Real-time formalism

We have so far used exclusively the imaginary-time formalism for thermal field theory. This followed from straightforward expression of the trace over thermal density operator as a path integral. This method allows us to compute all static observables in thermal equilibrium. However, one can not use it to study any t-dependent phenomena. An alternative formulation exists, that introduces real time variables. This formulation is more general, and allows extension to dynamical, out-of-equilibrium systems. At the least it provides an alternative formulation, which is often easier to use even in thermal limit.

Two time histories

We will again use the scalar theory as our test case, and start by considering the two-point function:

$$\langle \hat\phi(x_1)\,\hat\phi(x_2)\rangle \equiv \frac{1}{\mathrm{Tr}\,\hat\rho}\,\mathrm{Tr}\big[\overbrace{\hat\rho\,\hat\phi(x_1)\,\hat\phi(x_2)}^{\text{Heisenberg picture operators}}\big]$$

For now, let us assume that $\hat\rho$ is not known. We then can not write the correlator as path integral in imaginary time as we did before. Instead, we write it using Schrödinger picture operators $\hat\rho_s$; $\hat\phi_s$.

$$\langle \hat\phi(x_1)\,\hat\phi(x_2)\rangle = \frac{1}{Z}\,\mathrm{Tr}\Big[\hat\rho_{in}\,\underbrace{\hat U(t_{in},t_1)\,\hat\phi_S(\vec x_1)\,\mathrlap{\overbrace{\phantom{\hat U(t_1,t_{in})\,\hat U(t_{in},t_2)}}^{\hat U(t_1,t_2)}}\hat U(t_1,t_{in})}_{\hat\phi_H(t_1,\vec x)}\,\hat U(t_{in},t_2)\,\hat\phi_S(\vec x_2)\,U(t_2,t_{in})\Big]$$

$Z = \mathrm{Tr}\,\hat\rho$

where $\hat{U}(t_1,t_2)$ is the full unitary time-evolution operator of the Schrödinger States and $\hat{\rho}_{in} = \hat{\rho}(t_{in})$ is the density operator prepared at time t_{in}.
As usual we can write,

$$\langle \phi_b | U(t_2,t_1) | \phi_a \rangle = \int [\mathcal{D}\varphi]_{\substack{\varphi(t_1)=\varphi_a \\ \varphi(t_2)=\varphi_b}} \exp\left\{ i \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L} \right\}$$

$$\equiv \int_{\varphi_a}^{\varphi_b} [\mathcal{D}\varphi] \exp\left[i S[\varphi] \right]_{t_1}^{t_2}$$

The field basis is complete : $\mathbb{1} = \int \mathcal{D}\varphi |\varphi\rangle\langle\varphi|$ and $\langle\varphi_a|\varphi_b\rangle = \delta[\varphi_a - \varphi_b]$ as well as $\hat{\phi}_S(\vec{x})|\varphi\rangle = \varphi(\vec{x})|\varphi\rangle$. Inserting unit operators between each evolution operator, we can turn $\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\rangle$ into a path integral. Before doing that, note that

$$\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\rangle = \frac{1}{Z} \int \mathcal{D}\varphi_a \mathcal{D}\varphi_b \langle\varphi_a|\rho_{in}|\varphi_b\rangle\langle\varphi_b|\mathbb{O}_2|\varphi_a\rangle$$

with following three equivalent expressions for $\langle\varphi_2|\mathbb{O}_2|\varphi_1\rangle$

$$\mathbb{1} = \int \mathcal{D}\varphi_c |\varphi_c\rangle\langle\varphi_c|$$

$$\langle\varphi_b|\mathbb{O}_2|\varphi_a\rangle = \langle\varphi_b| \hat{U}(t_{in},t_1)\hat{\phi}_S(\vec{x}_1)\hat{U}(t_1,t_{in}) \hat{U}(t_{in},t_2) \hat{\phi}_S(\vec{x}_2) U(t_2,t_{in}) |\varphi_a\rangle$$

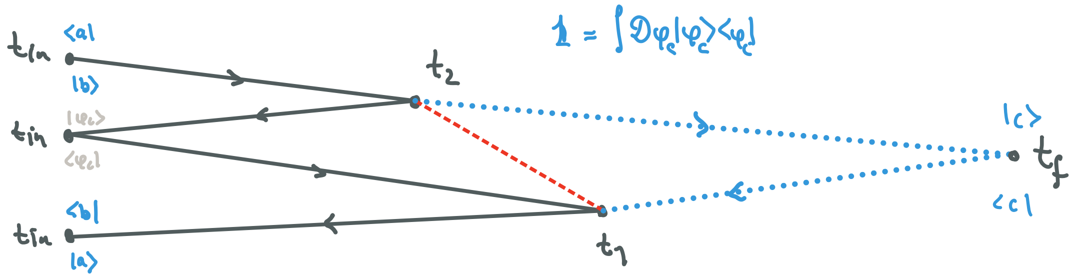
$$= \langle\varphi_b| \hat{U}(t_{in},t_1)\hat{\phi}_S(\vec{x}_1) \hat{U}(t_1,t_2) \hat{\phi}_S(\vec{x}_2) U(t_2,t_{in}) |\varphi_a\rangle$$

$$\longrightarrow \langle\varphi_b| \hat{U}(t_{in},t_1)\hat{\phi}_S(\vec{x}_1) \hat{U}(t_1,t_f) \hat{U}(t_f,t_2) \hat{\phi}_S(\vec{x}_2) \hat{U}(t_2,t_{in}) |\varphi_a\rangle$$

$$\uparrow$$

$$\mathbb{1} = \int \mathcal{D}\varphi_c |\varphi_c\rangle\langle\varphi_c|$$

(whether $t_1 > t_2$ or $t_2 < t_1$ does not matter)



We can write the path integral in several different ways. One possibility is to use the last expression, inserting 1 between $\hat{U}(t_1, t_f)$ and $\hat{U}(t_f, t_2)$:

$$\langle \varphi_b | \mathbb{O}_2 | \varphi_a \rangle = \int \mathcal{D}\varphi_c \langle \varphi_a | \hat{U}(t_{in}, t_1)\, \hat{\phi}_S(\vec{x}_1)\, \hat{U}(t_1, t_f) | \varphi_c \rangle \langle \varphi_c | U(t_f, t_2)\, \hat{\phi}_S(\vec{x}_2)\, U(t_2, t_{in} | \varphi_b \rangle$$

Here:

- $$\langle \varphi_c | U(t_f, t_2)\, \hat{\phi}_S(\vec{x}_2) \underset{\overset{\wedge}{\int \mathcal{D}\phi_d |\varphi_d\rangle\langle\varphi_d|}}{\,} U(t_2, t_{in}) | \varphi_b \rangle = \int \mathcal{D}\varphi_d \int_{\varphi_b}^{\varphi_d} [\mathcal{D}\varphi_+] \exp\left[iS[\varphi_+]\right]_{t_{in}}^{t_2} \varphi_d(\vec{x}) \times \int_{\varphi_d}^{\varphi_c} [\mathcal{D}\varphi_+] \exp\left[iS[\varphi_+]\right]_{t_2}^{t_f}$$
  $$= \int_{\varphi_b}^{\varphi_c} [\mathcal{D}\varphi_+]\, \varphi_+(x_2) \exp\left[iS[\varphi_+]\right]_{t_{in}}^{t_f}$$
- $$\langle \varphi_a | \hat{U}(t_{in}, t_1)\, \hat{\phi}_S(\vec{x}_1)\, \hat{U}(t_1, t_f) | \varphi_c \rangle = \int_{\varphi_c}^{\varphi_a} [\mathcal{D}\varphi_-]\, \varphi_-(x_1) \exp\left[iS(\varphi_-)\right]_{t_f}^{t_{in}}$$

$$\Rightarrow \langle \varphi_b | \mathbb{O}_2 | \varphi_a \rangle = \int_{\varphi_a}^{\varphi_b} [\mathcal{D}\varphi_-][\mathcal{D}\varphi_+]\, \varphi_+(x_2)\varphi_-(x_1) \exp\left\{ [iS(\varphi_+)]_{t_{in}}^{t_f} - [iS^*[\varphi_-]]_{t_{in}}^{t_f} \right\}$$

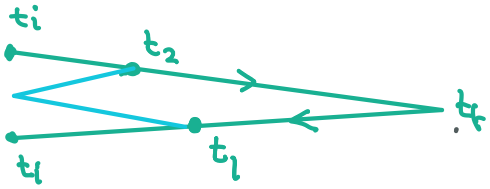
And then finally

$$\langle \hat{\phi}(x_1)\, \hat{\phi}(x_2) \rangle = \frac{1}{Z} \int [\mathcal{D}\varphi_+][\mathcal{D}\varphi_-] \overset{\text{unknown}}{\langle \phi_+ | \hat{\rho}_{in} | \phi_- \rangle}\, \overset{\text{on different branches. Essential.}}{\varphi_-(x_1)\, \varphi_+(x_2)} \exp\left\{ [iS(\varphi_+)]_{t_{in}}^{t_f} - [iS^*[\varphi_-]]_{t_{in}}^{t_f} \right\}$$

$(D^<)$

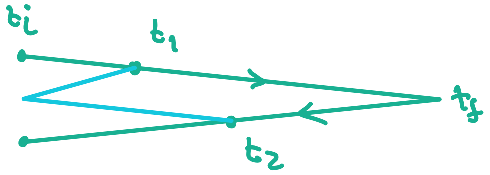
- The final time $t_f \geq \max(t_1, t_2)$, but otherwise t_f is arbitrary.
- States $|\phi_\pm\rangle$ can be understood as corresponding to $\varphi(t_{in}, \vec{x})$.
- Generalization to arbitrary n-point functions is obvious!
- Where we inserted t_f actually matters!

