_N))F(h',-i(\beta-\Delta\tau_N);h,0)$$

.

1.7 Prove the identity $\frac{\sinh\pi x}{\pi x} = \prod_{n=1}^{\infty}(1+\frac{x^2}{n^2})$.

1.8 We observed that the finite-temperature part of the pressure $P = (T/V)\log\mathcal{Z}$ can be written as:

$$P = \int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\boldsymbol{p}^2}{3T\omega_{\boldsymbol{p}}}\frac{1}{e^{\beta\omega_{\boldsymbol{p}}}-1}.$$

Show that this form is consistent with the kinetic pressure exerted by a gas of particles on a (imaginary) partitition wall embedded into the system.

2 Bosonic Field theory

Clearly computing the partition function for the simple harmonic oscillator from path integral was harder than the statistical approach. The PI method shows its power when we consider field theories and compute quantum (loop) corrections. We will now move to the field theory applications. We start by studying the simple non-interacting scalar field. Then we move to a complex scalar field, which can also carry a conserved charge and manifest the phenomenon of Bose condensation. After this we move to consider interacting bosonic field theories in section 4.

Keywords: *Quantization, partition function, generating function, chemical potential, KMS-relation, condensate.*

2.1 Non-interacting singlet scalar field

We first consider a free singlet scalar field (Klein-Gordon field) theory, which is defined by the Lagrangian

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2. \tag{2.1}$$

We quantize this theory by the usual canonical quantization rules

$$\begin{aligned} &[\hat{\phi}(\boldsymbol{x},t), \hat{\pi}(\boldsymbol{x}',t)] = i\delta^3(\boldsymbol{x}-\boldsymbol{x}') \\ &[\hat{\phi}(\boldsymbol{x},t), \hat{\phi}(\boldsymbol{x}',t)] = [\hat{\pi}(\boldsymbol{x},t), \hat{\pi}(\boldsymbol{x}',t)] = 0. \end{aligned} \tag{2.2}$$

We now introduce the field operators ($\pi \equiv \delta\mathcal{L}_\phi/\delta\dot{\phi} = \dot{\phi}$):

$$\begin{aligned} \hat{\phi}(\boldsymbol{x},t) &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} D_{\boldsymbol{p}} \big(\hat{a}_{\boldsymbol{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}} e^{ip\cdot x}\big) \\ \hat{\pi}(\boldsymbol{x},t) &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} D_{\boldsymbol{p}} \big[-i\omega_{\boldsymbol{p}}\big(\hat{a}_{\boldsymbol{p}} e^{-ip\cdot x} - \hat{a}^\dagger_{\boldsymbol{p}} e^{ip\cdot x}\big)\big], \end{aligned} \tag{2.3}$$

where $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and $D_{\boldsymbol{p}}$ is some still unspecified normalization of the density of the phase space. Canonical commutation relations (2.53) imply that

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = C_{\boldsymbol{p}}\delta^3(\boldsymbol{p}-\boldsymbol{p}') \tag{2.4}$$

and $[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}'}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = 0$, but the normalization factor $C_{\boldsymbol{p}}$ is only defined with respect to $D_{\boldsymbol{p}}$:

$$D^2_{\boldsymbol{p}} C_{\boldsymbol{p}} = \frac{(2\pi)^3}{2\omega_{\boldsymbol{p}}}. \tag{2.5}$$

We can now write the Hamiltonian operator for the system, introducing Hamiltonian density $\mathcal{H}_\phi = \hat{\pi}\hat{\dot{\phi}} - \mathcal{L}_\phi$ and raising all fields to operators. After a little algebra we find the Hamiltonian operator normalized to unit volume:

$$\frac{\hat{H}}{V} = \frac{1}{V}\int \mathrm{d}^3\boldsymbol{x}\big(\hat{\pi}\hat{\dot{\phi}} - \hat{\mathcal{L}}\big) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{2\omega^2_{\boldsymbol{p}} D^2_{\boldsymbol{p}}}{V}\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \frac{1}{2}[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}}]\Big). \tag{2.6}$$

In the box-normalization the momentum space delta-function is $\delta^3(0) = V/(2\pi)^3$, so the vacuum energy density appearing in (2.6) becomes, indepdent of the normalization choices:

$$\frac{H_{\rm vac}}{V} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{2\omega_{\boldsymbol{p}}^2 D_{\boldsymbol{p}}^2}{V}\frac{1}{2}C_{\boldsymbol{p}}\frac{V}{(2\pi)^3} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\omega_{\boldsymbol{p}}}{2}. \tag{2.7}$$

Of course the trace over the full Hamiltonian $\hat{H}/V$ is similarly independent of the normalizations. The operator form however, does show normalization dependece. If one wants to keep the simple realation between the Hamiltonian mode-function and the number operator (this also implies that 1-particle states are simply normalized: $\langle p|p\rangle = 1$), setting $\hat{H}_{\boldsymbol{p}} \equiv \omega_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}$, one has to set in addition:

$$\frac{2\omega_{\boldsymbol{p}}^2 D_{\boldsymbol{p}}^2}{V} \equiv \omega_{\boldsymbol{p}} \quad \Rightarrow \quad D_{\boldsymbol{p}} = \sqrt{\frac{V}{2\omega_{\boldsymbol{p}}}}, \quad C_{\boldsymbol{p}} = \frac{(2\pi)^3}{V}. \tag{2.8}$$

With this normalization then

$$\frac{\hat{H}}{V} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\omega_{\boldsymbol{p}}\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \frac{1}{2}\Big). \tag{2.9}$$

In field theory one often adopts, as we are doing here, the covariant normalization, setting $C_{\boldsymbol{p}} \equiv (2\pi)^3 2\omega_{\boldsymbol{p}}$:

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] \equiv (2\pi)^2(2\omega_{\boldsymbol{p}})\delta^3(\boldsymbol{p}-\boldsymbol{p}'). \tag{2.10}$$

This then implies that $D_{\boldsymbol{p}} = 1/(2\omega_{\boldsymbol{p}})$ and the Hamiltonian operator becomes:

$$\hat{H} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\Big(\omega_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + n_{0\boldsymbol{p}}\frac{\omega_{\boldsymbol{p}}}{2}\Big), \tag{2.11}$$

where we wrote $n_{0\boldsymbol{p}} = 2\omega_{\boldsymbol{p}}V$, which corresponds to number of states in vacuum in volume V in continuous normalization. Obviously the two Hamiltonians are physically equivalent. At any rate, the expression (2.9) is clearly the same that we obtained for a collection of simple harmonic oscillators confined into a box, except for the generalization to a massive field. This proves that quantized free field really is made out of simple quantum oscillators. From now on, we again switch our treatment to the path integral language.

Partition function Extending the previous path-integral results for the SHO to the case of an non-interacting scalar field is straightforward. We again remind the path-integral form of the real time transition amplitude:

$$\langle\phi_b(\boldsymbol{x})|e^{-i\hat{H}(t-t')}|\phi_a(\boldsymbol{x})\rangle = \int [\mathcal{D}\phi]_{\substack{\phi(x,t')=\phi_b(x)\\ \phi(x,t)=\phi_a(x)}} \exp\Big[\int_t^{t'}\mathrm{d}t L_M[\phi,\partial_\mu\phi]\Big], \tag{2.12}$$

where the Minkowski space action is

$$L_M[\phi,\partial_\mu\phi] = \int \mathrm{d}^3\boldsymbol{x}\Big(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2\Big). \tag{2.13}$$

Here the position of the particle has been replaced by a three dimensional field configuration $q \to \phi(\boldsymbol{x})$. This is a major complication in principle, which in practice can be handled by merely interpreting $\boldsymbol{x}$ as a label for a large (infninite) number of independent variables. As before the partition function can be computed as a path integral in Euclidean time:

$$\begin{aligned}\mathcal{Z}(\beta) &= \mathrm{Tr}[e^{-\beta\hat{H}}] \\ &= \int [\mathcal{D}\phi_a] \langle \phi_a(\boldsymbol{x})| e^{-\beta\hat{H}} |\phi_a(\boldsymbol{x})\rangle \\ &= \int [\mathcal{D}\phi]_{\phi(\boldsymbol{x},0)=\phi(\boldsymbol{x},\beta)=\phi_a(\boldsymbol{x})} \exp\Big[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\, \mathcal{L}_\mathrm{E}(\phi,\partial_\mu\phi)\Big]. \end{aligned} \tag{2.14}$$

The partition function is still a trace, now computed over a complete set of eigenstates of the field operator $\hat{\phi}(\tau,\boldsymbol{x})|\phi\rangle = \phi(\tau,\boldsymbol{x})|\phi\rangle$, so the result is a path integral over all 3D-field configurations periodic in τ with a period β. The Euclidean space Lagrangian density is

$$\mathcal{L}_\mathrm{E}[\phi,\partial_\mu\phi] = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\Big(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\Big) \tag{2.15}$$

where the derivative is again with respect to the complex time: $\dot{\phi} = \partial_\tau\phi$.

An alternative form for the partition function can be written in terms of the (Euclidean) Hamiltonian, which is understood to be a function of π:

$$\mathcal{Z}(\beta) = \int [\mathcal{D}\pi][\mathcal{D}\phi]_\beta \exp\Big[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\big(i\pi\dot{\phi} - \mathcal{H}[\phi,\pi]\big)\Big]. \tag{2.16}$$

where the free Hamiltonian density is the same as Euclidean lagrangian with $\dot{\phi} \to \pi$:

$$\mathcal{H}[\phi,\pi] = \frac{1}{2}\pi^2 + (\nabla\phi)^2 + m^2\phi^2. \tag{2.17}$$

Note that here the π-integration is not constrained by the periodicity requirement. This integration is Gaussian and integrating over π, using relation (B.12) equation (2.16) reduces to (2.14) up to an irrelevant constant.

Generating functional and propagator We can again define a generating functional by introducing external sources, which now are field configurations $j(\tau,\boldsymbol{x})$:

$$\mathcal{Z}[\beta,j] = \int [\mathcal{D}\phi]_\beta \exp\Big[-S_E[\phi] + \int_{X_\mathrm{E}} j(\tau,\boldsymbol{x})\phi(\tau,\boldsymbol{x})\Big]. \tag{2.18}$$

where we introduced a shorthand notation

$$\int_{X_\mathrm{E}^\beta} \equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}. \tag{2.19}$$

In the case of free theory action with the Euclidian Lagrangian density (2.15) this can be integrated, now already in an obvious way, to give

$$\mathcal{Z}[\beta,j] = \mathcal{Z}(\beta)\exp\Big[-\frac{1}{2}\int_{X_\mathrm{E}^\beta,X_\mathrm{E}^{\beta\prime}} j(\tau,\boldsymbol{x})\Delta_0(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')j(\tau',\boldsymbol{x}')\Big]. \tag{2.20}$$

We identify $\Delta_0(\tau, \boldsymbol{x})$ as the propagator, because it clearly satisfies

$$\Delta_0(\tau, \boldsymbol{x}) = \frac{1}{\mathcal{Z}(\beta)} \frac{\delta^2 \mathcal{Z}[\beta, j]}{\delta j(0) \delta j(\tau, \boldsymbol{x})}\Big|_{j=0} = \langle \mathcal{T}\big[\hat{\phi}(\tau, \boldsymbol{x})\hat{\phi}(0)\big]\rangle_\beta. \tag{2.21}$$

The Gaussien integral was done formally in going from (2.18) to(2.20). It is consistent with the requirement that the propagator $\Delta_0(\tau, \boldsymbol{x})$ satisfies the equation

$$\Big(- \partial_\tau^2 - \nabla^2 + m^2 \Big) \Delta_0(\tau, \boldsymbol{x}) = \delta(\tau)\delta^3(\boldsymbol{x}). \tag{2.22}$$

Going to the momentum space, by Fourier transforming with respect to $\boldsymbol{x}$ one easily gets:

$$\Big(- \partial_\tau^2 + \omega_{\boldsymbol{p}}^2 \Big) \Delta_0(\tau, \boldsymbol{x}) = \delta(\tau). \tag{2.23}$$

This is the same equation as (1.59), now with an energy corresponding to a massive particle and momentum $\boldsymbol{p}$: $\omega \to \omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. The explicit solution for the propagator in the Fourier representation[5] then is

$$\Delta_0(\omega_n, \boldsymbol{p}) = \frac{1}{\omega_n^2 + \omega_{\boldsymbol{p}}^2}, \tag{2.24}$$

where $\omega_n = 2\pi n T$. One can again solve the τ-dependent propagator by performing the complex integration. The calculation does not depend on the dispersion relation and we can directly write:

$$\Delta_0(\tau, \boldsymbol{p}) = \frac{1}{2\omega_{\boldsymbol{p}}} \Big((1 + n_{\rm BE}(\omega_{\boldsymbol{p}})) e^{-\omega_{\boldsymbol{p}}|\tau|} + n_{\rm BE}(\omega_{\boldsymbol{p}}) e^{\omega_{\boldsymbol{p}}|\tau|} \Big). \tag{2.25}$$

which generalized the old result (1.68) to a massive field.

Evaluating Klein-Gordon partitition function. To compute the partition function we write the Euclidean action in the Fourier representation:

$$\begin{aligned} S_E[\phi] &= \frac{1}{2} \int_{X_{\rm E}^\beta} \phi(x) \big(- \partial_\tau^2 - \nabla^2 + m^2 \big) \phi(x) \\ &= \beta \sum\!\!\!\!\!\!\int_B \frac{1}{2} \phi_{n\boldsymbol{p}}^* \beta^2 \big[\big(\omega_n^2 + \omega_{\boldsymbol{p}}^2 \big) \big] \phi_{n\boldsymbol{p}}, \end{aligned} \tag{2.26}$$

where $\phi_{n\boldsymbol{p}}$ is the Fourier transform of $\phi(\tau, \boldsymbol{x})$ (see appendix A) suitably scaled by β and we defined the shorthand notation, which we will be using throughout:

$$\sum\!\!\!\!\!\!\int_B \equiv T \sum_n \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3}. \tag{2.27}$$

[5]The Fourier transform is discrete with respect to frequency, which is conjugate to $\tau \in S_1$ and continuous with respect to momentum, which is conjugate to $\boldsymbol{x} \in \mathbb{R}^3$. That is, $\phi(\tau, \boldsymbol{x})$ is defined in Euclidean space[6] $S_1 \otimes \mathbb{R}^3$.