Above we made a point of putting x_1 & x_2 to different parts of the time path. This was necessary to compute the correlation function we wanted:



$$D^{-+}(x_1,x_2) = D^{>}(x_1,x_2) \equiv \langle \hat{\phi}(x_1)\hat{\phi}(x_2) \rangle$$

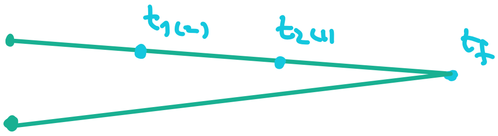
The order of operators is essential here. Path always meets the operator $\hat{\phi}(x_2)$ first, no matter whether $t_1 > t_2$ or not. Another function of interest is



$$D^{+-}(x_1,x_2) = D^{<}(x_1,x_2) = \langle \hat{\phi}(x_2)\hat{\phi}(x_1) \rangle$$

This can be obtained from $D^{>}(x_1,x_2)$ by reversing $\varphi_-(x_1)\varphi_+(x_2) \rightarrow \varphi_-(x_2)\varphi_+(x_1)$ in the integrand in the PI.

There are still two possibilities that could arise. We could extend the path to t_f not between the times t_1 & t_2, but either after or before. The former choice would give

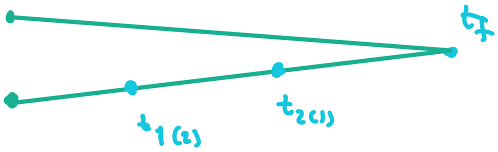


$$D^{++}(x_1,x_2) = D_F(x_1,x_2)$$

$$= \frac{1}{Z}\int[\mathcal{D}\varphi_+][\mathcal{D}\varphi_-]\langle \phi_+|\hat{\rho}_{in}|\phi_-\rangle\, \varphi_+(x_1)\varphi_+(x_2)\, \exp\,[iS[\varphi_+]]_{t_{in}}^{t_f} - [iS^*[\varphi_-]]_{t_{in}}^{t_f}\}$$

$$= \langle T(\hat{\phi}(x_1)\hat{\phi}(x_2)) \rangle$$ Time ordered out-of-equilibrium propagator.

And the latter:

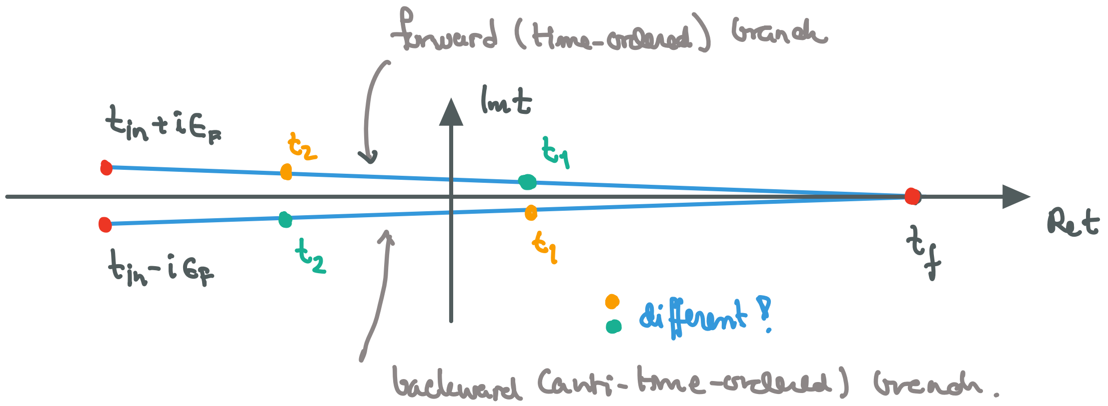


$$D^{--}(x_1,x_2) = D_{\bar{F}}(x_1,x_2) = \langle \bar{T}(\phi(x_1)\phi(x_2)) \rangle$$

where the arguments in the PI now are $\varphi_-(x_1)\varphi_-(x_2)$. This is anti-time-ordered propagator because the time-path runs in the negative direction.

Keldysh path

- Because the paths have the same final point, we can interpret the two braches as a single path going from $t_i \to t_f \to t_i$.



- The 4 different possibilities to place the time coordinates on the Keldysh path gives rise to 4 different (but related!) propagators.
- The convergence of the PI requires that both paths are tilted down

Indeed: consider free theory partition function, contribution from forward branch

$$Z \supset \int \mathcal{D}\varphi_+ \exp\left\{ \frac{i}{2} \int_{-t_i(1-i\epsilon)}^{t_f(1-i\epsilon)} dt_c \int_{\vec{x}} \phi\left(-\partial_t^2 + \nabla^2 - m^2\right)\phi \right\}$$

$$= \int \mathcal{D}\varphi_+ \exp\left\{ \frac{i}{2}(1-i\epsilon) \int_{-t_i}^{t_f} dt_R \int_{\vec{x}} \phi\left(-(1+2i\epsilon)\,\partial_t^2 + \nabla^2 - m^2\right)\phi \right\}$$

$$\overset{|t_i|,|t_f|\to\infty}{=} \int \mathcal{D}\hat{\varphi}_+ \exp\left\{ \frac{i}{2} \int_k (k^2-m^2)|\varphi_+(k)|^2 - \underbrace{\epsilon \int_k}_{\text{convergence}} (k^2-m^2+2k_0^2)|\varphi_+(k)|^2 \right\}$$

For the backward branch the time runs in opposite direction, and so the ϵ-prescription has to be opposite $t_C = t_R(1+i\epsilon)$.

For partition function the real part of the path function is not essential. It is just a fancy way to write $\delta[\varphi_a - \varphi_c]$. Indeed, removing the fields in the PI-expression for the propagator:

$$Z = \int \mathcal{D}\varphi_a \mathcal{D}\varphi_b \langle \varphi_a | \rho_{in} | \varphi_b \rangle \int_{\varphi_a}^{\varphi_b} [\mathcal{D}\varphi_-][\mathcal{D}\varphi_+] \exp\left\{ [iS[\varphi_+]]_{t_{in}}^{t_f} - [iS^*[\varphi_-]]_{t_{in}}^{t_f} \right\}$$

$$= \int \mathcal{D}\varphi_a \mathcal{D}\varphi_b \langle \varphi_a | \rho_{in} | \varphi_b \rangle \langle \varphi_b | \varphi_a \rangle = \int \mathcal{D}\varphi_a \langle \varphi_a | \hat{\rho}_{in} | \varphi_a \rangle$$

$$\xrightarrow{\text{Th. limit}} \int \mathcal{D}\varphi_a \langle \varphi_a | e^{-\beta \hat{H}} | \varphi_a \rangle$$

Generating functional

The non-equilibrium generating function is nontrivial however. We can define it as follows:

$$Z[J_\pm; \hat{\rho}] = \mathrm{Tr}\left\{ \hat{\rho}_{in} \overline{\mathcal{T}}\left[\exp\left(-i\int_{t_{in}}^{t_f} d^4x\, J_-(x)\hat{\phi}_-(x)\right)\right] \mathcal{T}\left[\exp\left(i\int_{t_{in}}^{t_f} d^4x\, J_+(x)\hat{\phi}_+(x)\right)\right] \right\}$$

$$= \int [\mathcal{D}\varphi_+][\mathcal{D}\varphi_-] \langle \phi_+ | \hat{\rho}_{in} | \phi_- \rangle \exp\left\{ \sum_{\pm} \pm i\left(S[\varphi^\pm] + J_\pm \phi^\pm\right) \right\}$$

$$= \int [\mathcal{D}\varphi_+][\mathcal{D}\varphi_-] \langle \phi_+ | \hat{\rho}_{in} | \phi_- \rangle \exp\left\{ \epsilon_{ab}\, i\left(S[\varphi^b] + J_b \varphi^b\right) \right\}$$

$$\uparrow \; \epsilon \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

where the shorthand $J_a \varphi^a \equiv \int d^4x\, J_a(x) \phi^a(x)$ was used, and $\epsilon_{11} = -\epsilon_{22} = 1$ and $\epsilon_{12} = \epsilon_{21} = 0$ (ϵ_g as matrix $\epsilon = \sigma_3$).

In terms of the complex time-variable on Keldysh path:

$$Z[J;\hat{\rho}] = \int [\mathcal{D}\varphi] \langle \phi_+ | \hat{\rho}_{in} | \phi_- \rangle \exp\left\{ i \int dt_c \int d^3x \, (\mathcal{L}_c + J_c \phi_c) \right\}$$

$$= \mathrm{Tr}\left\{ \hat{\rho}_{in} \, \mathcal{T}_c \left[\exp\left(i \int_C dt_c \int d^3x \, J_c \hat{\phi}_c \right) \right] \right\}.$$

We derived our expression for the 2-point function assuming the coordinates were on separate branches. If we define

$$W[J_a, \hat{\rho}] = -i \log Z[J_a; \hat{\rho}],$$

we can compute

$$-\frac{\partial W}{\partial J_-}\bigg|_{J=0} = \frac{\partial W}{\partial J_+}\bigg|_{J=0} = \langle \hat{\varphi} \rangle = \varphi_{cl}$$

$$D^{ab}(x_1, x_2) = \frac{(-i)^2 \delta^2 W}{\delta J_b(x_2) \delta J_a(x_1)}$$

where

$$\begin{aligned} D^{-+}(x_1,x_2) &\equiv D^{>}(x_1,x_2) = \langle \hat{\phi}(x_1) \hat{\phi}(x_2) \rangle, \qquad (p7.1) \\ D^{+-}(x_1,x_2) &\equiv D^{<}(x_1,x_2) = \langle \hat{\phi}(x_2) \hat{\phi}(x_1) \rangle \\ D^{++}(x_1,x_2) &\equiv D_F(x_1,x_2) = \langle T(\hat{\phi}(x_1) \hat{\phi}(x_2)) \rangle \\ D^{--}(x_1,x_2) &= D_{\bar{F}}(x_1,x_2) = \langle \bar{T}(\hat{\phi}(x_1) \hat{\phi}(x_2)) \rangle \end{aligned}$$

are general out-of-equilibrium two point functions. They are related, but hold different types of information.