We can then write (dropping constants on the way)

$$\begin{aligned}\log \mathcal{Z}(\beta) &= \log\Big\{\int [\mathcal{D}\phi]_\beta \exp\Big[-\frac{1}{2}\beta\sum\!\!\!\!\!\!\int \beta^2(\omega_n^2+\omega_{\boldsymbol{p}}^2))|\phi_{n\boldsymbol{p}}|^2\Big]\Big\} \\ &= \log\Big\{\prod_{n,\boldsymbol{p}}\int \mathrm{d}\phi_{in\boldsymbol{p}} \exp\Big[-\frac{1}{2}\beta^2(\omega_n^2+\omega_{\boldsymbol{p}}^2)|\phi_{n\boldsymbol{p}}|^2\Big]\Big\} \\ &= \frac{1}{2}\log\Big\{\prod_{n,\boldsymbol{p}}\frac{2\pi}{\beta^2(\omega_n^2+\omega_{\boldsymbol{p}}^2)}\Big\} \;=\; -\frac{1}{2}\sum_{n,\boldsymbol{p}}\log\beta^2(\omega_n^2+\omega_{\boldsymbol{p}}^2). \end{aligned} \tag{2.28}$$

So we see that up to a constant $\log \mathcal{Z}(\beta) = \frac{1}{2}\mathrm{Tr}(\log\Delta_0) = \frac{1}{2}\log\det(\Delta_0)$. We can first discretize $\boldsymbol{p}$ within a box so that the continuous product of determinants gives a discrete sum over logarithms, and then take the box size to infinity to regain the integral over momenta (but leave the infinite volume factor in front just as we did in (1.34)). For each $\boldsymbol{p}$ mode, the calculation is identical to that performed in section 1.5, and we immediately get:

$$P = \frac{T}{V}\log\mathcal{Z}(\beta) = \int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\Big(-\frac{\omega_{\boldsymbol{p}}}{2} - T\log\big(1-e^{-\beta\omega_{\boldsymbol{p}}}\big)\Big). \tag{2.29}$$

That is: $P = J_0(m) + J_T^-(m,T)$, with thermal integrals defined in (1.41) and (1.42). Let us remind, that while we derived (2.29) schematically, not keeping track of constant terms, the results of 1.5 show that this is an exact result.

2.2 Noninteracting complex scalar field

Let us now consider a sligthly more complicated structure, which also can support conserved charges and hence nonzero chemical potentials. The simplest QFT-system with this property is the free complex scalar field theory with the Lagrangian:

$$\mathcal{L} = |\partial_\mu\phi|^2 - m^2|\phi^2|. \tag{2.30}$$

Decomposing the complex field as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ where $\phi_{1,2}$ are real scalar fields, we can write the Lagrangian as $\mathcal{L} = \sum_i(\frac{1}{2}(\partial_\mu\phi_i)^2 - \frac{m^2}{2}\phi_i^2)$. Thus the free complex field can be seen as a combination of two scalar fields. Its partition function then is just the square of the one for the real scalar field and the pressure just twice that of the real scalar field. What makes this model a little more interesting is that it is invariant under global U(1)-transofmations $\phi \to e^{i\alpha}\phi$. Noether's theorem then implies that there is a conserved current and a conserve charge. They can be easily worked out from the Lagrangian:

$$\begin{aligned} j^\mu(x) &\equiv \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}i\phi + h.c. = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \\ Q &= \int \mathrm{d}^3\boldsymbol{x} j^0(x) = \int \mathrm{d}^3\boldsymbol{x}\; i(\phi\pi - \phi^*\pi^*) \equiv \int \mathrm{d}^3\boldsymbol{x}\mathcal{Q}(\pi,\phi). \end{aligned} \tag{2.31}$$

where we used $\pi = \delta\mathcal{L}_M/\delta(\partial_t\phi) = \partial^t\phi^* = \partial_t\phi^* = -i\partial_\tau\phi^*$. Note that in the component notation, where $\pi_i = \partial_t\phi_i^*$ one has $\pi = \pi_i - i\pi_2$. You can find the calculations in the following section performed fully, or partially using component basis in [2] and [1]. We shall stick to the complex representation however.

Partition function with chemical potential Writing down the partition function of a system with a conserved charge requires some additional structure. We already know from (1.1) that in a system with a conserved particle number, the quantum density operator contains the number operator $\hat{N}$ multiplied by the chemical potential: $\hat{\rho} = e^{-\beta(\hat{H}-\mu\hat{N})} = e^{-\beta\hat{F}}$. Here we are not interested in conserved number density, but of a conserved charge, so to get the relevant free energy we have to replace $\hat{N} \to \hat{Q}$ in our phase space density operator, which then becomes $\hat{\rho} = e^{-\beta(\hat{H}-\mu\hat{Q})}$. It is easiest to write the path integral for this partition function using the Hamiltonian form (2.16):

$$\begin{aligned}\mathcal{Z}(\beta,\mu) &= \mathrm{Tr}[e^{-\beta(\hat{H}-\mu\hat{Q})}] \\ &= \int \mathcal{D}\pi\mathcal{D}\pi^*[\mathcal{D}\phi\mathcal{D}\phi^*]_\beta \exp\Big[\int_{X_E^\beta}\big(\pi\dot{\phi}+\pi^*\dot{\phi}^*-\mathcal{H}+\mu\mathcal{Q}\big)\Big].\end{aligned} \tag{2.32}$$

where we still kept the notation $\dot{\phi} \equiv \partial_t\phi$ in terms of real time for the moment. The charge density $\mathcal{Q}$ was defined in (2.31) and

$$\mathcal{H} = |\pi|^2 + [\nabla\phi|^2 + m^2|\phi|^2. \tag{2.33}$$

To perform the π-integrations in (2.32), we first manipulate all terms containing conjugate momenta in the integrand in (2.32) as follows:

$$\begin{aligned}&-\pi\pi^* + \pi\dot{\phi} + \pi^*\dot{\phi}* + i\mu(\phi\pi - \phi^*\pi^*) \\ &-\pi\pi^* + \pi(\dot{\phi}+i\mu\phi) + \pi^*(\dot{\phi}^* - i\mu\phi^*) \\ &-(\pi - \dot{\phi}^* + i\mu\phi^*)(\pi^* - \dot{\phi} - i\mu\phi) + (\dot{\phi}+i\mu\phi)(\dot{\phi}^* - i\mu\phi^*).\end{aligned} \tag{2.34}$$

we can now shift $\pi \to \pi + \dot{\phi}^* - i\mu\phi^*$ and $\pi^* \to \pi + \dot{\phi} + i\mu\phi$, after which the π-integrals decouple and can be peformed to give an overall constant. Then noting that $\dot{\phi} = i\partial_\tau\phi$, we get, up to an irrelevant constant:

$$\mathcal{Z}(\beta,\mu) = \int[\mathcal{D}\phi]_\beta \exp\Big[-\int_{X_E^\beta}\big(\,[(\partial_\tau+\mu)\phi][(\partial_\tau-\mu)\phi^*] + |\nabla\phi|^2 + m^2|\phi|^2\big)\Big]. \tag{2.35}$$

Moving to Fourier space we write:

$$\phi(\tau,\boldsymbol{x}) = \sum\!\!\!\!\!\!\int \phi_n(\boldsymbol{p})e^{-i\omega_n\tau + i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{2.36}$$

Periodicity of configurations $\phi(0,\boldsymbol{x}) = \phi(\beta,\boldsymbol{x})$ requires that $\omega_n = 2\pi nT$. In Fourier space space we essentially replace: $\partial_\tau\phi \to -i\omega_n\phi_n$, $\partial_\tau\phi^* \to i\omega_n\phi_n^*$ and $\nabla\phi \to -i\boldsymbol{p}\phi_n$, so that

$$\mathcal{Z}(\beta,\mu) = \int[\mathcal{D}\phi_n]\exp\Big[-\sum\!\!\!\!\!\!\int\big(\phi_n^*(\boldsymbol{p})\big((\omega_n+i\mu)^2 + m^2 + \boldsymbol{p}^2\big)\phi_n(\boldsymbol{p})\big)\Big]. \tag{2.37}$$

Note that here We can now simply read off the propagator in the complex representation

$$\Delta_\phi(\omega_n,\boldsymbol{p};\mu) = \frac{1}{(\omega_n+i\mu)^2 + \omega_{\boldsymbol{p}}^2}. \tag{2.38}$$

So, the chemical potential appears as a complex shift in frequency $\omega_n \to \omega_n + i\mu$.

KMS-relation for the complex scalar field. We digress from evaluating $\mathcal{Z}$ to prove the KMS-relation for the complex scalar field with chemical potential using the operator formalism. We denote $\hat{H} - \mu\hat{Q} \equiv \hat{K}$ so that $\hat{\rho} = e^{-\beta\hat{K}}$ and assume $0 < \tau < \beta$. We then get:

$$\begin{aligned}
\Delta_\phi(\tau, \boldsymbol{x}) &= \frac{1}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[\hat{\rho}\mathcal{T}\big(\hat{\phi}(\tau,\boldsymbol{x})\hat{\phi}^*(0)\big)\big] \\
&= \frac{1}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[e^{-\beta\hat{K}}\hat{\phi}(\tau,\boldsymbol{x})e^{\beta\hat{K}}e^{\beta\hat{K}}\hat{\phi}^*(0)\big] \\
&= \frac{e^{\beta\mu}}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[\hat{\phi}(\tau-\beta,\boldsymbol{x})e^{-\beta\hat{K}}\hat{\phi}^*(0)\big] \\
&= \frac{e^{\beta\mu}}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[\hat{\rho}\mathcal{T}\big(\hat{\phi}(\tau-\beta,\boldsymbol{x})\hat{\phi}^*(0)\big)\big] = e^{\mu\beta}\Delta_\phi(\tau-\beta,\boldsymbol{x}).
\end{aligned} \tag{2.39}$$

In the second equality we used the cyclicity of the trace and introduced a unity operator, and in third $e^{-\beta\hat{H}}\hat{\phi}(\tau,\boldsymbol{x})e^{\beta\hat{H}} = \hat{\phi}(\tau-\beta,\boldsymbol{x})$ and again $e^{\beta\mu\hat{Q}}\hat{\phi}e^{-\beta\mu\hat{Q}} = e^{\beta\mu}\hat{\phi}$ (see excercise **2.5**). Fourier transforming the propagator

$$\Delta(\tau,\boldsymbol{x}) = \sum\!\!\!\!\!\!\!\int \Delta(\omega_n,\boldsymbol{p})e^{-ip_0\tau + i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{2.40}$$

and imposing the KMS-relation (2.39) to (2.40) we get the condition

$$e^{\mu\beta}e^{i\beta p_0} = 1 \quad \Rightarrow \quad p_0 = 2n\pi T + i\mu \equiv \omega_n + i\mu. \tag{2.41}$$

This is the same result that we obtained above from the path integral expression using the periodicty of the configuration $\phi(\tau,\boldsymbol{x})$.

Evaluating the partition function. Despite the chemical potential, we can proceed with (2.46) in similar fashion as with equation (2.28), except that here $\phi_n(p)$ and $\phi_n(p)^*$ are independent variables. This means we have a product of two identical partition functions in $\mathcal{Z}$ or twice the scalar term for $\log\mathcal{Z}$:

$$\begin{aligned}
\log\mathcal{Z}(\beta,\mu) &= \mathrm{Tr}[\log\Delta_0] \\
&= -\sum_{n,\boldsymbol{p}}\log((\omega_n + i\mu)^2 + \omega_{\boldsymbol{p}}^2) \\
&= -V\int_{\mathbf{p}}\int_0^{\omega_{\mathbf{p}}} \mathrm{d}\omega' \sum_{n=-\infty}^{\infty}\frac{2\omega'}{(\omega_n + i\mu)^2 + \omega'^2}.
\end{aligned} \tag{2.42}$$

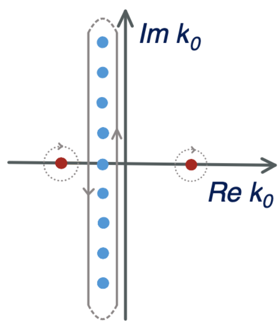

Figure 5: Integration contour $\mathcal{C}_1$ for complex scalar field Matsubara sums.

The last form has been written to resemble the Matsubara sum over the propagator in equation (1.65) and we can use the same contour integral trick as there, with the modified scontour shown in figure 5. We get:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\omega_n + i\mu)^2 + \omega'^2} = \frac{1}{2\pi i} \oint_{\mathcal{C}_1} \frac{1}{\omega'^2 - z^2} \frac{\beta}{e^{\beta(z+\mu)} - 1}$$
$$= \frac{1}{2\omega'} \Big(\frac{1}{e^{\beta(\omega'+\mu)} - 1} + \frac{1}{e^{-\beta(\omega'+\mu)} - 1} \Big). \tag{2.43}$$

Inserting this back to (2.42) and integrating over ω' one finds the expected result:

$$P(\beta, \mu) = \frac{1}{\beta V} \log \mathcal{Z}(\beta, \mu)$$
$$= -\int_{\mathbf{p}} \int^{\omega_{\mathbf{p}}} \mathrm{d}\omega' \Big(1 + \sum_{\pm} \frac{1}{e^{\beta(\omega' \pm \mu)} - 1} \Big)$$
$$= -\sum_{\pm} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \Big(\frac{\omega_{\boldsymbol{p}}}{2} + \frac{1}{\beta} \log\big(1 - e^{-\beta(\omega_{\boldsymbol{p}} \pm \mu)}\big) \Big). \tag{2.44}$$

This agrees with the result (1.34) obtained by simple quantum statistical analysis in section 1.2, except that there we only included the particle branch solution, proportional to $+\mu$, whereas here we find also the antiparticle solutoin corresponding to $-\mu$ in the exponent.