The initial density matrix

$\hat{\rho}_{in}$ is in general not known, but the expectation value $\langle \hat{\varphi}_+ | \hat{\rho}_{in} | \hat{\varphi}_- \rangle$ must be a functional of field configurations, and since $|\varphi\rangle$ form a basis, it can be expanded as

non-gaussian terms

$$\langle \hat{\varphi}_+ | \hat{\rho}_{in} | \hat{\varphi}_- \rangle = \exp\left\{ iK + iK_a \varphi^a + \frac{1}{2!} K_{ab} \varphi^a \varphi^b + \overset{\downarrow}{\ldots} \right\}$$

The functions K, K_a, K_{ab}, which parametrize $\langle \hat{\varphi}_+ | \hat{\rho}_{in} | \hat{\varphi}_- \rangle$ can be seen as initial sources.

K_a can be absorbed to the source J_a, but K_{ab} is something genuinely new. Introduction of K_{ab}, and higher order tensors K_{abc}, would allow us to construct non-equilibrium FTFT in form of the 2PI- and nPI- effective actions. These are generalizations of our familiar 1PI-effective action, with the difference, that eg. the effective 2PI-action is a function not only of the classical field, but also of classical 2-point function:

2-point function

$$\Gamma_{2PI} = \Gamma_{2PI}[\varphi_c, \overset{\downarrow}{\Delta_c}]$$

Instead of a perturbative expansion for Δ_c, we get a dynamical equation for it.

This is not the only way to construct out-of-eq. field theory. One can generalize the RTF to out-of-eq. situation more heuristically, using Dyson equation. nPI-methods are needed to consistently compute eg. the self-energy functions appearing in those equations. In some limits this can be done without nPI, though.

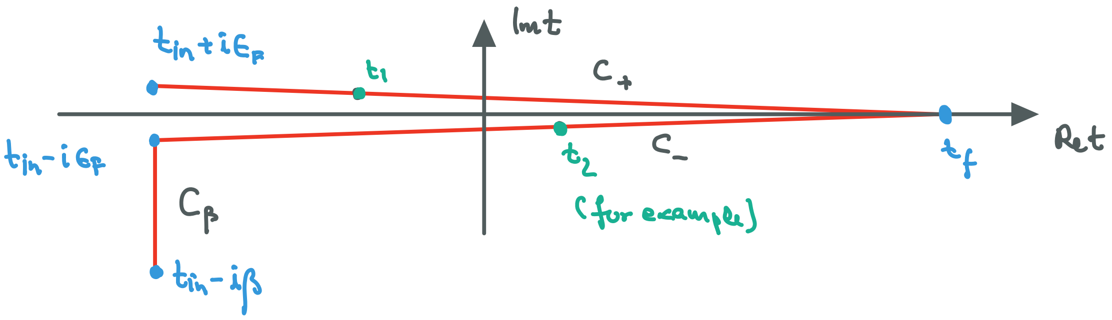
Thermal limit Here we are less ambitious and content us to find real-time formulation for thermal system. In thermal system $\hat{\rho} = \hat{\rho}_{in} = e^{-\beta \hat{H}}$ is known and time-independent.

write as a Euclidean PI as before

$$Z[J; \hat{\rho}] = \int \mathcal{D}\varphi_- \mathcal{D}\varphi_+ \langle \phi_- | e^{-\beta \hat{H}} | \varphi_+ \rangle \int [\mathcal{D}\phi]_{\substack{\phi_- \\ \phi_+}} \exp\left\{ \epsilon_{ab}\, i \left(S[\varphi^b] + J_b \varphi^b \right) \right\}$$

$$= \int [\mathcal{D}\varphi]_\beta \exp\left\{ \int_C \left(S[\varphi] + J\varphi \right) \right\} \qquad (1)$$

Here we end up integrating over all paths, periodic over the full extent of the extended Keldysh path.



From (1) we can derive all results necessary to set up real time FTFT. (And our old ITF, if we restrict sources to vertical complex part of the path.)

Other paths Equilibrium real time rules can be based also on different complex paths. Indeed, when system is thermal, we can "split the matrix element $\langle \varphi_+ | \hat{\rho} | \varphi_- \rangle$ between t_i and t_f, because $\hat{\rho}$ is t-independent. Formally

$= e^{-(\beta-\sigma)\hat{H}} \quad e^{-\sigma \hat{H}}$

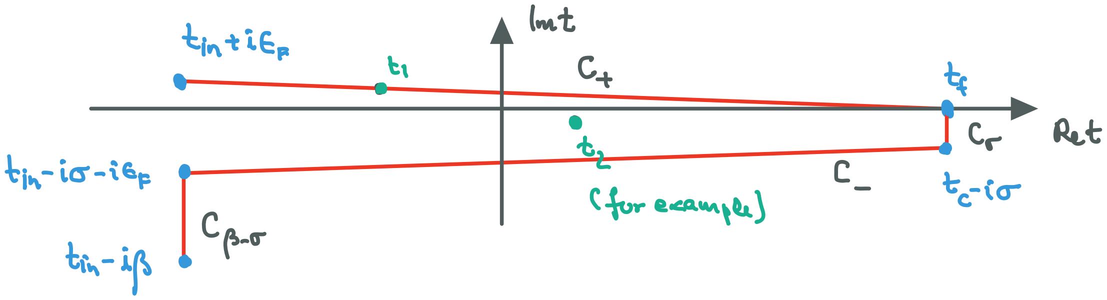
$$\langle \hat{\phi}(x_1) \hat{\phi}(x_2) \rangle_\beta = \mathrm{Tr}\left[e^{-\beta \hat{H}} \hat{U}(t_{in}, t_1) \hat{\phi}_S(\vec{x}_1) \hat{U}(t_1, t_f) \hat{U}(t_f, t_2) \hat{\phi}_S(\vec{x}_2) U(t_2, t_{in}) \right]$$

Now write $\hat{\rho} = e^{-\beta\hat{H}} = e^{-\sigma\hat{H}} e^{-(\beta-\sigma)\hat{H}}$ and move the $e^{-\sigma\hat{H}}$ between $U(t_1,t_f)$ and $U(t_f,t_2)$ Then going through the PI-construction as before we get:

$$Z[J;\hat{\rho}] = \int \mathcal{D}\varphi_{in}\mathcal{D}\varphi_b\mathcal{D}\varphi_c\mathcal{D}\varphi_d \, \overbrace{\langle \phi_{in}| e^{-(\beta-\sigma)\hat{H}} |\phi_b\rangle}^{\text{Euclidean PI}} \int_{\varphi_b}^{\varphi_c} [\mathcal{D}\phi] \exp\left\{ i\left(S[\varphi_+] + J_+\varphi_+\right)_{t_{in}}^{t_f} \right\}$$

$$\underbrace{\langle \phi_c | e^{-\sigma\hat{H}} |\varphi_d\rangle}_{\text{Euclidean PI}} \int_{\varphi_d}^{\varphi_{in}} [\mathcal{D}\phi] \exp\left\{ -i\left(S[\varphi_-] + J_-\varphi_-\right)_{t_{in}}^{t_f} \right\}$$

Again, this can be expressed in the form (1), where the complex path now is



Each different path choice leads to different real-time FTFT Feynman rules. Commonly used choice in the past was the "symmetric" $\sigma = \beta/2$-path. Nowadays one uses almost exclusively the Keldysh form $\sigma = 0$.

Vacuum-limit Amusingly, this can be seen directly from the PI in the last form. For any $0 < x < 1$ both $\sigma \equiv x\beta$ and $\beta - \sigma = (1-x)\beta$ go to ∞ as $T \to 0$. As a result the matrix elements $\langle \phi_{in}| e^{-(\beta-\sigma)\hat{H}} |\varphi_b\rangle$ and $\langle \phi_d | e^{-\sigma\hat{H}} |\varphi_c\rangle$ pick up only the vacuum state at t_1 and t_f.
(Eg: $\langle \varphi_d | e^{-\sigma\hat{H}} |\varphi_c\rangle \to \langle\Omega|\Omega\rangle \, \delta[\varphi_d - \Omega]\,\delta[\varphi_c-\Omega] = \delta[\varphi_d-\Omega]\,\delta[\varphi_c-\Omega]$.)
Restricting our sources to the forward branch, we recover the usual $T=0$ generating function for vacuum-vacuum transitions.

Real time Feynman rules Given the above, it is reasonable to take the generic complex time-path seriously and construct the Feynman rules for a generic σ, on path $C = C_+ \cup C_\sigma \cup C_- \cup C_{\beta-\sigma}$. We have already shown that we may write

$$Z_C[j,\beta] = \mathrm{Tr}\left\{ e^{-\beta\hat{H}} T_c\left[\exp\left(i\int_c j\phi\right)\right]\right\}$$

$$= \int [\mathcal{D}\varphi]_\beta \exp\left\{ i\int_c \left(S[\varphi] + J\varphi\right)\right\}$$

$$= \int [\mathcal{D}\varphi]_\beta \exp\left\{ i\int_c \left[\varphi(-\Box - m^2)\varphi - V(\varphi) + j_c\varphi\right]\right\}$$

$$= e^{-i\int_c V\left(-\frac{\delta}{\delta j}\right)} \int [\mathcal{D}\varphi]_\beta \exp\left\{ i\int_c \left[\varphi \overbrace{(-\Box - m^2)}^{D_c^{-1}}\varphi + j_c\varphi\right]\right\}$$

$$= Z_0[\beta]\, e^{-i\int_c V\left(-\frac{\delta}{\delta j}\right)} \exp\left\{ -\frac{1}{2}\int_c\int_{c'} j_c(x)\, D_c(x-x')\, j_c(x')\right\}$$

$$-\int_c V\left(-\frac{\delta}{\delta j_c}\right) = \int dt \left(V\left(-\frac{\delta}{\delta j_1}\right) - V\left(-\frac{\delta}{\delta j_2}\right)\right)$$

$$= \epsilon_{ab}\int dt\, V\left(-\frac{\delta}{\delta j_b}\right)$$

Arbitrary Greens functions can now obtained as follows:

$$G_c(x_1,\dots,x_n) = \frac{1}{Z[\beta]} \frac{(-i)^n \delta^n Z_c[\beta,j]}{\delta j_c(x_1)\cdots \delta j_c(x_n)}\Bigg|_{j=0}$$

In what follows we will restrict sources to C_1 & C_2 only. C_σ and $C_{\beta-\sigma}$ are not needed in RTF: That is

$$Z[\beta,j]_{\mathrm{eff}} \equiv Z[\beta,j]_{C_1\cup C_2} = \frac{1}{Z_0^{\mathrm{eff}}[\beta]}\, e^{-i\int_{C_{12}} V\left(-\frac{\delta}{\delta j}\right)}\, e^{-\frac{1}{2}\iint_{C_{12}} j D_c j}.$$

(This is all one has in Keldysh out-of-eq. theory)

Defining $j_1(\vec{x},t) \equiv j(\vec{x},t_R)$ and $j_2(\vec{x},t) \equiv j(\vec{x},t_R-i\sigma)$ we can write:

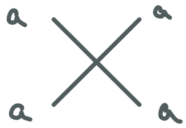
direction of time:

real times

$$Z[\beta,j]_{\text{eff}} = \frac{1}{Z_0^{\text{eff}}[\beta]}\, e^{-i\epsilon_{ab}\int d^4x\, V\left(-\frac{\delta}{\delta j_b}\right)}\, e^{-\frac{1}{2}\int d^4x\, d^4x'\, j_c(x)\, D_{cd}(x-x')\, j_d(x')}$$

2×2 matrix as expected

We shall compute the free propagators shortly. First note however that **The vertex Feynman rule** is



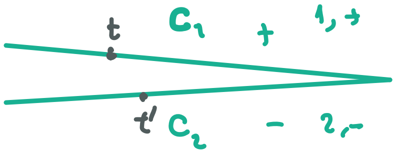
$$= -\frac{i\lambda}{4!}\epsilon_{ab}$$

- All fields must be the same
- 2222-vertex has opposite sign to 1111-vertex

In RTF all external lines have the usual time-evolution, eg they are 1-fields. Internal lines can be 2-fields however. For example

Obviously the off-diagonal parts of the propagator are relevant.

Propagator $D_c(x-x')$ is clearly the complex time argument Greens function, which can be written as C-time ordered function:



$$D_c(x,x') = \theta_c(t-t')\, D^{>}(x-x') + \theta_c(t'-t)\, D^{<}(x-x').$$

$\langle\hat{\phi}(t,\vec{x})\, \phi(t',\vec{x}')\rangle_\beta$ $\qquad$ $\langle\hat{\phi}(t',\vec{x}')\,\hat{\phi}(t,\vec{x})\rangle$

Writing θ-functions in real time variable one easily finds

$$\Theta_c(t_1-t_2) \longrightarrow \theta(t_1-t_2), \text{ when } t_1 \,\&\, t_2 \in C_1 \quad \Rightarrow \quad D_C(t_1,t_2) \longrightarrow D_{11}(t_1,t_2) = D_F(t_1,t_2)$$

$$\Theta_c(t_1-t_2) \longrightarrow \theta(t_2-t_1) \quad \text{"} \quad t_1 \,\&\, t_2 \in C_2 \quad \Rightarrow \quad D_C(t_1,t_2) \longrightarrow D_{22}(t_1,t_2) = D_{\bar{F}}(t_1,t_2)$$

$$\Theta_c(t_1-t_2) \longrightarrow 1 \quad \text{"} \quad t_1 \in C_1,\, t_2 \in C_2 \quad \Rightarrow \quad D_C(t_1,t_2) \longrightarrow \underset{+-}{D_{12}}(t_1,t_2) = D^<(t_1,t_2)$$

$$\Theta_c(t_1-t_2) \longrightarrow 0 \quad \text{"} \quad t_1 \in C_2,\, t_2 \in C_1 \quad \Rightarrow \quad D_C(t_1,t_2) \longrightarrow \underset{-+}{D_{21}}(t_1,t_2) = D^>(t_1,t_2)$$

Eg one recovers propagators listed on p.7. To work out explicit expressions, we need

the RTF KMS-relation. We work it out in operator form:

$$\left(D^<(t,t';\vec{x},\vec{x}') = \langle \hat\phi(t,\vec{x})\, \hat\phi(t',\vec{x}') \rangle_\beta \right)$$

$$\begin{aligned}
D^>(t,t';\vec{x},\vec{x}') &= \langle \hat\phi(t',\vec{x}')\, \hat\phi(t,\vec{x}) \rangle_\beta \\
&= \mathrm{Tr}\left[e^{-\beta\hat{H}}\, \hat\phi(t',\vec{x}')\, \hat\phi(t,\vec{x}) \right] \\
&= \mathrm{Tr}\left[e^{-\beta\hat{H}}\, \hat\phi(t',\vec{x}')\, e^{\beta\hat{H}}\, e^{-\beta\hat{H}}\, \hat\phi(t,\vec{x}) \right] \\
&= \mathrm{Tr}\left[\hat\phi(t'+i\beta,\vec{x}')\, e^{-\beta\hat{H}}\, \hat\phi(t,\vec{x}) \right] \\
&= \mathrm{Tr}\left[e^{-\beta\hat{H}}\, \hat\phi(t,\vec{x})\, \hat\phi(t'+i\beta,\vec{x}') \right] = D^<(t+i\beta,t';\vec{x},\vec{x}')
\end{aligned}$$

$$\begin{pmatrix} D^{++} & D^{+-} \\ D^{-+} & D^{--} \end{pmatrix} = \begin{pmatrix} D_F & D^< \\ D^> & D_{\bar{F}} \end{pmatrix}$$

- Note that if we move to pure complex time $\Rightarrow$ $D^>(-i\tau,0) = \Delta(\tau)$ and the KMS condition reduces to $\Delta(\tau) = \Delta(\tau-\beta)$.

suppress $\bar{x},\bar{x}'$

- $$D^>(t,t') = \sum_{n,m} \langle n|\, e^{-\beta\hat{H}}\, e^{i\hat{H}t}\, \hat\phi(0)\, e^{-i\hat{H}t} |m\rangle\langle m|\, e^{i\hat{H}t'}\, \hat\phi(0)\, e^{-i\hat{H}t'} |n\rangle$$

$$= \sum_{n,m} e^{-iE_m(t-t') + iE_n(t-t'+i\beta)}\, |\langle n|\hat\phi(0)|m\rangle|^2$$

This expression converges for $-\beta < \mathrm{Im}(t-t') < 0$. Similarly, one can show that $D^<(t,t')$ exists for $0 < \mathrm{Im}(t-t') < \beta$, when proves that $D_<(t,t')$ converges when $-\beta < \mathrm{Im}(t-t') < 0$. On the boundary it can be defined as a distribution (limiting sense).

Explicit free propagator satisfies

$$(\Box_c + m^2) D_c(x-x') = -i\,\delta_c(t-t')\,\delta^3(x-x')$$

$\delta_c(t-t') = \epsilon_{ab}\,\delta(t-t')$

We can go directly to Fourier representation wrt $\vec{k}$-coordinate:

$$(\partial_t^2 + \omega_{\vec{k}}^2)\, D^c_{\vec{k}}(t-t';\vec{k}) = -i\,\delta_c(t-t') \qquad ; \quad t \subset t' \in C_1 \cup C_2 .$$

Integrating this equation over $t-t'$ gives

$D_c(x,x') = \theta_c(t-t')\,D^>(x-x') + \theta_c(t'-t)\,D^<(x-x')$.

$$1 = \lim_{\delta\to 0} \int_{-\delta}^{\delta} dt\,\delta_c(t) = i \lim_{\delta\to 0}\left(\partial_t D^>_{\vec{k}}(\delta) - \partial_t D^<_{\vec{k}}(-\delta)\right)$$

Now make the ansatz: $D^{<,>} \equiv a^{<,>} e^{i\omega t} + b^{<,>} e^{-i\omega t}$ (suppress $\vec{k}$-dependence)

and use

- KMS $\quad D^>(t) = D^<(t+i\beta) \Rightarrow a^> e^{i\omega t} + b^> e^{-i\omega t} = a^< e^{-\beta\omega} e^{i\omega t} + b^< e^{\beta\omega} e^{-i\omega t}$

  $\Rightarrow a^> = a^< e^{-\beta\omega}$ and $b^> = b^< e^{\beta\omega}$
- Symmetry: $D^>(-t) = D^<(t) \Rightarrow b^< = a^>$ and $b^> = b^< e^{\beta\omega} = a^> e^{\beta\omega}$
- $1 = -\omega(a^> - b^> - a^< + b^<) = -\omega(1 - e^{\beta\omega} - e^{\beta\omega} + 1)a^> = 2\omega(e^{\beta\omega} - 1)a^>$

$$\Rightarrow \quad a^{>} = b^{<} = \frac{1}{2\omega}\,\frac{1}{e^{\beta\omega}-1} = \frac{1}{2\omega} f(\omega) \quad \text{and} \quad a^{<} = b^{>} = \frac{1}{2\omega} f(\omega)\, e^{\beta\omega}$$

Putting everything together we get

$$D_c(x-x') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, \frac{1}{2\omega_k}\, \frac{1}{e^{\beta\omega_k}-1} \Big\{ \Big[e^{i\omega_k(t-t')} + e^{\beta\omega_k - i\omega_k(t-t')} \Big] \theta_c(t-t')$$

$$+ \Big[e^{i\omega_k(t'-t)} + e^{\beta\omega_k - i\omega_k(t'-t)} \Big] \theta_c(t'-t) \Big\}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, 2\pi \overset{\text{sign}(k_0)}{\epsilon(k_0)}\, f(k_0)\, \delta(k^2-m^2) \Big\{ e^{ik\cdot(x-x')} \theta_c(t-t') + e^{-ik\cdot(x-x')} \underbrace{\theta_c(t'-t)}_{1-\theta(t-t')} \Big\}$$

complex time arguments

$$\Rightarrow \quad \boxed{D_c(x-x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')_c}\, 2\pi\epsilon(k_0)\, \delta(k^2-m^2) \Big\{ \theta_c(t-t') + f(k_0) \Big\}}$$

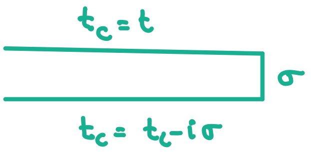
This is the Mills-representation.