2.2.1 Bose condensation

The free complex scalar field is the simplest QFT that displays Bose condensation. The physical cause for Bose condensation is that in a system with a conserved chage, all particles may not fit into the available phase space at very low temperatures. Charge then begins to accumulate to ground state and a collective state of zero modes is formed. We clearly missed this phenomenon in our calculation of the partition function above, but how? The problem is that ground state is not correctly represented in the transition from a discrete system to a continuous variables, where it is a state of zero measure. We can fix the issue by adding a condensate by hand into the Fourier decomposition of the field:

$$\phi(\tau, \boldsymbol{x}) = \phi_c + \sum\!\!\!\!\!\!\int \phi_n(\boldsymbol{p}) e^{-i\omega_n \tau + i\boldsymbol{p}\cdot\boldsymbol{x}}, \tag{2.45}$$

where $\phi_c \equiv \xi e^{i\theta}$ represents the condensate with $\omega_n = 0$. Even though the condensate has zero energy, it does carry a charge, which will allow to preserve the charge conservation at low temperatures. In momentum space we again get rid of time and space gradients: $\dot{\phi}_i(x) \to -i\omega_n \phi_{i,n}(\boldsymbol{p})$ and $\nabla\phi_i(x) \to i\boldsymbol{p}\phi_{in\boldsymbol{p}}$. The partition function then becomes

$$\mathcal{Z}(\beta, \mu, \xi) = \int [\mathcal{D}\phi_{in\boldsymbol{p}}] \exp \Big[\beta V(\mu^2 - m^2)\xi^2$$
$$- \sum\!\!\!\!\!\!\int \big(\phi_n^*(\boldsymbol{p}) \big((\omega_n + i\mu)^2 + m^2 + \boldsymbol{p}^2 \big) \phi_n(\boldsymbol{p}) \big) \Big]. \tag{2.46}$$

The first term in (2.46) comes from the condensate. It was taken to be constant so that its derivatives vanish and the integral over its action is proportional to the volume βV of the space $S_1 \otimes \mathbb{R}^3$. The rest of the partition function agrees with what we already computed above and we can immediately write down the final result:

$$\frac{1}{V}\log \mathcal{Z}(\beta,\mu,\xi) = \beta(\mu^2 - m^2)\xi^2 - \sum_{\pm}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\Big[\frac{1}{2}\beta\omega_{\boldsymbol{p}} + \log(1 - e^{\beta(\omega_{\boldsymbol{p}}\pm\mu)})\Big]. \tag{2.47}$$

Note that we did not integrate over the condensate variable in the partition function. Indeed, we are not treating the condensate as a quantum fluctuation, but as an external background field that is used to impose the charge conservation.

Critical temperature T_c and formation of condensate at $T < T_c$. Let us now treat ξ as a variational parameter, which should be minimized for a given temperature $\beta = 1/T$ and chemical potential μ

$$\frac{1}{V}\Big(\frac{\partial \log \mathcal{Z}}{\partial \xi}\Big)_{\beta,\mu} = 2\beta(\mu^2 - m^2)\xi \equiv 0 \quad \Rightarrow \quad \xi = 0 \quad \text{if} \quad |\mu| \neq m, \tag{2.48}$$

In fact the condition for μ will be $\xi = 0$ if $|\mu| < m$. However for $|\mu| = m$ the condensate ξ can be nonzero. As has already been suggested, we determine ξ form charge conservation at $|\mu| = m$. Assume now that the total charge per volume Q/V in the system is fixed. At high temperaturs, where there are many states available for the fluctuations, all charge resides in fluctuations:

$$\begin{aligned} q \equiv \frac{Q}{V} &= -\frac{eT}{V}\Big(\frac{\partial \log \mathcal{Z}}{\partial \mu}\Big) \\ &\xrightarrow{\text{high } T} q(\mu,T)_{\text{particles}} = -e\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\Big(\frac{1}{e^{\beta(\omega_{\boldsymbol{p}}-\mu)} - 1} - \frac{1}{e^{\beta(\omega_{\boldsymbol{p}}+\mu)} - 1}\Big). \end{aligned} \tag{2.49}$$

Here we assumed that particles have charge $-e$ and antiparticles a charge e (see equation (1.39)). From this expression we see that the charge density in particles decreases as temperature is lowered for fixed μ and it is increased when $|\mu|$ is increased (either positive or negative depending on the sign of Q) for a fixed T. However, $|\mu|$ cannot be raised above m, as then the particle number would not well defined (non-integrable singularity would develop in the integral). Instead, there is a critical temperature corresponding to $|\mu| = m$, which is the lowest temperature in which all excitations can reside in fluctuations:

$$q = q(m, T_c)_{\text{particles}} \tag{2.50}$$

Below T_c, a $|\mu|$ cannot be increased, and a condesate is formed instead. The charge

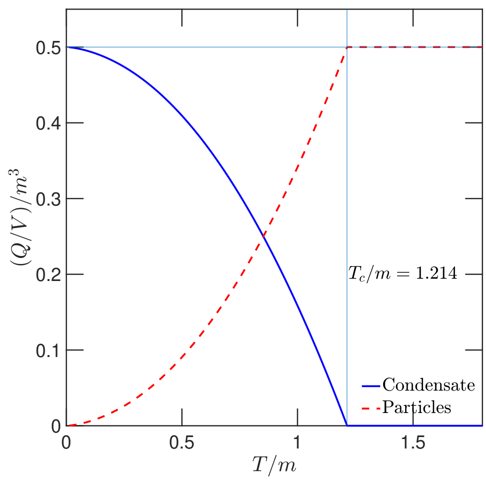

Figure 6: Charge density in the residing in particles and in the condensate. The critical temperature in this case is $T_c/m \approx 0.1214$, corresponding to $Q/V \equiv 0.5m^3$.

distribution in the low temperatures $T < T_c$ comes again from (2.47) by differentiation with respect to μ and then setting $\mu = m$ (note that based on high-T limit $\text{sgn}(q) = -\text{sgn}(e\mu)$):

$$|q| = -2m|e|\xi(T)^2 - |q(m,T)_{\text{particles}}| \quad \Leftrightarrow \quad \xi(T)^2 = \frac{1}{2m}(|q| - |q(m,T)_{\text{particles}}|). \tag{2.51}$$

and then

$$q_{\text{cond}}(T) = e\,\text{sgn}(\mu)\big(\,|q| - |q(m,T)_{\text{particles}}|\,\big), \quad T < T_c. \tag{2.52}$$

This solution is displayed in figure 6 (blue solid line) for the case $Q/V = 0.5m^3$, along with the particle contribution to the charge density (red dashed line). Bose condensation was observed in dilute atomic gases in 1995 by Eric Cornell and Carl Wieman at the NIST–JILA lab, University of Colorado. After this the phenomenon has been repeated by hundreds of laboratories worldwide.

Finally let us point out that one could formally include the condensate into the density of the state function $f(p)$, as an independent spectral contribution at zero momenta. However, not all systems with an IR-peaked distribution qualify as Bose condensates. For example during inflation the mode-freezing causes a pile-up of modes with wavelengths longer than the horizon. From the point of view of a local causal observer these states can not be distinguished from a zero-mode and a construction of the form (2.45) becomes meaningful. The subsequent analysis connecting the condensate and chemical potential does not apply however, because the "condensate" in this case does not carry a charge.

Excercises to section 2

2.1 Show by direct calculation that the canonical commutation relations (2.53) imply commutation the mode operator commutation relations (2.10), given the field operators (2.3). Show also that expression for the single scalar field Hamiltonian takes the simple form (2.6) in terms of raising an lowering oprators.

2.2 Show that the critical temperature for forming the condensate in the non-relativistic limit ($Q/V \ll m^3$, this is only an approximative condition) can be expressed as

$$T_c = \frac{2\pi}{m}\left(\frac{Q}{V}\right)^{2/3}\zeta(\tfrac{3}{2})^{-2/3}$$

and in the relativistic limit ($Q/V \gg m^3$) as

$$T_c = \left(\frac{3}{m}\frac{Q}{V}\right)^{1/2}.$$

2.3 Quantize the complex scalar field theory by the usual canonical quantization rules

$$\begin{aligned}
&[\hat\phi(\boldsymbol{x},t), \hat\pi(\boldsymbol{x}',t)] = i\delta^3(\boldsymbol{x}-\boldsymbol{x}') \\
&[\hat\phi(\boldsymbol{x},t), \hat\phi(\boldsymbol{x}',t)] = [\hat\pi(\boldsymbol{x},t), \hat\pi(\boldsymbol{x}',t)] = 0.
\end{aligned} \tag{2.53}$$

where the field operator $\hat{\phi}(t,\boldsymbol{x})$ is

$$\hat{\phi}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \left(\hat{a}_{\boldsymbol{p}} e^{-ip\cdot x} + \hat{b}^\dagger_{\boldsymbol{p}} e^{ip\cdot x}\right) \tag{2.54}$$

and $\pi \equiv \delta\mathcal{L}_\phi/\delta\dot{\phi} = \dot{\phi}^*$.

2.4 Write the quantized charge operator $\hat{Q}$ of the charged scalar field in terms of the rising and lowering operators: $\hat{Q} = \int_{\boldsymbol{p}}(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}})$. What do you need to do to make this result physically sensible?

2.5 Using the expresssios for the Hamiltonian density (2.33) and charge (2.31), show that $e^{-\beta\hat{H}}\hat{\phi}(\tau,\boldsymbol{x})e^{\beta\hat{H}} = \hat{\phi}(\tau-\beta,\boldsymbol{x})$ and $e^{\beta\mu\hat{Q}}\hat{\phi}e^{-\beta\mu\hat{Q}} = e^{\beta\mu}\hat{\phi}$. Hint: show first that $[\hat{H},\hat{\phi}(x)] = i\partial_t\hat{\phi}$ and $[\hat{Q},\hat{\phi}(x)] = -\hat{\phi}$.

2.6 Compute the energy density and pressure for the Klein-Gordon field form the energy momentum tensor $E = {T_0}^0$ $p = {T_i}^i$. For this purpose quantize the energy momentum tensor for the field:

$$\hat{T}_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

Compare your results to the energy-density and pressure obtained from quantum-statistical methods. Show that if you use cut-off regularization, then the vacuum energy equation of the state does not satisfy $w = p/\rho = 1$ in the energy-momentum tensor picture, while in the statistcal definition it does. Show then that assuming dimensional regularization both methods agree on the eos. Spend some time thinking what this is teling about the reality of the vacuum energy divergence?

2.7 Show that the thermal part of the bosonic J_T^--function can be expanded as:

$$J_T^-(m,T) = -\frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2\left(\frac{nm}{T}\right)$$

where $K_2(x)$ is the Bessel function of the second kind.

2.8 Compute also the related integral:

$$I^-(m,T) \equiv \SumInt \Delta_0(\omega_n,\omega_{\boldsymbol{p}}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega}\big(1+2n_B(\omega_{\boldsymbol{p}})\big).$$

Show that the thermal part of this integral has the expression

$$I_T^-(m,T) = \frac{mT}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}K_1\left(\frac{nm}{T}\right).$$

Show also that $mI_T^-(m,T) = \partial_m J_T^-(m,T)$

3 Higher spin fields

Now we expand our library of quantum systems to higher spin fields. We begin from fermions, which we quantize both using canonical and path integral quantization, followed up by the quantum statistcis analysis of a free femion theory. We then move on to gauge fields, which we quantize using the path integral method. We discuss the gauge invariance problem and find the black body radiation formulae for free gauge theory.

Keywords: *Anticommutation rules, grassmann valued fields, KMS-relation, gauge fixing, Abelian gauge fields, non-Abelian gauge fields.*

3.1 Fermions

We first quickly go through the canonical quantization of a free fermion field theory, which is is described by the Lagrangian

$$\mathcal{L}_\psi = i\bar{\psi}\,\partial\!\!\!/\,\psi - m\bar{\psi}\psi. \tag{3.1}$$

The conjugate momentum to ψ is $\pi = \delta\mathcal{L}/\delta\dot{\psi} = i\psi^\dagger$. Theory is quantized by canonical *anticommutation* rules:

$$\begin{aligned}&\{\hat{\psi}_\alpha(t,\boldsymbol{x}), i\hat{\psi}^\dagger_\alpha(t,\boldsymbol{x}')\} = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{x}'),\\ &\{\hat{\psi}_\alpha(t,\boldsymbol{x}), i\hat{\psi}_\alpha(t,\boldsymbol{x}')\} = \{i\hat{\psi}^\dagger_\alpha(t,\boldsymbol{x}), i\hat{\psi}^\dagger_\alpha(t,\boldsymbol{x}')\} = 0.\end{aligned} \tag{3.2}$$

We can expand the fermionic field operator in terms of the particle and anti-particle creation and annihilation operators:[7]

$$\hat{\psi}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}(2\pi)^3}\sum_s \Big(\hat{a}^s_{\boldsymbol{p}} u(s,\boldsymbol{p})e^{-ip\cdot x} + \hat{b}^{s\dagger}_{\boldsymbol{p}} v(s,\boldsymbol{p})e^{ip\cdot x}\Big) \tag{3.3}$$

We also choose to normalize the particle and antiparticle spinors such that $u^\dagger(s,\boldsymbol{p})u(s',\boldsymbol{p}) = v^\dagger(s,\boldsymbol{p})v(s',\boldsymbol{p}) = 2\omega_{\boldsymbol{p}}\delta_{s,s'}$. With these normalizations the canonical anticommutation relations (3.2) imply

$$\{\hat{a}^s_{\boldsymbol{p}}, \hat{a}^{s'\dagger}_{\boldsymbol{p}'}\} = \{\hat{b}^s_{\boldsymbol{p}}, \hat{b}^{s'\dagger}_{\boldsymbol{p}'}\} = (2\pi)^2(2\omega_{\boldsymbol{p}})\delta_{s,s'}\delta^3(\boldsymbol{p}-\boldsymbol{p}'), \tag{3.4}$$

while all other commutators vanish. The Hamiltonian density of the Dirac field is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L}_\psi = \bar{\psi}\big(-i\gamma\cdot\nabla + m\big)\psi = i\psi^\dagger\partial_t\psi, \tag{3.5}$$

where in the last line we used the Dirac equation $(i\partial\!\!\!/+m)\psi = 0$. It is then straightforward to show that the Hamiltonian operator can be written as:

$$\hat{H} = \sum_s \int \frac{\mathrm{d}^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}(2\pi)^3}\Big[\omega_{\boldsymbol{p}}\big(\hat{a}^{s\dagger}_{\boldsymbol{p}}\hat{a}^s_{\boldsymbol{p}} + \hat{b}^{s\dagger}_{\boldsymbol{p}}\hat{b}^s_{\boldsymbol{p}}\big) - n_{0\boldsymbol{p}}\omega_{\boldsymbol{p}}\Big]. \tag{3.6}$$

Where again $n_{0\boldsymbol{p}} = 2\omega_{\boldsymbol{p}}V$ and the strange appearance of the number-operator terms is due to the chosen normalization of states.

[7] We use the same covariant normalization here that was introduced for scalar fields in (2.10).

Conserved charge and partition function with chemical potential The free Dirac theory (3.1) is invariant in the global U(1)-transformation

$$\psi \to e^{i\alpha}\psi, \tag{3.7}$$

which implies that there is a conserved current and a conserved charge given by

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x); \qquad Q = \int \mathrm{d}^3\boldsymbol{x} j^0 = \int \mathrm{d}^3\boldsymbol{x}\psi^\dagger\psi. \tag{3.8}$$

Just as complex scalar field, fermions can also carry a conserved charge. In the fermonic case one again uses the Hamiltonian representation of the path integral, so that

$$\begin{aligned}
\mathcal{Z}(\beta,\mu) &= \mathrm{Tr}[e^{-\beta(\hat{H}-\mu\hat{Q})}] \\
&= \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]_{\bar{\beta}} \exp\Big[\int_{X_\mathrm{E}^\beta}\Big(i\pi\partial_\tau\psi - \mathcal{H}(\pi,\psi) + \mu Q(\pi,\psi)\Big)\Big] \\
&= \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]_{\bar{\beta}} \exp\Big[-\int_{X_\mathrm{E}^\beta}\bar{\psi}\Big(\gamma^0(\partial_\tau-\mu) - i\boldsymbol{\gamma}\cdot\nabla + m\Big)\psi\Big] \\
&\equiv \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]_{\bar{\beta}} \exp\Big[-\int_{X_\mathrm{E}^\beta}\bar{\psi}\Delta_F^{-1}\psi\Big],
\end{aligned} \tag{3.9}$$

where $\bar{\beta}$ in the integration measure represents the fact that fermion configurations are *antiperiodic* over the interval β: $\psi(\beta,\boldsymbol{x}) = -\psi(0,\boldsymbol{x})$ and $\bar{\psi}(\beta,\boldsymbol{x}) = -\bar{\psi}(0,\boldsymbol{x})$, as explained in **??**. The fields ψ and $i\psi^\dagger$ are independent variables which, due to anticommutativity of the fermion operators $\hat{\psi}$ are Grassman valued, anticommuting numbers. The basic properties of Grassman numbers are given in appendix C.

Fermionic generating functional and propagator We can build a Generating function for Fermions simlarly to the Bosonic case, introduce Grassmann valued sources η and $\bar{\eta}$ for $\bar{\psi}$ and ψ and

$$\begin{aligned}
\mathcal{Z}[\beta,\mu,\eta,\bar{\eta}] &= \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]_\beta \exp\Big[-\int_{X_\mathrm{E}^\beta}\Big(\bar{\psi}\Delta_F^{-1}\psi + \bar{\psi}\eta + \bar{\eta}\psi\Big)\Big] \\
&= \mathcal{Z}(\beta,\mu)\exp\Big[\int_{X_\mathrm{E}^\beta}\int_{X_\mathrm{E}^{\beta\prime}}\bar{\eta}(\tau',\boldsymbol{x}')\Delta_F(\tau,\tau';\boldsymbol{x},\boldsymbol{x}')\eta(\tau,\boldsymbol{x})\Big].
\end{aligned} \tag{3.10}$$

Here we allowed the normalization factor also depend on the chemical potential. To get to the second line, one shifts $\psi \to \psi' \equiv \psi - \Delta_F\eta$ and notes the invariance of the integration measure on the shift, so that the gaussian integral over ψ' and $\bar{\psi}'$ still is equal to $\mathcal{Z}(\beta,\mu)$. We can see that Δ_F is the free fermion propagator:

$$\begin{aligned}
\Delta_F(\tau,\boldsymbol{x}) &= \frac{1}{\mathcal{Z}(\beta,\mu)}\frac{\delta^2\mathcal{Z}[\beta,\mu;j]}{\delta\bar{\eta}(\tau,\boldsymbol{x})\delta\eta(0)}\Big|_{j=0} \\
&= \frac{1}{\mathcal{Z}(\beta,\mu)}\int[\mathcal{D}\bar{\psi}\mathcal{D}\psi]_\beta\ \psi(\tau,\boldsymbol{x})\bar{\psi}(0)\ \exp\Big[-\int_{X_\mathrm{E}^\beta}\bar{\psi}\Delta_F^{-1}\psi\ \Big] \\
&= \frac{1}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[\hat{\rho}\mathcal{T}\big(\hat{\psi}(\tau,\boldsymbol{x})\hat{\bar{\psi}}(0)\big)\big] = \langle\mathcal{T}\big[\psi(\tau,\boldsymbol{x})\bar{\psi}(0)\big]\rangle.
\end{aligned} \tag{3.11}$$

Because ψ and $\bar{\psi}$ are anticommuting variables, the path integral expectation value corresponds (as it should by design) to an *anti time-ordered* product:

$$\mathcal{T}\big[\hat{\psi}(\tau,\boldsymbol{x})\hat{\bar{\psi}}(0)\big] = \theta(\tau)\hat{\psi}(\tau,\boldsymbol{x})\hat{\bar{\psi}}(0) - \theta(-\tau)\hat{\bar{\psi}}(0)\hat{\psi}(\tau,\boldsymbol{x}). \tag{3.12}$$

Fermionic KMS-relation We can now derive the fermionic KMS-relation from the operator formalism. We again denote $\hat{\rho} = e^{-\beta(\hat{H}-\mu\hat{Q})} \equiv e^{-\beta\hat{K}}$, one can now show that for $0 < \tau < \beta$:

$$\begin{aligned}
\Delta_F(\tau,\boldsymbol{x}) = \;& \frac{1}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[\hat{\rho}\mathcal{T}\big(\hat{\psi}(\tau,\boldsymbol{x})\hat{\bar{\psi}}(0)\big)\big] \\
= \;& \frac{1}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[e^{-\beta\hat{K}}\hat{\psi}(\tau,\boldsymbol{x})e^{\beta\hat{K}}e^{-\beta\hat{K}}\hat{\bar{\psi}}(0)\big] \\
= \;& \frac{e^{-\beta\mu}}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[\hat{\psi}(\tau-\beta,\boldsymbol{x})e^{-\beta\hat{K}}\hat{\bar{\psi}}(0)\big] \\
= \;& -\frac{e^{-\beta\mu}}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\big[\hat{\rho}\mathcal{T}\big(\hat{\psi}(\tau-\beta,\boldsymbol{x})\hat{\bar{\psi}}(0)\big)\big] = -e^{-\mu\beta}\Delta_F(\tau-\beta,\boldsymbol{x}).
\end{aligned} \tag{3.13}$$

The proof is entirely simlar to the one we did in (2.39) for the charged scalar field. In particular, we used $e^{-\beta\hat{H}}\hat{\psi}(\tau,\boldsymbol{x})e^{\beta\hat{H}} = \hat{\psi}(\tau-\beta,\boldsymbol{x})$ and $e^{\beta\mu\hat{Q}}\hat{\psi}e^{-\beta\mu\hat{Q}} = e^{-\beta\mu}\hat{\psi}$ and in third equality we used (3.12). This is the KMS-realation for fermions in a thermal equilibirium system. It implies that if $\mu = 0$, the fermionic propagator is *antiperiodic* in τ. Introducing a Fourier transformation,

$$\Delta_F(\tau,\boldsymbol{x}) = \sum\!\!\!\!\!\!\!\int \Delta_F(p_0,\boldsymbol{p})e^{-ip_0\tau - i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{3.14}$$

and imposing the KMS-condition (3.13) now directly implies:

$$e^{-\beta\mu}e^{-i\beta p_0} = -1 \quad\Rightarrow\quad p_0 = (2n+1)\pi T - i\mu \equiv \omega_{\mathrm{F}n} - i\mu. \tag{3.15}$$

so that $\sum\!\!\!\!\!\!\int \to \sum\!\!\!\!\!\!\int_F$ in (3.14). The same result can be obtained also by introducing the Fourier decomposition for the field and requiring the antiperiodicity

Fermionic propagator in momentum space We can read off the fermionic propagator in momentum space (we use the same signature for fermions and bosons in the definition of the Fourier transforms, given in appendix A) including chemical potential from (3.9):

$$\Delta_F(p_0,\boldsymbol{p}) = \frac{1}{\gamma^0(i\omega_{\mathrm{F}n}+\mu) + \boldsymbol{\gamma}\cdot\boldsymbol{p} + m}. \tag{3.16}$$

The restriction to fermionic Matsubara frequencies could have been inferred also from the antiperiodicity of the field configurations within the definition of (3.9), without the use of the KMS-relation. But it is good to know more than one avenue to a given result.

An important result which follows from the antiperiodicity is that there are no fermionic zero-modes: even the lowest fermionic Matsubara mode has a finite frequency πT. This result has far reaching consequences. For example, there can be no fermionic condensates. Also, all fermionic degrees of freedom are heavy at finite temperatures, and they can often be integrated out if one is not interested in the short range properties of a given system.

3.1.1 Free fermi gas pressure

We now perform the gaussian integration over the fermionic degrees of freedom to evaluate the partition function and the pressure of the fermionic gas. The most relevant property of Grassmann variables here is the Gaussian integral (C.10) over a discrete set of Grassmann variables η_i:

$$\int \Pi_i \mathrm{d}\eta_i^\dagger \mathrm{d}\eta_i e^{-\eta^\dagger D \eta} = \det D.$$

The path integral over fermion fields is again understood as a product of integrals over a dense grid of $\boldsymbol{x}$-values, brought to the continuum limit at the end of the calculation. We then get:

$$\begin{aligned} P(\beta,\mu) &= \frac{1}{\beta V}\log \mathcal{Z}(\beta,\mu) \\ &= \frac{1}{\beta V}\log\int[\mathcal{D}\bar{\psi}_{n,\boldsymbol{p}}][\mathcal{D}\psi_{n,\boldsymbol{p}}]\exp\Big[-\sum\!\!\!\!\!\!\int_F \bar{\psi}_n(\boldsymbol{p})[\Delta_F^{-1}]\psi_n(\boldsymbol{p})\Big] \\ &= \frac{1}{\beta V}\log\prod_{n,\boldsymbol{p}}\mathrm{Det}\big(\Delta_F^{-1}(n,\boldsymbol{p})\big) \\ &= \sum\!\!\!\!\!\!\int_F \log\mathrm{Det}\big(-\omega_{\mathrm{F}n}+i\mu+i\boldsymbol{\alpha}\cdot\boldsymbol{p}+i\gamma^0 m\big) \\ &= \sum\!\!\!\!\!\!\int_F \log\mathrm{Det}\begin{pmatrix}-\omega_{\mathrm{F}n}+i\mu+i\boldsymbol{\sigma}\cdot\boldsymbol{p} & im \\ im & -\omega_{\mathrm{F}n}+i\mu-i\boldsymbol{\sigma}\cdot\boldsymbol{p}\end{pmatrix} \\ &= 2\sum\!\!\!\!\!\!\int_F \log\Big[(\omega_{\mathrm{F}n}-i\mu)^2+\boldsymbol{p}^2+m^2\Big] \\ &= 2\sum\!\!\!\!\!\!\int_F \int^{\omega_{\boldsymbol{p}}}\mathrm{d}\omega'\frac{2\omega'}{(\omega_{\mathrm{F}n}-i\mu)^2+\omega'^2}, \end{aligned} \tag{3.17}$$

where $\sum\!\!\!\!\!\!\int_F$ refers to combined sum and integral over fermionic Matsubara frequencies. Note the factor of 2, which comes from the 4×4 structure of the Dirac matrices, for which we used the Weyl representation (A.2) (we did not display $\mathbb{1}_2$-factors explicitly). Note that the sign of μ is not physical in (3.17). Because we are summing over both all $n\in\mathbb{Z}$, we can switch the sign of μ by changing $n\to -n$.

Fermionic Matsubara sum We again compute the Matsubara sum using complex integration. This time we need just a different function to pick up the Fermionic frequencies.

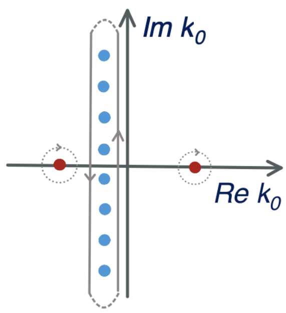

Figure 7: Complex integration contour $\mathcal{C}_1$ for appropriate for evaluating the sum over Fermionic Matsubara frequences with a chemical potential μ.

Given our experience with bosonic sums, the choice is obvious:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\omega_{Fn}-i\mu)^2+\omega'^2} = \frac{1}{2\pi i}\oint_{\mathcal{C}_1} \frac{1}{\omega'^2 - z^2}\frac{-\beta}{e^{\beta(z-\mu)}+1}$$
$$= \frac{\beta}{2\omega'}\Big(-\frac{1}{e^{\beta(\omega'-\mu)}+1} + \frac{1}{e^{-\beta(\omega'+\mu)}+1}\Big). \tag{3.18}$$

Inserting this back to (2.42) and integrating over ω' one finds the expected result:

$$P(\beta,\mu) = 2\int_{\boldsymbol{p}}\int_0^{\omega_{\boldsymbol{p}}} \mathrm{d}\omega'\Big(1-\sum_{\pm}\frac{1}{e^{\beta(\omega'\pm\mu)}-1}\Big)$$
$$= 2\sum_{\pm}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\Big(\frac{\omega_{\boldsymbol{p}}}{2}+\frac{1}{\beta}\log\big(1+e^{-\beta(\omega_{\boldsymbol{p}}\pm\mu)}\big)\Big). \tag{3.19}$$

Again, we found out old result for fermoionic system with a chemical potential, but now extended to inclulde antiparticles as well. The factor of 2 counts the spin degree of freedom.