- On the second to last line we used $\epsilon(k_0) f(k_0) e^{ik_0 t} = \begin{cases} f(\omega) e^{i\omega t} & ; k_0 = \omega \\ e^{\beta\omega} f(\omega) e^{-i\omega t} & ; k_0 = -\omega \end{cases}$

  and on the last line

- $$\int dk_0\, \epsilon(k_0) f(k_0) \left(e^{ik_0 t} - e^{-ik_0 t} \right) = \int dk_0\, e^{-ik_0 t} \left(\epsilon(-k_0) f(-k_0) - \epsilon(k_0) f(k_0) \right)$$
  $$= \int dk_0\, e^{-ik_0 t} \epsilon(k_0) \underbrace{\left(-f(k_0) + f(-k_0) \right)}_{= -\frac{1}{e^x-1} + \frac{1}{e^{-x}-1} = -\frac{1}{e^x-1} + \frac{e^x}{e^x-1} = 1}$$
  $$= \int dk_0\, e^{-ik_0 t} \epsilon(k_0)$$

From the Mills representation we can directly read the different propagators



1] Assume $t \in C_1$ & $t' \in C_2$ in σ-path.

Then $t = t_R$ and $t' = t'_R - i\sigma$ and $\theta_c(t-t') = 0$

real times

$$D^{+-}(x-x') \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} e^{\sigma k_0} 2\pi \epsilon(k_0) f(k_0) \delta(k^2-m^2)$$

$$\Rightarrow \underline{D^{+-}(k) = D^<(k) = e^{\sigma k_0} 2\pi \epsilon(k_0) f(k_0) \delta(k^2-m^2)}$$

2] $t \in C_2$ & $t' \in C_1$. Now $t = t_R - i\sigma$ and $t' = t'_R$, and $\theta_c(t-t') = 1$

$$D^{-+}(x-x') \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} e^{-\sigma k_0} 2\pi \epsilon(k_0) (1+f(k_0)) \delta(k^2-m^2)$$

$$\Rightarrow \underline{D^{-+}(k) = D^>(k) = e^{-\sigma k_0} 2\pi \epsilon(k_0) (1+f(k_0)) \delta(k^2-m^2)}$$

The ++ & -- propagators require a little more work:

3] t & $t' \in C_1$. Now t&$t' \in \mathbb{R}$ and $\theta_c(t-t') \rightarrow \theta(t-t') = i\int \frac{dq_0}{2\pi} \frac{e^{-iq_0(t-t')}}{q_0+i\eta}$

Then $\int \frac{dk_0}{2\pi} e^{-ik_0 t} \theta(t) 2\pi \epsilon(k_0) \delta(k^2-m^2) = \frac{1}{2\omega_k} \theta(t) \left(e^{-i\omega_k t} - e^{i\omega_k t}\right)$

$$= \frac{i}{2\omega_k} \int \frac{dq_0}{2\pi} \frac{e^{-iq_0 t}}{q_0+i\eta} \left(e^{-i\omega_k t} - e^{i\omega_k t}\right) = i\int \frac{dq_0}{2\pi} \frac{1}{2\omega_k} \sum_{\pm} \frac{\pm e^{-i(q_0 \pm \omega_k)t}}{q_0+i\eta}$$

$$= \int \frac{dq_0}{2\pi} \frac{1}{2\omega_k} e^{-iq_0 t} \left(\frac{i}{q_0-\omega+i\eta} - \frac{i}{q_0+\omega+i\eta} \right)$$

$$= \int \frac{dq_0}{2\pi} \frac{1}{2\omega_k} e^{-iq_0 t} \left(\frac{i}{q_0-(\omega-i\eta)} - \frac{i}{q_0+(\omega-i\eta)} + \frac{i}{q_0+\omega-i\eta} - \frac{i}{q_0+\omega+i\eta} \right)$$

$$= \int \frac{dq_0}{2\pi} e^{-iq_0 t} \left(\frac{i}{q^2-m^2+i\eta} - \frac{1}{2\omega_k} 2\pi \delta(q_0+\omega) \right)$$

Renaming $q \to k$ we can now combine this term with the $f(k_0)$-part:

$$D^{++}(k) = D_F(k) = \frac{i}{k^2-m^2+i\eta} + 2\pi \left[\overset{\frac{1}{2\omega}\left[\delta(k_0-\omega) - \delta(k_0+\omega)\right]}{f(k_0)\,\epsilon(k_0)\,\delta(k^2-m^2)} - \frac{1}{2\omega_k} \delta(q_0+\omega) \right]$$

$$= \frac{i}{k^2-m^2+i\eta} + \frac{2\pi}{2\omega} \left(f(k_0)\delta(k_0-\omega) - \underbrace{(1+f(k_0))}_{=-f(-k_0) = -f(|k_0|)} \delta(k_0+\omega) \right)$$

$$= \frac{i}{k^2-m^2+i\eta} + 2\pi f(|k_0|)\,\delta(k^2-m^2)$$

↳ thermal part!

4) Similarly, if $t_1 \,\&\, t_2 \in C_2$ we still have $t_1 - t_2 \in \mathbb{R}$ and $\Theta_C(t-t') \to \theta(t'-t)$ in real arguments. So the only change in the above derivation is $\theta(t) \to \theta(-t)$. This immediately leads to

$$D^{--}(k) = D_{\bar{F}}(k) = \frac{-i}{k^2-m^2-i\eta} + 2\pi f(|k_0|)\,\delta(k^2-m^2) = \left(D^{++}(k)\right)^*$$

- Note that D_F and $D_{\bar{F}}$ do not depend on σ.

- In early times one often used $\sigma = \frac{\beta}{2}$, which leads to symmetrical $D^{<,>}$:

$$D^>(k) = D^<(k) = 2\pi\, e^{\beta|k_0|/2} f(|k_0|)\,\delta(k^2-m^2) \quad ; \quad \sigma = \frac{\beta}{2}.$$

- Nowadays one uses almost exclusively the Keldysh-path $\sigma = 0$, for which the full set is

$$\overset{11}{D^{++}}(k) = \frac{i}{k^2-m^2+i\eta} + 2\pi f(|k_0|)\,\delta(k^2-m^2)$$

$$\overset{22}{D^{--}}(k) = \left(D^{++}(k)\right)^*$$

$$\overset{12}{D^{<}}(k) = \underline{2\pi\epsilon(k_0)}\, f(k_0)\, \underline{\delta(k^2-m^2)}$$

$$\overset{21}{D^{>}}(k) = \underline{2\pi\epsilon(k_0)}\,\left(1+f(k_0)\right)\, \underline{\delta(k^2-m^2)}$$

Keldysh-path RTF thermal propagators for a real scalar field

$$f(k_0) = \frac{1}{e^{\beta k_0}-1}$$

$$-\frac{i\lambda}{4!}\epsilon_{ab}$$

The spectral function

The Mills representation can be derived alternatively by noting that

$$\begin{aligned} D_c(t) &= \theta_c(t)\, D^>(t) + \theta_c(-t)\, D^<(t) \\ &= D^<(t) + \theta_c(t)\left[D^>(t) - D^<(t)\right] \\ &= D^<(t) + \theta_c(t)\, \rho_c(t) \end{aligned}$$

On Keldysh path time-arguments are real ($\sigma=0$), and we can immediately see the connection between $D^{>,<}$ and the spectral function ρ, which is defined as the commutator with real time arguments:

$$\rho(t,\vec{x}) \equiv \langle[\hat{\phi}(t,\vec{x}),\hat{\phi}(0)]\rangle_\beta \quad ; t\in\mathbb{R}$$

Indeed (for $\sigma=0$) $\rho(k) = D^>(k) - D^<(k) = 2\pi\epsilon(k_0)\delta(k^2-m^2)$, and then $D^<(k) = f(k_0)\rho(k)$ and $D^>(k) = (1+f(k_0))\rho(k)$.

- we can verify the spectral function directly from the field operator:

$$\hat{\phi}(t,\vec{x}) = \int_{\vec{q}} \left(a_{\vec{q}} e^{iq\cdot x} + a^\dagger_{\vec{q}} e^{-iq\cdot x} \right) \qquad \int_{\vec{q}} = \int \frac{d^3q}{(2\pi)^3 2\omega_q}$$

$$\Rightarrow \int d^4x\, e^{-ik\cdot x} \langle [\hat{\phi}(t,\vec{x}), \hat{\phi}(0)] \rangle_\beta$$

$$= \int d^4x\, e^{-ik\cdot x} \int_{\vec{q},\vec{q}'} \Big(\langle [\underbrace{a_{\vec{q}}, a^\dagger_{\vec{q}'}}_{=(2\pi)^3 2\omega_{\vec{q}} \delta^3(\vec{q}-\vec{q}')}] \rangle_\beta e^{iq\cdot x} + \langle [a^\dagger_{\vec{q}}, a_{\vec{q}'}] \rangle_\beta e^{-iq\cdot x} \Big)$$

$$= \int d^4x\, e^{-ik\cdot x} \int \frac{d^4q}{(2\pi)^4} 2\pi \theta(q_0) \delta(q^2-m^2) \left(e^{iq\cdot x} - e^{-iq\cdot x} \right) \overbrace{\langle \Omega|\Omega \rangle_\beta}^{=1}$$

$$= 2\pi\, \epsilon(k_0)\, \delta(k^2-m^2).$$

- Note that $\rho(k_0) = -\rho(-k_0)$ and $\epsilon(k_0)\rho(k_0) > 0$.