3.2 Free Abelian gauge field

We now add the electromagnetic field to our repertoir. We quantize gauge fields only using the path integral quantization, using the Faddev-Popov method. Canonical quantization can be found for example in Laine and Vuorinen [1]. The Lagrangian density for the free electromagnetic field in the Minkowski space is

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\boldsymbol{E}^2-\boldsymbol{B}^2) = -\mathcal{L}_A^{\mathrm{E}}. \tag{3.20}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the physical electric an magnetic fields are

$$\begin{aligned} E_i &= F_{io} = \partial_0 A_i - \partial_i A_0 \\ B_i &= \tfrac{1}{2}\epsilon_{ijk}F_{jk} = (\nabla\times\boldsymbol{A})_i \end{aligned} \tag{3.21}$$

Going to Euclidean space A_μ transforms as x_μ:

$$A_\mu^{\mathrm{E}} = (-iA_0;\boldsymbol{A}). \tag{3.22}$$

Since the gauge field is just a vector composed of four scalar fields, one could naively expect that the partition function can be expressed as

$$\mathcal{Z}_{\text{naive}}(\beta) \overset{?}{=} \int[\mathcal{D}A_\mu]_\beta \exp\Big[\int_{X_{\mathrm{E}}^\beta}\mathcal{L}_A^{\mathrm{E}}\Big]. \tag{3.23}$$

As is well known from zero-temperature QFT, the integral (3.23) does not even exist. It is an ill defined quantity because one attempts to integrate over an infinite volume of physically equivalent gauge configurations. Indeed, the physical electric and magnetic fields (and the QED-Lagrangian) remain invariant in the gauge transformation:

$$A_\mu \to A_\mu^\alpha = A_\mu + \partial_\mu\alpha. \tag{3.24}$$

Here $\alpha(\tau, \boldsymbol{x})$ is an arbitrary scalar field configuration. The set of all gauge-fields that can be obtained from each others by a gauge transformation form a *gauge orbit*. In order to find a physically meaningful partition (or generating) function, we need to get rid of the identical copies and instead pick only one member of each orbit. To this end one introduces a constraint in the form

$$G[A_\mu^\alpha] \ = 0 \tag{3.25}$$

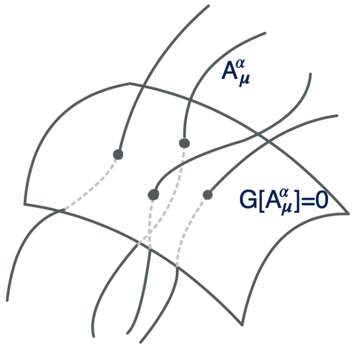

Figure 8: Schematic picture showing gauge orbits (curves) that describe the same physics and a gauge fixing surface that picks only one member of each orbit.

This idea is schematically illustrated in figure 8. In practice, one has to be very careful in imposing the constraint. The path integral quantization is based on the full equalty of all possible quantum configurations, and the gauge fixing has to be done such that one does not spoil this democracy. This is precisely what the Faddeev-Popov method does: instead of simply inserting a functional δ-distribution, one extracts the infinite gauge-volume factor from the path integral (3.23). We start by inserting a functional unit-operator into (3.23) in the form:

$$\mathbb{1} = \Delta_{\rm FP}[A_\mu] \int [\mathcal{D}\alpha]_\beta \delta(G[A_\mu^\alpha]), \tag{3.26}$$

where

$$\Delta_{\rm FP}[A_\mu] = \det\Big(\frac{\delta G[A_\mu^\alpha]}{\delta\alpha}\Big) \equiv \det(M_G). \tag{3.27}$$

is the functional Faddeev-Popove determinant. One can easily show that $\Delta_{\rm FP}[A_\mu]$ is Gauge-invariant:

$$\begin{aligned} \Delta_{\rm FP}^{-1}[A_\mu^{\alpha'}] &= \int [\mathcal{D}\alpha]_\beta \delta(G[A_\mu^{\alpha'\alpha}]) \\ &= \int [\mathcal{D}(\alpha'\alpha)]_\beta \delta(G[A_\mu^{\alpha'\alpha}]) \ = \ \Delta_{\rm FP}^{-1}[A_\mu]. \end{aligned} \tag{3.28}$$

It is then easy to show that

$$\begin{aligned} \mathcal{Z}_{\rm naive}(\beta) &= \int [\mathcal{D}A_\mu]_\beta \Big(\Delta_{\rm FP}[A_\mu] \int [\mathcal{D}\alpha]_\beta \delta(G[A_\mu^\alpha])\Big) \exp\Big[\int_{X_{\rm E}^\beta} \mathcal{L}_A^{\rm E}\Big] \\ &= \Big(\int [\mathcal{D}\alpha]_\beta\Big) \int [\mathcal{D}A_\mu]_\beta \Delta_{\rm FP}[A_\mu] \delta(G[A_\mu]) \exp\Big[\int_{X_{\rm E}^\beta} \mathcal{L}_A^{\rm E}\Big]. \end{aligned} \tag{3.29}$$

The front factor is an infinite volume integral following from integration along the gauge orbit. The remaining part is the physical partition fucnction:

$$\mathcal{Z}(\beta) = \int [\mathcal{D}A_\mu]_\beta \Delta_{\rm FP}[A_\mu] \delta(G[A_\mu]) \exp\Big[\int_{X_{\rm E}^\beta} \mathcal{L}_A^{\rm E}\Big]. \tag{3.30}$$

3.2.1 Black body radiation

It is instructive to evaluate (3.33) directly in the free field case. To do this, we have to fix the gauge. A particularily convenient choice here is to use the axial gauge condition:

$$A_3 \equiv 0. \tag{3.31}$$

In this gauge the FP-determinant takes the form:

$$\Delta_{\rm FP}[A_\mu] = \det\Big(\frac{\delta(A_3 + \partial_3\alpha)}{\delta\alpha}\Big) = \det(\partial_3). \tag{3.32}$$

Using this result and integrating over the A_3 field configurations we get:

$$\mathcal{Z}(\beta) = \det(\partial_3) \int [\mathcal{D}A_0\mathcal{D}A_1\mathcal{D}A_2]_\beta \exp\Big[\int_{X^\beta_{\rm E}} \mathcal{L}^{\rm E}_A\Big]. \tag{3.33}$$

Normally in the zero-T QFT one throws the $\det(\partial_3)$-term out, because it does not couple to dynamical fields and hence reduces to an (there) irrelevant constant. At finite T this term contributes to the pressure, cancelling a residual gauge-contribution that remains after integrating out the delta-function constraint. We now evaluate in the axial gauge (3.32):

$$\begin{aligned}\int_{X^\beta_{\rm E}} -\frac{1}{4}F^{\rm E}_{\mu\nu}F^{\mu\nu}_{\rm E}\Big|_{A_3=0} &= \frac{1}{2}\int_{X^\beta_{\rm E}} A^{\rm E}_\mu\big(\delta_{\mu\nu}\Box_{\rm E} - \partial^{\rm E}_\mu\partial^{\rm E}_\nu\big)A^\nu_{\rm E}\Big|_{A_3=0} \\ &= \frac{1}{2}\int_{X^\beta_{\rm E}} (A_0, A_1, A_2)\begin{pmatrix} \nabla^2 & -\partial_\tau\partial_1 & -\partial_\tau\partial_2 \\ -\partial_1\partial_\tau & \partial_\tau^2+\partial_2^2+\partial_3^2 & -\partial_2\partial_1 \\ -\partial_2\partial_\tau & -\partial_1\partial_2 & \partial_\tau^2+\partial_1^2+\partial_3^2\end{pmatrix}\begin{pmatrix}A_0\\A_1\\A_2\end{pmatrix}.\end{aligned} \tag{3.34}$$

Going to the Fourier space using our usual signature for $p\cdot x$ one can write this as

$$\begin{aligned}\log\mathcal{Z} &= \log\det(\partial_3) - \frac{1}{2}\log\det\begin{pmatrix}\boldsymbol{p}^2 & \omega_n p_1 & \omega_n p_2 \\ \omega_n p_1 & \omega_n^2+\boldsymbol{p}^2-p_1^2 & -p_2p_1 \\ \omega_n p_2 & -p_1p_2 & \omega_n^2+\boldsymbol{p}^2-p_2^2\end{pmatrix} \\ &= \frac{1}{2}{\rm Tr}\big[\log p_3^2\big] - \frac{1}{2}{\rm Tr}\big[\log(p_3^2(\omega_n^2+\boldsymbol{p}^2))\big] \;=\; -{\rm Tr}\big[\log(\omega_n^2+\boldsymbol{p}^2)\big]\end{aligned} \tag{3.35}$$

The $\log\det(\partial_3)$-term can be seen as arising from ghost loop in the axial gauge as indicated in figure 9. We can now read off the pressure the last line in (3.35) is already in the standard form, and we know the final result of the Matsubara summation. The black body radiation pressure then becomes:

$$\begin{aligned}P &= \frac{1}{\beta V}\log\mathcal{Z} \\ &= -2\int\frac{{\rm d}^3\boldsymbol{p}}{(2\pi)^3}\Big[\frac{|\boldsymbol{p}|}{2} + T\log(1-e^{\beta|\boldsymbol{p}|})\Big].\end{aligned} \tag{3.36}$$

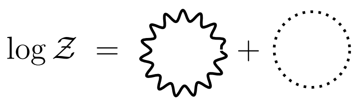

Figure 9: Gaussian determinant contributions to $\mathcal{Z}$ are visualized by vacuum loops. Wavy line describes photons and dotted line ghosts.

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If we throw out the vacuum part and make one partial integration in the thermal part, we can write this result as

$$P_T = -2J^-_T(0) = \frac{T^4}{3\pi^2}\int_0^\infty {\rm d}y\,\frac{y^3}{e^y\mp 1} = \frac{\pi^2}{45}T^4. \tag{3.37}$$

3.3 Photon propagator

Axial gauge may be somewhat cumbersome to use in perturbative calculations. A more useful gauge condition is the covariant gauge

$$G_\omega[A_\mu] = \partial^\mu A_\mu - \omega(x) = 0. \tag{3.38}$$

In this gauge

$$\Delta_{\text{FP}} = \det_\beta(\partial^2). \tag{3.39}$$

The final trick in the FP-procedure is to integrate over all possible covariant gauge-fixing functions ω with a gaussian weight $\exp[\frac{1}{2\xi}\int_{X_{\text{E}}^\beta}\omega^2]$. After this integration the partition function becomes, up to a constant:

$$\begin{aligned}\mathcal{Z}(\beta) &= \int[\mathcal{D}\omega]_\beta \exp[-\frac{1}{2\xi}\int_{X_{\text{E}}^\beta}\omega^2]\int[\mathcal{D}A_\mu]_\beta\Delta_{\text{FP}}[A_\mu]\delta(G[A_\mu])\exp\Big[\int_{X_{\text{E}}^\beta}\mathcal{L}_A^{\text{E}}\Big] \\ &= \det_\beta(\partial^2)\int[\mathcal{D}A_\mu]_\beta\exp\Big[\int_{X_{\text{E}}^\beta}\underbrace{\Big(\mathcal{L}_A^{\text{E}} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\Big)}_{\equiv\mathcal{L}_{\text{eff}}}\Big]\end{aligned} \tag{3.40}$$

WE are now back to integrating over all gauge-field contributions, but now the ξ-dependent part imposes a penalty for gauge copies corresponding to high frequencies, and the path integral is finite. This also manifests with the fact that the greens function $(\Delta_{\mu\nu}^{\text{E}})^{-1}$ in $\mathcal{L}_{\text{eff}}$

$$\int_{X_{\text{E}}^\beta}\mathcal{L}_{\text{eff}} \;=\; -\frac{1}{2}\int_{X_{\text{E}}^\beta}A_\mu^{\text{E}}\underbrace{\big(\,\delta_{\mu\nu}\Box_{\text{E}} + (1-\frac{1}{\xi})\partial_\mu^{\text{E}}\partial_\nu^{\text{E}}\,\big)}_{\equiv(\Delta_{\mu\nu}^{\text{E}})^{-1}}A_{\text{E}}^\nu, \tag{3.41}$$

can be inverted. Going to momentum space $\partial_\mu^{\text{E}} \to -ip_\mu$ and $\Box_{\text{E}} \to -\delta_{\mu\nu}ip^\mu ip^\nu = p^2$. Then $(\Delta_{\mu\nu}^{\text{E}})^{-1} \to p^2\delta^{\mu\nu} - (1-\frac{1}{\xi})p^\mu p^\nu$ and hence the Euclidean propagator in the R_ξ-gauge is:

$$\Delta_{\mu\nu}^{\text{E}} = \frac{1}{p^2}\Big(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\Big), \tag{3.42}$$

where p^μ is the bosonic 4-momentum on $S_1 \otimes \mathbb{R}^3$: $p^\mu = (\omega_n; \boldsymbol{p})$ with $\omega_n = 2\pi nT$.

Photon pressure, again We can now again compute the photon pressure using the R_ξ-gauge propagator. Indeed,

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Excercises to section 3

2.1 Show that

$$e^{-\beta(\hat{H}-\mu\hat{Q})}\hat{\psi}(0,\boldsymbol{x})e^{\beta(\hat{H}-\mu\hat{Q})} = e^{-\beta\mu}\hat{\psi}(\beta,\boldsymbol{x}).$$

2.2 Fill in all the gaps in the derivation of the gauge field action (3.35) and partition function (3.34).

2.3 Show that in the low temperature limit $m/T \gg 1$ the thermal integrals

$$J_T^{\mp}(m,T) \equiv \mp T \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \log(1 \mp e^{-\beta\omega_{\boldsymbol{p}}}) \to J_T^{\mp} = T n(m,T),$$

where $\omega_{\boldsymbol{p}}^2 = \boldsymbol{p}^2 + m^2$ and $n(m,T)$ is the Maxwell-Bolzmann number density.

2.4 Show the following identity for bosonic fields

$$\sum\!\!\!\!\!\!\int' \frac{1}{Q^{2p}} \equiv \sum_{n\neq 0} \int \frac{\mathrm{d}^3 \boldsymbol{Q}}{(2\pi)^d} \frac{\mu^{3-d}}{(\omega_n^2 + \boldsymbol{Q}^2)^p} = \frac{2\pi^{d/2}}{(2\pi)^{2p}} T^{d-2p+1} \mu^{3-d} \frac{\Gamma(p-\frac{d}{2})}{\Gamma(p)} \zeta(2p-d),$$

where $\zeta(s)$ is the Riemann zeta function.

2.5 Show that in the high-temperature limit the bosonic thermal integral has the expansion:

$$J_T^{-}(m,T) = \frac{\pi^2 T^4}{90} - \frac{m^2 T^2}{24} + \frac{m^3 T}{12\pi} + \frac{m^4}{2(4\pi)^2}\left[\log(\frac{m e^{\gamma_E}}{4\pi T}) - \frac{3}{4}\right] - \frac{m^6 \zeta(3)}{3(4\pi)^4 T^2} + \cdots$$

Note that the third term arises purely from the zero mode. Similarly for Fermions show that

$$J_T^{+}(m,T) = \frac{7}{8}\frac{\pi^2 T^4}{90} - \frac{m^2 T^2}{48} - \frac{m^4}{2(4\pi)^2}\left[\log(\frac{m e^{\gamma_E}}{\pi T}) - \frac{3}{4}\right] + \frac{7 m^6 \zeta(3)}{3(4\pi)^4 T^2} + \cdots$$

The two first 2-3 terms in these expansions are easy but the logarithmic corrections are much harder. The difficulty comes from the fact that J's are not analytic around $m=0$. The section 2.3 of Laine and Vuorinen or the classic article by Doland and Jackw [4].

2.6 Show carefully the following identitites on fermionic integrals

$$\int \prod_{i=1}^{N} \mathrm{d}\theta_i^* \mathrm{d}\theta e^{-\theta_i^* A_{ij} \theta_j} = \det A$$

$$\int \prod_{i=1}^{N} \mathrm{d}\theta_i^* \mathrm{d}\theta_k \theta_l^* e^{-\theta_i^* A_{ij} \theta_j} = \det A (A^{-1})_{kl}.$$

4 Interacting bosonic field theory

Sof far we have mostly derived known results from relativistic statistical physics, starting from the path integral representation. We will now put the machinery we have developed to good use, to study interacting theories. The explicit rotation $t \to -i\tau$ requires no changes to the formal perturbative zero-temperature QFT machinery. The only difference appears in the evaluation of loop integrals: whereas in the zero temperatures the integrals are, after the Wick-rotation defined in $\mathbb{R}^4$, the finite temperature field theory integrals are living in the semicompact space $S_1 \otimes \mathbb{R}^3$, which results in frequency integrals being replaced by frequency sums over the Matsubara frequencies. In particular the UV-structure of the theory, which is sensitive only to the short distances $|\Delta x| \ll 1/T$, is not at all altered by the compactification of one of the spatial dimensions.

Let us very briefly

4.1 Self-interacting scalar field

Consider an interacing real scalar field theory with a Lagrangian:

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + V_I(\phi) \;\equiv\; \mathcal{L}_{E0} + V_I(\phi). \tag{4.1}$$

The generating function for this system in the path integral formulation can be written as:

$$\begin{aligned}
\mathcal{Z}[\beta, j] &= \int [\mathcal{D}\phi]_\beta \exp\Big[-\int_{X_{\rm E}^\beta}\big(\mathcal{L}_{E0} + V_I(\phi) - j\phi\big)\Big] \\
&= \exp\Big[-\int_{X_{\rm E}^\beta} V_I\Big(\frac{\delta}{\delta j}\Big)\Big] \int [\mathcal{D}\phi]_\beta \exp\Big[\int_{X_{\rm E}^\beta}(\mathcal{L}_{E0} - j\phi)\Big] \\
&= \mathcal{Z}_0(\beta) \exp\Big[-\int_{\tau,\boldsymbol{x}} V_I\Big(\frac{\delta}{\delta j}\Big)\Big] \exp\Big[\frac{1}{2}\int_{X_{\rm E}^\beta}\int_{X_E^{\beta\prime}} j_{\tau'\boldsymbol{x}'}\Delta_0(\tau-\tau', \boldsymbol{x}-\boldsymbol{x}')j_{\tau,\boldsymbol{x}})\Big] \\
&= \mathcal{Z}_0(\beta)\mathcal{Z}_1[\beta, j].
\end{aligned} \tag{4.2}$$

The second step appears trivial: if one peforms all functional differentations with respcet to the source term, the second line expression reproduces the first. The equality is only formal however, as we can never perform the expansion to all all orders in practice. In the last step we used the known result for the free theory generating function and consequently Δ_0 is the free theory propagator and $\mathcal{Z}_0(\beta)$ is the free theory partition function. In terms of the Grand potential this result becomes:

$$\Omega = -\frac{1}{\beta}\log \mathcal{Z}[\beta, j] = -\frac{1}{\beta}\log \mathcal{Z}_0(\beta) - \frac{1}{\beta}\log \mathcal{Z}_1(\beta) = \Omega_0 + \delta\Omega. \tag{4.3}$$

Expanding the opertor $V_I(-\delta/\delta j)$ consistently in powers of coupling constants, creates the *perturbative expansion* for the partition function and physical quantitites that can be derived from it. Perturbative expansions for thermal Greens functions can then be derived taking appropriate number of derivatives with respect to the source.

Themal expansion for the partition function Let us study this explicitly in the case with quartic self-interaction:

$$V_I(\phi) = \frac{\lambda}{4!}\phi^4. \tag{4.4}$$

To compute the partition function we must first perform the partial derivatives with respect to j and then set $j \equiv 0$.

$$\begin{aligned}\log \mathcal{Z}_1(\beta) &= \log\left[e^{-\int V_I(\delta/\delta j)} e^{\frac{1}{2}\iint j\Delta_0 j}\right]_{j=0} \\ &= \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\Big[\int V_I\Big(\frac{\delta}{\delta j}\Big)\Big]^k e^{\frac{1}{2}\iint j\Delta_0 j}\Big|_{j=0}^{\text{connected}} \\ &= -\int V_I\Big(\frac{\delta}{\delta j}\Big)e^{\frac{1}{2}\iint j\Delta_0 j}\Big|_{j=0} + \dots, \end{aligned} \tag{4.5}$$

where we were used an even more compact notation $\int \equiv \int_{X_E^\beta}$. This produces the well known perturbative expansion of all vacuum diagrams contributing to $\mathcal{Z}(\beta)$. As is well known, the similar series for the Grand potential, which is the logarithm of $\mathcal{Z}$, contains only the subset of *connected* diagrams. We can now compute the lowest order pertrubative correction to $\delta\Omega$:

$$\begin{aligned}\delta\Omega_{(1)} &= \frac{1}{\beta}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\,\frac{\lambda}{4!}\frac{\delta^4}{\delta j(\tau,\boldsymbol{x})^4}e^{\frac{1}{2}\iint j\Delta_0 j}\Big|_{j=0,\text{connected}} \\ &= \frac{1}{\beta}\frac{\lambda}{4!}\frac{1}{2!}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\frac{\delta^4}{\delta j(\tau,\boldsymbol{x})^4}\Big[\frac{1}{2}\iint j\Delta_0 j\Big]^2 \\ &= \frac{\lambda}{8\beta}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\left[\Delta_0(0)\right]^2 \\ &= \frac{\lambda}{8\beta}\beta V\left[\Delta_0(0)\right]^2 = \frac{\lambda V}{8}\Big[\sum\!\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_{\boldsymbol{p}}^2}\Big]^2 \end{aligned} \tag{4.6}$$

In the third line we noted that the propagator Δ_0 gets to be evaluated at coinciding space-time points. This quantity is indpendent of the coordinates, which results in the integral giving out the volume factor βV and finally we wrote the propagator in the momentum space representation. We thus get a perturbative correction to the pressure:

$$\delta P = -\frac{\delta\Omega_{(1)}}{V} = -\frac{\lambda}{8}\Big[\sum\!\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_{\boldsymbol{p}}^2}\Big]^2. \tag{4.7}$$

this result corresponds to the vacuum diagram “eight”, displayed in figure 10 and we could have derirved the expression (4.7) directly using the Feynman rules given in the figure 11. Our definition is to define vertices without any combinatoric factors, so for each diagram the combinatoric factor must be comptuted including all non-equivalent ways to construct the diagram in question. The problem is of course that the correction (4.7) is, as expected, UV-divergent and needs to be renormalized before a physically sensible contribution to the pressure can be identified.

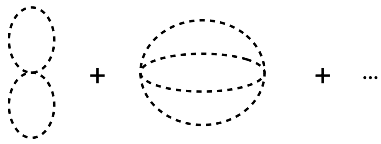

Figure 10: The first two perturbative corrections contributing to the grand potential $\delta\Omega$.

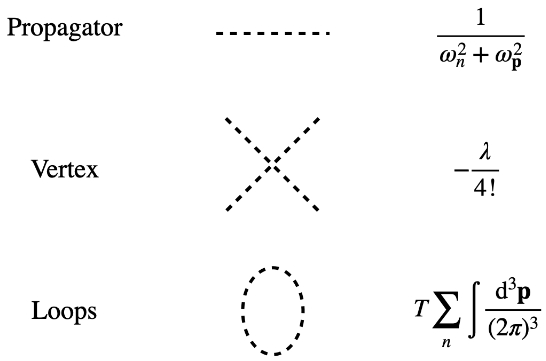

Figure 11: Finite temperature field theory Feynman rules for real scalar field with quartic self interaction, corresponding to imaginary time path shown in figure 3.

4.2 Renormalization in FTFT.

In finite temperature field theory calculations we enounter the same UV-divergences as the zero temperature field theory. On the other hand, the finite temperature does not induce any new UV-divergences. This is easy to undrestand qualitatively from the point of view of loop integrals, wher the finite-T field theory can be seen as a deformation of the vacuum 4d field theory from $\mathbb{R}^4$-the space to $S_1 \otimes \mathbb{R}^3$-space, with one of the spatial dimensions is compactified on a circle. This compactification obviously has no effect on the short distance physics of the theory, which is where the UV-divergences come from. From technical point the issue is less trivial. With higher order perturbative corrections some fine-tuned conspiracies are clearly needed so that no T-dependent sub-divergences emerge. The issue becomes even more complex when one introduces resummations of infinite subsets of diagrams, which is required to cancel some IR-divergences appearing in perturbative computations.

Fundamentally renormalization is needed because the Lagrangian parameters are all associated with *local* operators (with several fields evaluated at the same space-time point), which can never be measured directly, since all physical measurements have some finite resolution. Instead, we have to parametrize the theory based on some set of either measured or othewise defined quantitites, which neessarily depend on some finite spatial resolution or the associated energy scale. This unavoidably leads to scale dependence of all QFT parameters. This phenomenon of *running couplings* can be studied quantitatively by setting up renormalization group equations based on the scale dependence of some fundamental

Greens functions. We shall briefly go through these theory structures to be self contained in our analysis of the finite-T renormalization.

BPHZ-method We study the renormalization procedure using the simple scalar field theory of (4.1) and (4.4) as and example:

$$\mathcal{L}(\phi, \lambda, m) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{4.8}$$

Here the mass m^2 and coupling λ and the field ϕ itself are non-observable *bare* parameters. We introduce the physical parameters using the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) method, expressing the bare parameters in terms of some physical parameters, denoted by index R:

$$\phi \equiv Z_\phi^{1/2}\phi_{\rm R}, \qquad \lambda \equiv \lambda_{\rm R} + \delta\lambda, \qquad m^2 \equiv m_{\rm R}^2 - \delta m^2, \tag{4.9}$$

Rescalind the field and rewriting the bare mass and couplings in terms of the renormalized operators, one can now rewrite the original Lagrangian as

$$\begin{aligned}\mathcal{L}(\phi, \lambda, m) =\; & \mathcal{L}(\phi_R, \lambda_R, m_R) + \frac{\delta_\phi}{2}(\partial_\mu \phi_{\rm R})^2 - \frac{1}{2}\delta_m \phi_{\rm R}^2 - \frac{1}{4!}\delta_\lambda \phi_{\rm R}^4 \\ \equiv\; & \mathcal{L}_{\rm R} + \delta\mathcal{L} \;\; \equiv \mathcal{L}_{\rm R0} + V_R(\phi_R),\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}&\delta_\phi \equiv Z_\phi - 1, \\ &\delta_\mu \equiv Z_\phi\big(m_{\rm R}^2 + \delta m^2\big) - m_{\rm R}^2, \\ &\delta_\lambda \equiv Z_\phi^2\big(\lambda_{\rm R} + \delta\lambda\big) - \lambda_{\rm R},\end{aligned} \tag{4.11}$$

The terms containing δ_ϕ, δ_m and δ_λ are absorbed into the renormalized potential $V(\phi_R)$. This means they are treated perturbatively and they introduce the new interaction terms. The corresponding momentum space Feynman rules are shown in figure 12. Because δ_x's merely express relations between observable and non-observable parameters, they can be adjusted at will order by order in perturbation theory, as dictated by the chosen renormalization conditions.

Renormalization schemes Renormalized parameters depend on the chosen scheme. Different choices lead to different sets of renormalized parameters, which are related to each others by finite relations. For example, in the on-shell scheme, we require that the pole of the propagator is physical on-shell mass. This means setting

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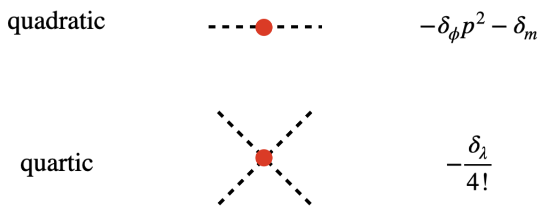

Figure 12: Additional FTFT Feynman rules for the counter-terms for the real scalar field in the BPHZ-schem.

Excercises to section 4

4.1 Show explicitly the following identities

$$I_0(m) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{k^2 - m^2}.$$

and

$$J_0(m) = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\omega}{2} = i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \log(k^2 - m^2)$$

Consider these from the point of view of the contour integration and the dimensional regularization. Can you prove these also using the cut-off regularization?

4.2 Consider a theory defined by the Lagrangian fucntion

$$\mathcal{L} = \sum_{i=1}^{2} \left[\frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{m_i^2}{2}\phi_i^2 - \frac{\lambda}{4!}\phi_i^4\right] - g(\phi_1\phi_2^2 + \phi_2\phi_1^2).$$

Compute the thermal self-energy functions at one-loop level. Compute the thermal integrals explicitly in the high temperature limit $T \gg m_i$. Perform also the vacuum reonrmalization (see appendix for details) assuming that $\lambda_i, g > 0$, such that the theory has a symmetric vacuum state.

4.3 Compute carefully the thermal integrals

5 IR-divergences and resummations

5.1 Superdaisy resummation, Gap equation

6 Other interacting field theories

6.1 Yukawa theory

6.2 Non-Abelian gauge fields

6.3 Standard model

7 Effective action

7.1 One-loop effective action

8 Applications

8.1 Phase transition parameters

8.2 Bubble nucleation

9 Applications

9.1 Bubble nucleation

9.2 Bubble growth and coalescence

9.3 Spahaleron rate

10 Real-time methods

10.1 Self-consistent renormalization and resummation at Hartree level

Appendices

In these appendices we collect a large number of notations and definitions made throughout these lecture notes. We also complement the discussion in the text by several appendices with highly relevant, but technical calculations.