- Spectral function is related to the canonical commutation relation

$$[\hat{\phi}(t,\vec{x}), \dot{\hat{\phi}}(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$

$$\Rightarrow \underline{1} = -i \frac{\partial}{\partial t} \langle [\hat{\phi}_{\vec{k}}(0), \hat{\phi}_{\vec{k}}(t)] \rangle_\beta \Big|_{t\to 0^+} = i \frac{\partial}{\partial t} \rho_{\vec{k}}(t) \Big|_{t\to 0^+}$$

$$= i \int \frac{dk_0}{2\pi} \frac{\partial}{\partial t} e^{-ik_0 t} \rho(k_0,\vec{k}) \Big|_{t\to 0^+} = \underline{\int \frac{dk_0}{2\pi} k_0 \rho(k_0,\vec{k})}$$

This sum-rule is a general relation that holds nonperturbatively. It is easy to see that it is obeyed by the free spectral function

Chemical potential For a charged scalar field $D_C(t,t') = \theta(t-t')D^>(x,x') + \theta(t'-t)D^<(x,x')$

with

$$D^>(x,x') = \langle \phi(x)\phi^\dagger(x')\rangle_\beta$$

$$D^<(x,x') = \langle \phi^\dagger(x)\phi(x')\rangle_\beta$$

With chemical potential the KMS-condition reads ($\hat{K} \equiv \hat{H} - \mu\hat{Q}$, I choose the sign of the current operator to be opposite of that in section 2 (fix there!) eg $\hat{Q} = \int_{\vec{x}} i(\dot{\hat{\phi}}\hat{\phi}^* - \dot{\hat{\phi}}^*\hat{\phi})$)

$$\underline{D^>(t,\vec{x};t',\vec{x}')} = \frac{1}{\mathrm{Tr}\hat{\rho}}\mathrm{Tr}\left(e^{-\beta\hat{K}}\hat{\phi}(t,\vec{x})\, e^{\beta\hat{K}} e^{-\beta\hat{K}}\hat{\phi}^\dagger(t',\vec{x}')\right)$$

$$= \frac{1}{\mathrm{Tr}\hat{\rho}}\mathrm{Tr}\left(e^{-\beta\hat{K}}\hat{\phi}^\dagger(t',\vec{x}')\, e^{+\beta\mu\hat{Q}}\hat{\phi}(t+i\beta)\, e^{-\beta\mu\hat{Q}}\right)$$

$$= \frac{e^{-\beta\mu}}{\mathrm{Tr}\hat{\rho}}\mathrm{Tr}\left(e^{-\beta\hat{K}}\hat{\phi}^\dagger(t',\vec{x}')\hat{\phi}(t+i\beta)\right) = \underline{e^{-\beta\mu}D^<(t+i\beta,\vec{x};t',\vec{x}')}$$

In momentum space this reads $D^>(k_0) = e^{\beta(k_0-\mu)}D^<(k_0)$. Then from

$$\rho = D^> - D^< = \left(e^{\beta(k_0-\mu)} - 1\right)D^< \quad \Rightarrow \quad \boxed{\begin{aligned} D^<(k_0) &= f(k_0-\mu)\,\rho(k_0) \\ D^>(k_0) &= (1+f(k_0-\mu))\,\rho(k_0)\end{aligned}}$$

$$f(x) = \frac{1}{e^x - 1}.$$

From these and $\rho(k_0) = 2\pi\epsilon(k_0)\delta(k^2-m^2)$ one easily finds (Ex.)

$$\boxed{\begin{aligned} \overset{11}{D^{++}}(k) &= \frac{i}{k^2-m^2+i\eta} + 2\pi f(|k_0| - \epsilon(k_0)\mu)\,\delta(k^2-m^2) \\ \overset{22}{D^{--}}(k) &= (D^{++}(k))^* \\ \overset{12}{D^<}(k) &= \underline{2\pi\epsilon(k_0)}\, f(k_0-\mu)\, \underline{\delta(k^2-m^2)} \\ \overset{21}{D^>}(k) &= \underline{2\pi\epsilon(k_0)}\,(1+f(k_0-\mu))\, \underline{\delta(k^2-m^2)} \end{aligned}}$$

Only the 11-rule needs a little work. Note that

$$f(|k_0| - \epsilon(k_0)\mu) = \theta(k_0)f(k_0-\mu) + \theta(-k_0)f(-k_0+\mu)$$

On pinch singularities When one employs RTF-rules in computations, one regularly encounters products of δ-functions. These pinch singularities cancel when one sums over all contributions, however. It should be noted that RTF-propagators are distributions. Sometimes it is useful to remember this in the calculations, and use regulated δ-functions

$$\delta_\eta(k^2-m^2) = \frac{1}{\pi}\frac{\eta}{k^2-m^2+\eta^2} = \frac{1}{2\pi}\left(\frac{i}{k^2+m^2+i\eta} - \frac{i}{k^2+m^2-i\eta}\right)$$

$$= \frac{1}{2\pi}\left(D_F^0(k) + D_F^0(k)^*\right)$$

Example: "Split" mass

$$\mathcal{L} = \overbrace{\left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2}\varphi^2\right]}^{\text{"free theory"}} - \overbrace{\frac{1}{2}\mu^2\varphi^2}^{\text{"interaction"}}$$

For the free part we use the Keldysh-propagators, while the interaction Lagrangian gives the "vertex"

$$= -i\mu^2\epsilon_{ab}$$

We of course know that full theory is a free theory with mass $m_f^2 = m^2+\mu^2$, but it is instructive to analyze the situation perturbatively:

$$\overset{\text{full}}{\bar{D}_{11}} = \overset{\text{free}}{D_{11}} - i\mu^2 D_{1a}\epsilon_{ab}D^0_{b1} + [-i\mu^2]^2 D_{1a}\epsilon_{ab}D_{bc}\epsilon_{cd}D_{d1} + \dots$$

let us compute to first nontrivial order. To this end we write

$$D_{11} = \overbrace{\frac{i}{k^2-m^2+i\eta}}^{D_F} + 2\pi n(k_0)\overbrace{\delta_\eta}^{\frac{1}{2}(D_F+D_F^*)} \qquad ; \qquad n(k_0) \equiv f(|k_0|)$$

$$D_{12} = f(k_0)\delta_\eta = 2\pi\left[n(k_0) + \theta(-k_0)\right]\delta_\eta$$

$$D_{21} = (1+f(k_0))\delta_\eta = 2\pi\left[n(k_0)+\theta(k_0)\right]\delta_\eta \qquad (\mathcal{E}x)$$

Then

$$\delta D_{11}^{(1)} = -i\mu^2\left(D_{11}^2 - D_{12}D_{21}\right)$$

$$= -i\mu^2\left(\left(D_F^0 + n\left(D_F^0+D_{F}^{0*}\right)\right)^2 - \overbrace{\left[\theta(k_0)+n\right]\left[\theta(-k_0)+n\right]}^{= n^2+n}\left(D_F^0+D_F^{0*}\right)^2\right.$$

$$= -i\mu^2\left(D_F^{0\,2} + 2nD_F^0\left(D_F^0+D_F^{0*}\right) - n\left(D_F^0+D_F^{0*}\right)^2\right)$$

$$= -i\mu^2\left(D_F^{0\,2} + n\left(D_F^{0\,2} - D_F^{0*\,2}\right)\right)$$

$$= \mu^2\frac{\partial}{\partial m^2}\left(D_F^0 + n\left(D_F^0 + D_F^{0*}\right)\right) = \mu^2\frac{\partial}{\partial m^2}D_{11}$$

Similarly, one can show that

$$\delta D_{11}^{(2)} = (-i\mu^2)^2\, D_{1a}\,\varepsilon_{ab}\,D_{bc}\,\varepsilon_{cd}\,D_{d1} = \frac{1}{2}\mu^4\frac{\partial^2}{\partial m^{2\,2}}$$

and eventually $D_{11} = D_{11}(m^2) + \mu^2\frac{\partial}{\partial m^2}D_{11}(m^2) + \frac{1}{2!}\mu^4\frac{\partial^2}{\partial m^{2\,2}}D_{11}(m^2) + \dots = D_{11}(m^2+\mu^2)$

as expected.

- In this calculation it was essential that we used the distribution $n(k_0)\delta(k^2-m_0^2)$ and not $n(\omega)\delta(k^2-m^2)$. These quantities are not the same for the regulated δ-function δ_η. Here the use of $n(\omega)$ would not work in the mass-derivative formula, because $\frac{\partial}{\partial m^2}n(\omega) \neq 0$, while $\frac{\partial}{\partial m^2}n(k_0) = 0$. The need for $n(k_0)\delta(k^2-m^2)$ is necessary for the KMS-condition to hold with regulated δ_η-function

Self-energy in RTF The above example can be handled more efficiently by moving to diagonal basis. Actually, we can handle a generic self energy function this way. We start by writing

Not unitary: $U^{-1} = \begin{pmatrix} \sqrt{1+n} & -\frac{\theta_+ + n}{\sqrt{1+n}} \\ -\frac{\theta_- + n}{\sqrt{1+n}} & \sqrt{1+n} \end{pmatrix}$.

$$D_{ab}(k) = U_{ac}(k) \begin{pmatrix} D^0_{11}(k) & 0 \\ 0 & D^0_{11}(k)^* \end{pmatrix}_{cd} U_{ab}(k)$$

$(\det U = 1)$

with

$$U(k) = \begin{pmatrix} \sqrt{1+n} & \frac{\theta_- + n}{\sqrt{1+n}} \\ \frac{\theta_+ + n}{\sqrt{1+n}} & \sqrt{1+n} \end{pmatrix}, \quad \text{where } n = n(k_0) \text{ and } \theta_\pm = \theta(\pm k_0).$$