A Notations and conventions

In these notes we are very often transforming between spaces with Minkowski and Euclidean metrics. All zero-temperature 4d-loop integrals are computed using a Wick rotation to $\mathbb{R}^4$-space, and the imaginary time formulation of thermal field theory is inherently living in partly compactified Euqlidean space $S_1 \otimes \mathbb{R}^3$. Here we collect some of our basic definitions regarding 4-momenta, inner products and so on. The Widk rotation from the Minkowski space M to Euclidean space E, is effected by $t \to -it$ and $k_0 \to ik_0$. This results in number of substitution rules:

$$
\begin{aligned}
\mathrm{d}^4 x_M &\to -i\mathrm{d}^4 x_E \\
\mathcal{L}_M &\to -\mathcal{L}_E \\
iS_M &\to -S_E \\
\eta_{\mu\nu} &\to -\delta_{\mu\nu} \\
(p\cdot q)_M &\to (p\cdot q)_E \\
\not{k}_M &\to -\not{k}_E \\
\not{\partial}_M &\to -\not{\partial}_E \\
\gamma^\mu_E &\equiv (i\gamma^0; \boldsymbol{\gamma}) \\
\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} &\to \{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu}
\end{aligned}
\qquad (A.1)
$$

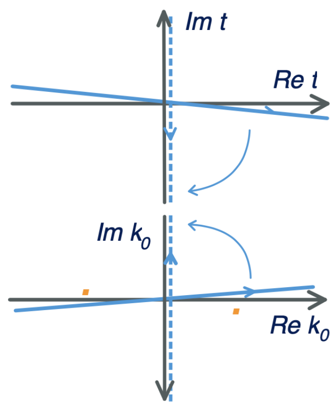

Figure 13: Wick rotations in the time and frequency planes.

We define the Euclidean metric with fully negative signature $g^E_{\mu\nu} \equiv -\delta_{\mu\nu}$ and the Minkowski metric with mostly negative signature $g^M_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. These rules imply that on-shell $m^2 = p_M^2 = -p_E^2$. Note that replacement rules do not constrain the definition of $x\cdot p$ and we keep this quantity the same in both space-times: $x \cdot p = x_0 p_o - \boldsymbol{x}\cdot\ \boldsymbol{p}$.

Dirac matrices Dirac matrices in the Weyl representation are given by

$$
\gamma^5 \equiv \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad (A.2)
$$

where $\mathbb{1}_2$ is the 2-dimensional unit matrix and the Pauli matrices are:

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (A.3)
$$

Integral shorthands We often need to integrate and sum over the space-time or fourier mode variables in different spaces $S_1^\beta \otimes \mathbb{R}^3$, $\mathbb{R}^4$ and Minkowski space. To simplify the notation we will be using a number of shorthands. For the $S_1^\beta \otimes \mathbb{R}^3$-spcace integral we often use the shorthands listed below:

$$\int_{X_{\rm E}^\beta} \equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x} \equiv \int_0^\beta \mathrm{d}\tau \int_{\boldsymbol{x}}, \qquad \int_{X_{\rm E}} \equiv \int \mathrm{d}^4 x_E, \qquad \int_{X_{\rm M}} \equiv \int \mathrm{d}^4 x_M. \tag{A.4}$$

where the index E refers to Euclidean and M to Minkowski space. Similarly, we will use

$$\int_{K_{\rm E}} \equiv \int \frac{\mathrm{d}^4 k_{\rm E}}{(2\pi)^4}, \qquad \int_{K_{\rm M}} \equiv \int \frac{\mathrm{d}^4 k_{\rm M}}{(2\pi)^4} \tag{A.5}$$

and in particular

$$T \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \equiv T \sum_{n=-\infty}^{\infty} \int_{\boldsymbol{k}} \equiv \not{\Sigma}\!\!\!\!\int. \tag{A.6}$$

Fourier transformations We often use normalization to a finite box, whose volume is then brought to infinity at the end of the calculation. We write some of the relevant formulae here to facilitate these calculations. First, consider a system in a finite box of length L centered at $x = 0$. Then the discrete Fourier transformation is defined as

$$\varphi(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \hat{\varphi}_n e^{i(2n+s)\frac{\pi x}{L}} \quad \Leftrightarrow \quad \hat{\varphi}_n = \int_{-L/2}^{L/2} \mathrm{d}x\, \varphi(x) e^{-i(2n+s)\frac{\pi x}{L}}. \tag{A.7}$$

where $s = 0$ for periodic (bosonic) and $s = 1$ for antiperiodic (fermionic) boundary conditions. If we now define $k_n = 2\pi n/L$, $\hat{\varphi}_n \equiv \hat{\phi}(k_n)\mathrm{d}k$ and $\mathrm{d}k = 2\pi/L$ and take the limit $L \to \infty$, we get the continuous Fourier transforms

$$\varphi(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \hat{\varphi}(k) e^{ikx} \quad \Leftrightarrow \quad \hat{\varphi}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \varphi(x) e^{-ikx}. \tag{A.8}$$

Going to more than one dimension is straightforward and we just list our conventions for different spacetimes starting from the 4d-Euclidean and the Minkowski spaces:

$$\varphi(x) = \int_{k_{\rm M,E}} \hat{\varphi}(k) e^{-ik\cdot x} \quad \Leftrightarrow \quad \hat{\varphi}(k) = \int_{x_{\rm M,E}} \varphi(x) e^{ik\cdot x}, \tag{A.9}$$

where $k \cdot x = k_0 t - \boldsymbol{k}\cdot\boldsymbol{x}$ in Minkowski and $k \cdot x = k_0\tau - \boldsymbol{k}\cdot\boldsymbol{x}$ in Euclidean space. We define the Fourier-transforms in the fully ($S_1^\beta \otimes (S_1^L)^3$) and partly ($S_1^\beta \otimes \mathbb{R}^3$) compactified spaces of interest following the same sign-convention:

$$\varphi(\tau, \boldsymbol{x}) = \frac{1}{\beta V} \sum_{n=-\infty}^{\infty} \sum_{\{m\}} \hat{\varphi}_{n,\{m\}} e^{-i\omega_n\tau + i\boldsymbol{k}_{\{m\}}\cdot\boldsymbol{x}} \quad \xrightarrow{L\to\infty} \quad \not{\Sigma}\!\!\!\!\int \hat{\varphi}_n(\boldsymbol{k}) e^{-i\omega_n\tau + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.10}$$

where $\omega_n = (2n+s)\pi/\beta$ and $\boldsymbol{k}_{\{m\}} \equiv (k_{m1}, k_{m2}, k_{m3})$ with $k_{mi} = (2m_i + s)\pi/L$, where $V = L^3$ and $m_i \in \mathbb{Z}$. Usually we omit the discrete indices in the box-normalized Fourier transform, denoting just $\boldsymbol{k}_{\{m\}} \to \boldsymbol{k}$. The inverse transformations are

$$\hat{\varphi}_{n,\{m\}} = \int_0^\beta \mathrm{d}\tau \int_{-L/2}^{L/2} \prod_{i=1}^{3} \mathrm{d}x_i \, \varphi(\tau, \boldsymbol{x}) e^{i\omega_n \tau - i\boldsymbol{k}_{\{m\}}\cdot\boldsymbol{x}}$$
$$\hat{\varphi}_n(\boldsymbol{k}) = \int_{X_\mathrm{E}^\beta} \varphi(\tau, \boldsymbol{x}) e^{i\omega_n \tau - i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{A.11}$$

B Transition amplitude and path integral integral

Path integral for transition amplitude is usually derived from the operator formalism. It is also possible to define the transition amplitude as a path integral and derive the operator formalism from it. We take this approach here. First consider a particle moving in 1-dimension in a potential $V(q)$. The probability for the particle to move from position q_a at time t_a to a position q_b at time $t_b$is defined as:

$$P(t_a, q_a; t_b, q_b) = |K(t_a, q_a; t_b, q_b)|^2, \tag{B.1}$$

where the transition amplitude $K(t_a, q_a; t_b, q_b)$ is a linear superposition of phase factors over all possible paths connecting the two points:

$$K(t_b, q_b; t_a, q_a) \equiv \sum_{\text{all paths}} k e^{iS[q,\dot{q}]/\hbar}, \tag{B.2}$$

where $S[q, \dot{q}]$ is the classical action for the system. The constant factor k is related to the path integral measure, and it is defined from the natural condition.

$$K(t_b, q_b; t_a, q_a) = \sum_c K(t_b, q_b; t_c, q_c) K(t_c, q_c; t_a, q_a) \tag{B.3}$$

All paths contribute a mere phase factor and are hence equally likely; the classical correspondence and even causality in relativistic field theory arise from constructive quantum interference effects. Emergence of the classical regime is then clearly and explicitly controlled by the size of the Planck constant $\hbar$.

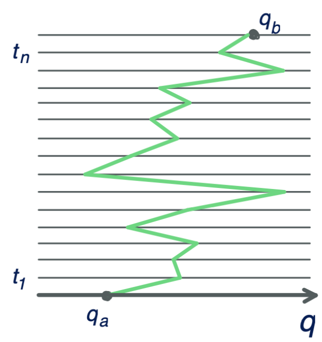

Figure 14: Example of a discrtized path between points q_a and q_b.

Path integral measure We start with the path integral for the simple harmonic oscillator in real time domain. In figure 14 we show a sample discretized path between points q_a and q_b. The discretized path integral measure is

$$\sum_{\text{paths}} = \lim_{N\to\infty} k_N \prod_{i=1}^{N} k_N \int \mathrm{d}q_i \equiv \int [\mathcal{D}q]. \tag{B.4}$$

The need for the one "extra k_N-factor becomes obvious shortly. Similarly, the discretized action becomes

$$S = \int \mathrm{d}t L = \int \mathrm{d}t\big(\tfrac{1}{2}m\dot{q}^2 - V(q)\big) \to \sum_i \Big(\frac{m}{2}\frac{(q_{i+1}-q_i)^2}{\epsilon} - \epsilon V\big(\frac{q_{i+1}+q_i}{2}\big)\Big). \quad \text{(B.5)}$$

Again, our choice of the symmetrical combination as the argument of the discreteized potential function become evident shortly. To fix k_N and to show that $K(t_b, q_b; t_a, q_a)$ obeys the expected equation for the time-evolution operator, we apply the completeness relation (B.3) choosing t_c as the last time-sclice in the discretized path:

$$\begin{aligned} K(t_b, q_b; t_a, q_a) &= k_N \int_{-\infty}^{\infty} \mathrm{d}q' \exp\Big[\frac{i}{\hbar}\Big(\frac{m}{2}\frac{(q_b-q')^2}{\epsilon} - \epsilon V\big(\frac{q_b+q'}{2}\big)\Big)\Big] K(t_b-\epsilon, q'; t_a, q_a) \\ &\approx k_N \int_{-\infty}^{\infty} \mathrm{d}\delta q \, \exp\Big(\frac{im}{2\hbar\epsilon}\delta q^2\Big)\Big[1 - \frac{i\epsilon}{\hbar}V(q_b) + \cdots\Big] \times \\ &\qquad \times \Big(1 + \delta q \frac{\partial}{\partial q_p} + \frac{1}{2}\delta q^2 \frac{\partial^2}{\partial q_b^2} + \cdots\Big) K(t_b-\epsilon, q_b; t_a, q_a). \quad \text{(B.6)} \end{aligned}$$

The first observation here is that the existence of the gaussian integral requires tilting of the time-path shown in the figure 13

$$t \to (1-i\delta)t \quad \Rightarrow \quad \epsilon \to (1-i\delta)\epsilon. \quad \text{(B.7)}$$

As a result of this tilt, gaussian integral in (B.6) is well defined and can be performed with the usual formula:

$$\int_{-\infty}^{\infty} \mathrm{d}y y^{2n} e^{-by^2} = (-1)^n \partial_b^n \sqrt{\frac{\pi}{b}}. \quad \text{(B.8)}$$

Integrals over odd powers vanish by symmety. Regulator can be removed after integrations and we obtain

$$K(t_b, q_b; t_a, q_a) = k_N \sqrt{\frac{2i\pi\hbar\epsilon}{m}} \Big(1 - \frac{i\epsilon}{\hbar}V(q_b) + \frac{i\epsilon\hbar}{2m}\frac{\partial^2}{\partial q_b^2}\Big) K(t_b-\epsilon, q_b; t_a, q_a), \quad \text{(B.9)}$$

where we dropped all terms that are higher than first order in ϵ. One first observes that to leading order in ϵ, when $\epsilon \to 0$, one must have

$$k_N = \sqrt{\frac{m}{2i\pi\hbar\epsilon}}. \quad \text{(B.10)}$$

After this identification, one can easily turn (B.11) into a differential equation

$$i\hbar\frac{\partial}{\partial t}K(t, q; t_a, q_a) = \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\Big)K(t, q_b; t_a, q_a). \quad \text{(B.11)}$$

Which is just the Schrodinger equation obeyed by the time-evolution operator. One can also apply the formula without any integration xxx

Connection to operator picture

$$\sqrt{\frac{ia}{\pi}} \int_{-\infty}^{\infty} e^{iap^2 + ibp} = e^{-ib^2/4a} \quad \text{(B.12)}$$

Imaginary time paths

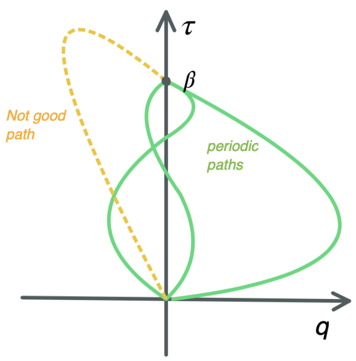

Figure 15: Examples of consistent periodic paths in τ (green) and a nonconsistent path (orange, dashed).

C Grassmann variables

For fermionic path-integral we needed to introduce anticommuting Grassmann variables. In the main text we assumed that the reader is familiar with these structures, but we give a brief review here as a reference and for the benefit of those less familiar with the subject. Suppose that θ_i and θ_j are Grassman valued variables, or G-numbers. The fundamental realation then is:

$$\theta_i\theta_j = -\theta_j\theta_i. \tag{C.1}$$

It then follows immediately that $\theta_i^2 = 0$. This has far reaching consequences. For example the most complicated function of G-number θ is

$$\phi(\theta) = a + b\theta \tag{C.2}$$

where $a, b \in \mathbb{C}$. Then many elementary functions acting on G-numbers are redundant. For example

$$e^{a\theta} = 1 + a\theta = \frac{1}{1 - a\theta} \tag{C.3}$$

and so on. Integration over G-numbers is defined to be translationally invariant:

$$\int \mathrm{d}\theta\phi(\theta) = \int \mathrm{d}\theta\phi(\theta + \xi), \tag{C.4}$$

where ξ is another G-number and $\mathrm{d}\theta$ is also anticommuting G-nubmer. Using (C.7) this implies that

$$a \int \mathrm{d}\theta + b \int \mathrm{d}\theta\theta = (a - b\xi) \int \mathrm{d}\theta + b \int \mathrm{d}\theta\theta \tag{C.5}$$

Since this must hold for all ϕ (a and b) and ξ, we must have $\int \mathrm{d}\theta = 0$. We furthermore set $\int \mathrm{d}\theta\theta \equiv 1$, so that

$$\int \mathrm{d}\theta(a + b\theta) = b. \tag{C.6}$$

Grassmann integral is then same as Grassmann derivative $\int \mathrm{d}\theta\phi(\theta) = \partial_\theta\phi(\theta)$. Note also that because of anticommutation rules:

$$\int \mathrm{d}\theta\mathrm{d}\eta\, \eta\theta = 1. \tag{C.7}$$

Any odd permutation of the four factors in the prodcut $\mathrm{d}\theta\mathrm{d}\eta\eta\theta$ changes the sign of the integral. Let us now define complex conjugated G-numbers.

$$\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2), \quad \theta^* = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \tag{C.8}$$

That is, complex conjugation does nothing (of course, it lives in different space) for G-numbers. Then from the above interation rules, we find $\int \mathrm{d}\theta\mathrm{d}\theta^*\theta^*\theta = 1$ and then

$$\int \mathrm{d}\theta^*\mathrm{d}\theta e^{-\theta^* b\theta} = b. \tag{C.9}$$

It is straightforward to extend this to an arbitrary N-vector of G-nubers θ_i and θ_i^*:

$$\begin{aligned}\int \prod_{i=1}^{N} \mathrm{d}\theta_i^*\mathrm{d}\theta_i e^{-\sum_{ij}\theta_i^* A_{ij}\theta_j} &= \int \prod_{i=1}^{N} \mathrm{d}\theta_i^*\mathrm{d}\theta_i \frac{(-1)^N}{N!}\big(\sum_{jl}\theta_j^* A_{jk}\theta k\big)^N \\ &= \int \prod_{i=1}^{N} \mathrm{d}\theta_i\mathrm{d}\theta_i^* \prod_{j=1}^{N}\theta_j^*\big(\sum_{k_j} A_{jk_j}\theta_{k_j}\big) \\ &= \int \prod_{i=1}^{N} \mathrm{d}\theta_i\mathrm{d}\theta_i^* \sum_{\mathrm{perm}} A_{1k_2}\cdots A_{Nk_N}\theta_1^*\theta_{k_1}\cdots\theta_N^*\theta_{k_N} \\ &= \epsilon_{k_1,\ldots,k_N} A_{1k_2}\cdots A_{Nk_N} \;=\; \det(A).\end{aligned} \tag{C.10}$$

Fermionic path integral To construct a fermionic path integral it is useful to define the eigenstates of Grassmann coordinates θ (one assumes that creation and annihilation operators anticommite with Grassmann numbers θ: $\{\theta, \hat{a}\} = 0$, *etc.*):

$$\begin{aligned}|\theta\rangle &= e^{-\theta\hat{a}^\dagger}|0\rangle = (1-\theta\hat{a}^\dagger)|0\rangle \;\Rightarrow\; \hat{a}|\theta\rangle = \theta|\theta\rangle = \theta|0\rangle \\ \langle\theta| &= \langle 0|e^{-\hat{a}\theta^*} = \langle 0|(1-\hat{a}\theta^*) \;\Rightarrow\; \langle\theta|\hat{a} = \langle\theta|\theta^* = \langle 0|\theta^*\end{aligned} \tag{C.11}$$

These are the fermionic equivalent of theof the position coordinate eigenstates for the bosonic system. These definitions imply the following normalization:

$$\langle\theta|\theta'\rangle = \langle 0|(1-\hat{a}\theta^*)(1-\theta'\hat{a}^\dagger)|0\rangle = 1+\theta^*\theta' = e^{\theta^*\theta'}. \tag{C.12}$$

With this normalization one finds the expected unit operator:

$$\begin{aligned}\int \mathrm{d}\theta^*\mathrm{d}\theta e^{-\theta^*\theta}|\theta\rangle\langle\theta| &= \int \mathrm{d}\theta^*\mathrm{d}\theta(1-\theta^*\theta)(1-\theta\hat{a}^\dagger)|0\rangle\langle 0|(1-\hat{a}\theta^*) \\ = \int \mathrm{d}\theta^*\mathrm{d}\theta\big(-\theta^*\theta|0\rangle\langle 0| + \theta\hat{a}^\dagger|0\rangle\langle 0|\hat{a}\theta^*\big) &\;=\; |0\rangle\langle 0| + |1\rangle\langle 1| \;=\; \mathbb{1}.\end{aligned} \tag{C.13}$$

Similarly, one can prove that the trace of a given operator $\hat{A}$ is

$$\begin{aligned}\int \mathrm{d}\theta^*\mathrm{d}\theta e^{-\theta^*\theta}\langle -\theta|\hat{A}|\theta\rangle &= \int \mathrm{d}\theta^*\mathrm{d}\theta(1-\theta^*\theta)\langle 0|(1+\theta\hat{a}^\dagger)\hat{A}(1-\hat{a}\theta^*)|0\rangle \\ = \int \mathrm{d}\theta^*\mathrm{d}\theta\big(-\theta^*\theta\langle 0|\hat{A}|0\rangle - \langle 0|\hat{a}\theta^*\theta\hat{a}^\dagger|0\rangle\big) &\;=\; \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle \;=\; \mathbb{1}.\end{aligned} \tag{C.14}$$

Here we assumed implicitly that $\hat{A}$ is bosonic, *i.e.* it commutes with the Grassmann variables θ and θ^*. It is important to note that the trace requires using *antiperiodic* configuration with respcect to the initial and final states.

We can now use these results to write the Partition function as a path integral by splicing the contour to N equal length intervals $\epsilon = \beta/N$:

$$\begin{aligned}
&\int \mathrm{d}\theta^* \mathrm{d}\theta e^{-\theta^*\theta} \langle -\theta | e^{-\beta \hat{H}} | \theta \rangle \\
&\int \mathrm{d}\theta^* \mathrm{d}\theta e^{-\theta^*\theta} \langle -\theta | e^{-\epsilon \hat{H}} \mathbb{1} e^{-\epsilon \hat{H}} \mathbb{1} \cdots \mathbb{1} e^{-\epsilon \hat{H}} | \theta \rangle. \qquad \text{(C.15)}
\end{aligned}$$

Let us label the unit operators from right to left with index $i = 1, ..., N$ and call $\theta_1 = \theta$ and $\theta_{N+1} = -\theta$. Now observe that in each position involving the unit operator, we encounter a term:

$$\begin{aligned}
e^{-\theta^*_{i+1}\theta_{i+1}} \langle \theta_{i+1} | e^{-\beta \hat{H}(\hat{a}^\dagger, \hat{a})} | \theta_i \rangle &= e^{-\theta^*_{i+1}\theta_{i+1}} \langle \theta_{i+1} | \theta_i \rangle e^{-\epsilon H(\theta^*_{i+1}, \theta_i)} \\
&= \exp \Big[-\theta^*_{i+1}\theta_i + \theta_{i+1}\theta_i - H(\theta^*_{i+1}, \theta_i) \Big] \\
&= \exp \Big(-\epsilon \Big[\theta^*_{i+1} \Big(\frac{\theta_{i+1} - \theta_i}{\epsilon} \Big) + H(\theta^*_{i+1}, \theta_i) \Big] \Big) \qquad \text{(C.16)}
\end{aligned}$$

The rightmost point obeys this rule with $\theta_1 = \theta$ and the leftmost point with $\theta_{N+1} = -\theta$. One then has

$$\begin{aligned}
\mathcal{Z}(\beta) &= \int \mathrm{d}\theta^* \mathrm{d}\theta e^{-\theta^*\theta} \langle -\theta | e^{-\beta \hat{H}} | \theta \rangle \\
&= \int \prod_{i=1}^{N} \mathrm{d}\theta^*_i \mathrm{d}\theta_i \exp \Big(-\epsilon \sum_{i=1}^{N} \Big[\theta^*_{i+1} \Big(\frac{\theta_{i+1} - \theta_i}{\epsilon} \Big) + H(\theta^*_{i+1}, \theta_i) \Big] \Big) \\
&\xrightarrow{N\to\infty} \int [\mathcal{D}\theta^* \mathcal{D}\theta]_{\bar{\beta}} \exp \Big(-\int_0^\beta \mathrm{d}\tau \Big(\theta^* \partial_\tau \theta + H(\theta^*, \theta) \Big) \qquad \text{(C.17)}
\end{aligned}$$

where $\theta(\tau)$ is a Grassmann valued field.

A Dirac field can be composed from four independent G-number fields in each space-time point. Extending the previous result to spatial variable is a simple question of first labeling the discrete spatial coordinates and then taking the continuum limit. Moreover, since $\det(\gamma^0) = 1$, we can replace $\psi^\dagger$ by $\bar{\psi}$ in the measure, and define the Dirac field path integral as in the text:

$$\mathcal{Z}(\beta) \equiv \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]_{\bar{\beta}} \exp \Big[-\int_{X_{\mathrm{E}}^\beta} \bar{\psi} \Delta_F^{-1} \psi \Big] = \det(\Delta_F^{-1}). \qquad \text{(C.18)}$$

Where we $\det(\Delta_F^{-1})$ is a functional determinant that can then defined as the continuum limit of (C.10).

D Euclidean space integrals in dimensional regularization

Dimensional regularization is the convenient way to isolate the singularities in the integrals arising from loop expansions and vacuum contributions to effective action. The idea is familiar from QFT-textbooks and we give minimum amount of details, mainly to fix our conventions. Given a n-dimensional UV-divergent integral, we continue the integral into $d = n - \epsilon$ dimensions. A standard integral that one encounters in thermal field theory is

$$\boxed{\Phi(m,d,\alpha) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \frac{1}{(m^2)^{\alpha - \frac{d}{2}}}.} \tag{D.1}$$

The right hand side can be expanded in a Laurent series in ϵ near any integer dimension using the properties of the Γ-functions. We list some of the relevant properties below:

$$\begin{aligned}
\Gamma(s+1) &= s\Gamma(s) \\
\Gamma(1/2) &= \sqrt{\pi}, \qquad \Gamma(-1/2) = -2\sqrt{\pi} \\
\Gamma(\epsilon) &\approx \frac{1}{\epsilon} - \gamma_E \\
\Gamma(-1+\epsilon) &\approx -\frac{1}{\epsilon} - 1 + \gamma_E \\
\Gamma(-2+\epsilon) &\approx \frac{1}{2\epsilon} + \frac{3}{4} - \frac{1}{2}\gamma_E
\end{aligned} \tag{D.2}$$

where the Euler-Mascheroni constant is $\gamma_E \approx 0.577215664901$. Moreover, one often needs to use the formula: $a^\epsilon = e^{\epsilon \log a} \approx 1 + \epsilon \log a$. For example, the vacuum contribution to the pressure, continued to dimension $3 - \epsilon$, can now be evaluated to give:

$$\begin{aligned}
J_0(m) \equiv \mu^\epsilon \int \frac{\mathrm{d}^{3-\epsilon} p}{(2\pi)^4} \frac{1}{2}\sqrt{p^2+m^2} &= \frac{1}{2}\mu^\epsilon \Phi(m, 3-\epsilon, -\tfrac{1}{2}) \\
&= \frac{m^4}{(4\pi)^{3/2}} (4\pi \frac{\mu^2}{m^2})^{\epsilon/2} \frac{\Gamma(-2+\frac{\epsilon}{2})}{\Gamma(-1/2)} \\
&\approx -\frac{m^4}{64\pi^2}\Big(\frac{2}{\epsilon} - \gamma_E + \log 4\pi + \log\frac{\mu^2}{m^2} + \frac{3}{2}\Big) \\
&\equiv -\frac{m^4}{64\pi^2}\Big(\frac{2}{\epsilon_{\overline{\mathrm{MS}}}} + \frac{3}{2} + \log\frac{\mu^2}{m^2}\Big).
\end{aligned} \tag{D.3}$$

where we defined a shorthand

$$1/\epsilon_{\overline{\mathrm{MS}}} \equiv 1/\epsilon + \log(4\pi) - \gamma_E, \tag{D.4}$$

This is the term subtracted from the infinite integral in the MS-bar scheme. A friendly advice is that it is best to perform all these expansions using Mathematica or an equivalent. Similarly, the 4d-vacuum scalar bubble diagram, can be evaluated

$$iA_0(m) \equiv \mu^\epsilon \int \frac{\mathrm{d}^{4-\epsilon} p}{(2\pi)^4} \frac{1}{p^2+m^2} = \mu^\epsilon \Phi(m, 4-\epsilon, 1) \tag{D.5}$$

$$\begin{aligned}
&= \frac{m^2}{(4\pi)^2} (4\pi \frac{\mu^2}{m^2})^{\epsilon/2} \frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(1)} \\
&\approx -\frac{m^2}{16\pi^2}\Big(\frac{2}{\epsilon} - \gamma_E + \log 4\pi + \log\frac{\mu^2}{m^2} + 1\Big) \\
&\equiv -\frac{m^2}{16\pi^2}\Big(\frac{2}{\epsilon_{\overline{\mathrm{MS}}}} + 1 + \log\frac{\mu^2}{m^2}\Big).
\end{aligned} \tag{D.6}$$

Add more stuff when needed...

E Thermal $J_T^{\pm}$ and $I_T^{\pm}$-integrals

In this appendix we derive and collect the main properties of the thermal integrals $J_T^{\pm}$ and $I_T^{\pm}$. In particular we derive their high and low temperature limits. We also extend these integrals to the complex plane, extending their use in the effective action to the unstable region.

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