Indeed:

$$\begin{pmatrix} \sqrt{\ } & \frac{\theta_- + n}{\sqrt{\ }} \\ \frac{\theta_+ + n}{\sqrt{\ }} & \sqrt{\ } \end{pmatrix} \begin{pmatrix} D_F & 0 \\ 0 & D_F^* \end{pmatrix} \begin{pmatrix} \sqrt{\ } & \frac{\theta_- + n}{\sqrt{\ }} \\ \frac{\theta_+ + n}{\sqrt{\ }} & \sqrt{\ } \end{pmatrix} = \begin{pmatrix} \sqrt{\ } & \frac{\theta_- + n}{\sqrt{\ }} \\ \frac{\theta_+ + n}{\sqrt{\ }} & \sqrt{\ } \end{pmatrix} \begin{pmatrix} \sqrt{\ }\, D_F & \frac{\theta_- + n}{\sqrt{\ }} D_F \\ \frac{\theta_+ + n}{\sqrt{\ }} D_F^* & \sqrt{\ }\, D_F^* \end{pmatrix}$$

$$= \begin{pmatrix} D_F & 0 \\ 0 & D_F^* \end{pmatrix} + \pi \begin{pmatrix} n & \theta_- + n \\ \theta_+ + n & n \end{pmatrix} \left(D_F + D_F^* \right) \quad \boxtimes \,. \qquad (P1)$$

This is not restricted to free propagator. U diagonalizes also the full propagator. To prove note first that Mills-representation is true for the full propagator, and based on it

$$\begin{aligned} \mathrm{Re}\, D_{11} &\propto \mathrm{Re} \int dk_0\, e^{-ik_0 t} \left(\theta(t) + f(k_0) \right) \rho(k_0) \\ &= \mathrm{Re} \int dk_0 \cos k_0 t \left(\theta(t) + f(k_0) \right) \rho(k_0) \\ &= \mathrm{Re} \int dk_0\, e^{-ik_0 t} \left(\tfrac{1}{2} + f(k_0) \right) \rho(k_0) \\ \mathrm{Re}\, D_{12} &\propto \mathrm{Re} \int dk_0\, e^{-ik_0 t} f(k_0) \rho(k_0) \end{aligned}$$

So the components of full propagator are related similarly to free one (only the common function $\rho(k_0)$ changes). So they are diagonalized by common rotation.

Now write the Dyson equation:

$\bar{\Pi}$ is diagonalized by U^{-1}

$$\overset{U^{-1}}{\vphantom{D}}D_{ab}\overset{U^{-1}}{\vphantom{D}} = \overset{U^{-1}}{\vphantom{D}}D^{\text{free}}_{ab}\overset{U^{-1}}{\vphantom{D}} + \overset{U^{-1}}{\vphantom{D}}D^{\text{free}}_{ac}\overset{U^{-1}U}{\vphantom{D}}(-i\Pi_{cd})\overset{UU^{-1}}{\vphantom{D}}D_{db}\overset{U}{\vphantom{D}} \qquad \Leftrightarrow \qquad D^{-1} = D^{-1}_{\text{free}} + i\bar{\Pi}$$

$$\begin{pmatrix} D & 0 \\ 0 & D^* \end{pmatrix} = \begin{pmatrix} D_{\text{free}} & 0 \\ 0 & D^*_{\text{free}} \end{pmatrix}\left[1 - i\begin{pmatrix} \bar{\Pi} & 0 \\ 0 & -\bar{\Pi}^* \end{pmatrix}\right]\begin{pmatrix} D & 0 \\ 0 & D_* \end{pmatrix}$$

Thus, the solutions D and D^* do not mix, and one finds:

$\bar{\Pi}$ defines the dispersion relation

$$D = \frac{i}{k^2 - m^2 - \bar{\Pi} + i\eta} \qquad \text{(D1)}$$

We now obtain

$$D_{ab} = U_{ac}\begin{pmatrix} \frac{i}{k^2 - m^2 - \bar{\Pi} + i\eta} & 0 \\ 0 & \frac{-i}{k^2 - m^2 - \bar{\Pi} - i\eta} \end{pmatrix}_{cd} U_{db}\,. \qquad \text{(D2)}$$

- Full resummation result in our 'split-mass' example now follows from setting $\bar{\Pi} = \mu^2$.

We can compute all Π_{ab} from just Π_{11}. First write $\bar{\Pi}$ in diagonal basis as

$$\begin{pmatrix} \bar{\Pi} & 0 \\ 0 & -\bar{\Pi}^* \end{pmatrix} = \mathrm{Re}\,\bar{\Pi}\,\sigma_3 + i\,\mathrm{Im}\,\bar{\Pi}\,\mathbb{1}_2 \qquad U^{-1} = \begin{pmatrix} \sqrt{1+n} & -\frac{\theta_+ + n}{\sqrt{1+n}} \\ -\frac{\theta_- + n}{\sqrt{1+n}} & \sqrt{1+n} \end{pmatrix}$$

and then use $U^{-1}\sigma^3 U^{-1} = \sigma_3$ and $U^{-1}U^{-1} = 2\begin{pmatrix} n+\frac{1}{2} & -(\theta_- + n) \\ -(\theta_+ + n) & n+\frac{1}{2} \end{pmatrix}$ to get

$$\Rightarrow \quad \Pi_{ab} = \mathrm{Re}\,\bar{\Pi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\,\mathrm{Im}\,\bar{\Pi}\begin{pmatrix} n+\frac{1}{2} & -(\theta_- + n) \\ -(\theta_+ + n) & n+\frac{1}{2} \end{pmatrix} \qquad (\Pi 1)$$

We can then read: $\left(\theta_- + n = \epsilon(k_0) f(k_0) \text{ and } \theta_+ + n = \epsilon(k_0) + \theta_- + n = \epsilon(k_0)(1+f(k_0)) \right)$

- $\mathrm{Re}\,\Pi_{11} = -\mathrm{Re}\,\Pi_{22} = \mathrm{Re}\,\overline{\Pi} \quad ; \quad \mathrm{Re}\,\Pi_{12} = \mathrm{Re}\,\Pi_{21} = 0$

$f(|k_0| \to |k_0| + \epsilon(k_0)\mu$

- $\mathrm{Im}\,\Pi_{11} = \mathrm{Im}\,\Pi_{22} = (2f(|k_0|)+1)\,\mathrm{Im}\,\overline{\Pi} = \coth\frac{\beta|k_0|}{2}\,\mathrm{Im}\,\overline{\Pi}$

- $\Pi_{12} = -2i\,\overbrace{\epsilon(k_0) f(k_0)}^{=\theta_-+n}\,\mathrm{Im}\,\overline{\Pi}$ $\qquad 2\epsilon(k_0)f(k_0) = \epsilon(k_0)\dfrac{2e^{-\beta k_0/2}}{e^{\beta k_0/2}-e^{-\beta k_0/2}} = \dfrac{e^{-\beta k_0/2}}{\sinh\frac{\beta|k_0|}{2}}$

$$= -2i\,\epsilon(k_0)f(k_0)\tanh\frac{\beta|k_0|}{2}\,\mathrm{Im}\,\Pi_{11} = -2i\,\epsilon(k_0)f(k_0)\,\mathrm{Im}\,\overline{\Pi} = \Pi^<$$

- $\Pi_{21} = -2i\,\underbrace{\epsilon(k_0)}_{\mathrm{sign}(k_0)}(1+f(k_0))\,\mathrm{Im}\,\overline{\Pi} = e^{\beta k_0}\,\Pi_{12} = e^{\beta k_0}\,\Pi^< \quad$ (KMS)

So, all complex parts follow from $\Pi^<$. To summarize:

$\mathrm{Re}\,\overline{\Pi} = \mathrm{Re}\,\Pi_{11}$ $\qquad$ DR & width

$\mathrm{Im}\,\overline{\Pi} = \frac{i}{2} e^{\beta k_0/2}\sinh\frac{\beta|k_0|}{2}\,\Pi^< = \frac{i}{2}\epsilon(k_0)(\Pi^> - \Pi^<)$

$\mathrm{Im}\,\Pi_{11} = \mathrm{Im}\,\Pi_{22} = \frac{i}{2} e^{\beta k_0/2}\cosh\frac{\beta|k_0|}{2}\,\Pi^<$

$\Pi^> = e^{\beta k_0}\Pi^<, \quad \mathrm{Re}\,\Pi^< = \mathrm{Re}\,\Pi^> = 0 \quad \text{and} \quad \mathrm{Re}\,\Pi_{22} = -\mathrm{Re}\,\Pi_{11}$

These relations are fully general and obviously reduce the required computational work significantly.

Relations between Greens functions

We parametrized D_C with $D^>$ and $D^<$, D_F and $D_{\bar F}$, defined as

$$D^>(x,y) = \langle \phi(y)\phi(x) \rangle$$
$$D^<(x,y) = \langle \phi(x)\phi(y) \rangle$$
$$D_F(x,y) = \langle T(\phi(x)\phi(y)) \rangle = \theta(x_0-y_0)\,D^>(x,y) + \theta(y_0-x_0)\,D^<(x,y)$$
$$D_{\bar F}(x,y) = \langle \bar T(\phi(x)\phi(y)) \rangle = \theta(y_0-x_0)\,D^>(x,y) + \theta(x_0-y_0)\,D^<(x,y)$$

We can also define Retarded and advanced functions

$$D_R(x,y) \equiv \theta(x_0-y_0)\langle[\phi(x),\phi(y)]\rangle = \theta(x_0-y_0)\,\rho(x,y)$$
$$D_A(x,y) \equiv -\theta(y_0-x_0)\langle[\phi(x),\phi(y)]\rangle = -\theta(y_0-x_0)\,\rho(x_0,y_0)$$

It is easy to see that $D_F = D_R + D^> = D_A + D^< = D_{11}$

$$D_{\bar F} = D^> - D_R = D^< - D_A = D_{22}$$

$$\rho = D^> - D^< = D_R - D_A$$

Also using $\theta(\pm t) = i\int \frac{dk_0}{2\pi}\frac{e^{\mp ik_0 t}}{k_0 + i\eta} = \pm i \int \frac{dk_0}{2\pi}\frac{e^{-ik_0 t}}{k_0 \pm i\eta} \equiv \int \frac{dk_0}{2\pi}\hat\theta_\pm(k_0)\,e^{ik_0 t}$

we get

$$iD_{\underset{(A)}{R}}(t,\vec x) = \underset{(-)}{+}\, i\,\theta(\underset{(-)}{+}t)\,\rho(t,\vec x)$$

and then

$$\underline{iD_{\underset{(A)}{R}}(p_0,\vec p)} = \underset{(-)}{+}\, i\int dt\;\theta(\pm t)\,\rho(t,\vec p)\,e^{-ip_0 t} = \underset{(-)}{+}\, i\int \frac{dq_0\,dq_0'}{(2\pi)^2}\overbrace{\int dt\; e^{-i(p_0-q_0-q_0')t}}^{2\pi\delta(q_0'-(p_0-q_0))}\hat\theta_{\underset{(-)}{+}}(q_0')\,\rho(q_0,\vec p)$$

$$= \underset{(-)}{+}\, i\int dq_0\,\hat\theta_{\underset{(-)}{\pm}}(p_0-q_0)\,\hat\rho(q_0,\vec p) = + \underline{\int\frac{dq_0}{2\pi}\frac{\rho(q_0,\vec p)}{q_0-p_0 \underset{(+)}{-} i\eta}}$$

spectral representations

These are spectral representations for the retarded and advanced functions.

> Free field: $\rho(q_0,\vec{p}) = 2\pi\,\epsilon(q_0)\,\delta(q_0^2-\omega_{\vec{p}}^2) = \frac{\pi}{\omega_{\vec{p}}}\left(\delta(q_0-\omega_{\vec{p}})-\delta(q_0+\omega_{\vec{p}})\right)$
>
> $$\Rightarrow\; iD_{\underset{(A)}{R}}(p_0,\vec{p}) = \frac{1}{2\omega_p}\left(\frac{i}{p_0 \pm i\eta - \omega_{\vec{p}}} - \frac{-i}{p_0 \pm i\eta + \omega_{\vec{p}}}\right)$$
>
> $$= \frac{i}{(p_0 \pm i\eta)^2 - \omega_{\vec{p}}^2} = \frac{i}{p_0^2 - \omega_{\vec{p}}^2 \pm i\epsilon(p_0)\eta} \quad \text{\%.}$$

Moreover: one defines $-iD_R = D_H - \frac{i}{2}\rho$ and $-iD_A = D_H + \frac{i}{2}\rho$ so that

$$D_H = -\frac{i}{2}(D_R + D_A) = -\frac{i}{2}\epsilon(t)\rho(t)$$

which gives another spectral relation

$$D_H(p_0,\vec{p}) = \frac{i}{2}\int\frac{dq_0}{2\pi}\left(\frac{1}{q_0-p_0-i\eta}+\frac{1}{q_0-p_0+i\eta}\right)\rho(q_0,\vec{p}) = \frac{i}{2}\int\frac{dq_0}{\pi}\,\mathbb{P}\left(\frac{1}{q_0-p_0}\right)\rho(q_0,\vec{p}).$$

Clearly, all Greens functions can be expressed in terms of ρ. This extends to ITF 2-point function via analytic continuation:

$$\Delta(\omega_n,\vec{p}) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\,\Delta(\tau,\vec{p}) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\int\frac{dq_0}{2\pi}\, e^{-iq_0 t} D^>(q_0,\vec{p})\Big|_{t=-i\tau}$$

$$= \int\frac{dq_0}{2\pi}\int_0^\beta d\tau\, e^{-(q_0-i\omega_n)\tau}\, e^{\beta q_0} f(q_0)\rho(q_0) = \int\frac{dq_0}{2\pi}\,\frac{1}{q_0-i\omega_n}\,\overbrace{(1-e^{-\beta q_0})\,e^{\beta q_0} f(q_0)}^{=1}\,\rho(q_0)$$

$$= \int\frac{dq_0}{2\pi}\,\frac{\rho(q_0)}{q_0-i\omega_n}$$

From this expression we can connect $\Delta(\omega_n, \vec{p})$ with D_R and other RTF-correlation functions:

$$\Delta(i\omega_n = p_0 + i\eta) = i D_R(p)$$
$$\Delta(i\omega_n = p_0 - i\eta) = i D_A(p)$$

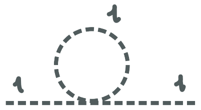
These are examples of analytic continuation rules. We can derive further connections by noting first that $\mathrm{Im}\, i D_R = \rho$, so that

$$D^< = 2 f(k_0)\, \mathrm{Im}\, i D_R$$
$$D^> = 2(1 + f(k_0))\, \mathrm{Im}\, i D_R$$
$$\mathrm{Im}\, D_n = \mathrm{Im}(i(D_R + D^<)) = 2(1 + 2f(k_0))\, \mathrm{Im}\, i D_R$$

Continue a little with self energy maybe.

Some examples of using RTF.

Thermal mass correction:

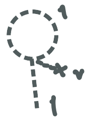
$\Pi_{11} = i \Big\{$  $\Big\}$ $\hat{=}\ i\left(\frac{-i\lambda}{4!}\right) 4.3 \int \frac{d^4k}{(2\pi)^4} D_{11}$

$$= \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{i}{k^2 - m^2 + i\eta} + 2\pi f(|k_0|)\, \delta(k^2 - m^2) \right\}$$

$$= \Pi_{vac} + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} f(\omega_k) = \Pi_{vac} + \Pi_T$$

Our old result.

1-loop effective potential

$\frac{\partial V}{\partial \varphi_1} = i$  $= i\left(-\frac{i\lambda}{6}\varphi\right) \cdot 3 \int \frac{d^4k}{(2\pi)^4} D_{11}$

$\mathcal{L} = -\frac{\lambda}{6} v_1 \phi_1^3$

$$= \frac{\lambda\varphi}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{i}{k^2 - m_\varphi^2} + \frac{1}{\omega_k} f(\omega_k) \right\}$$

$$= \frac{d}{d\varphi_1} \left\{ \frac{1}{2} \int \frac{d^4k_E}{(2\pi)^3} \log(k_E^2 + m^2) + T \int \frac{d^3k}{(2\pi)^3} \log\left(1 - e^{-\beta\omega}\right) \right\}$$

$$\Rightarrow \quad V_{1\text{-loop}}(\varphi_1) = \frac{1}{2} \int \frac{d^4k_E}{(2\pi)^3} \log(k_E^2 + m^2) + T \int \frac{d^3k}{(2\pi)^3} \log\left(1 - e^{-\beta\omega}\right).$$

Again our old result.

Special issue with the partition function:

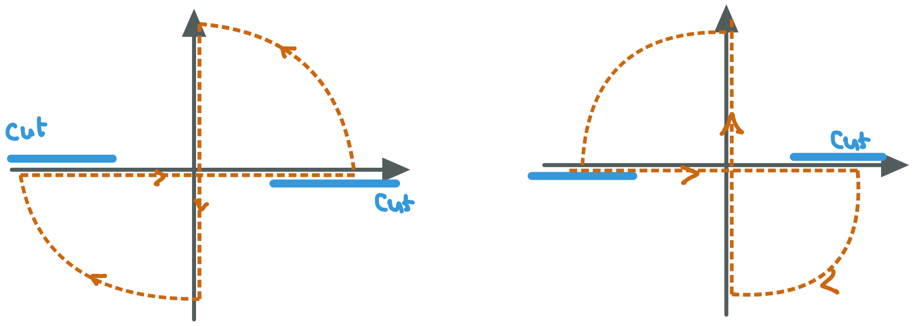
This example shows that $Z_{GVC_2}[\beta]$ is not the correct partition function. Indeed

$$\log Z_{GVC_2}^{free} = i W_{GVC_2}^{free} = i \log \int D\phi_1 D\phi_2\, e^{\frac{i}{2}\int \varphi_a D_{ab}^{-1} \varphi_b}$$

$$= \frac{i}{2}\log\det D_{ab} = \frac{i}{2}\log\left[\det U \begin{pmatrix} D & 0 \\ 0 & D^* \end{pmatrix} U\right] \qquad ; \det U = 1$$

$$= \frac{i}{2}\log\det D + \frac{i}{2}\log\det D^* = \frac{i}{2}\operatorname{Tr}\log D + \frac{i}{2}\operatorname{Tr}\log D^*$$

$$= \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(\frac{i}{k^2-m^2+i\eta}\right) + \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(\frac{-i}{k^2-m^2-i\eta}\right)$$



$$= \frac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\log\left(k_E^2+m^2\right) - \frac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\log\left(k_E^2+m^2\right) = 0$$

Thus the contributions to free energy from the combined path vanishes, with C_1 producing the correct vacuum limit, which C_2 then cancels. Yet, when we got the right V_{eff} by computing first $\partial V/\partial\varphi_1$. What is going on?

- If we include path C_3-contribution (extended Keldysh path), which is just the correct thermal result, we get the right result
  $Z[\beta]_{C_1 \cup C_2 \cup C_3} = Z[\beta]_{C_3} = Z[\beta]_{ITF}$. So $Z_{C_1 \cup C_2} = 0$ is correct!

- When we compute $\partial V/\partial\varphi_1$, we force the external leg to be 1-field. This restriction projects out only 11-component, setting
  $Z_{eff} = \int d\varphi_1 \frac{\partial V}{\partial\varphi_1} = \frac{i}{2}\log\det D_{11} \,. =$ .

Restriction to physical fields makes the difference!

Quasiparticles

Breaking $\overline{\Pi} = \mathrm{Re}\,\overline{\Pi} + i\,\mathrm{Im}\,\overline{\Pi}$, we can write first (D1 on p.24)

$$-iD_F = \frac{1}{k^2-m^2-\overline{\Pi}+i\eta} = \frac{1}{k^2-m^2-\mathrm{Re}\,\overline{\Pi}-i(\mathrm{Im}\,\overline{\Pi}-\eta)}$$

$$= \frac{(k^2-m^2-\mathrm{Re}\,\overline{\Pi})}{(k^2-m^2-\mathrm{Re}\,\overline{\Pi})^2+(\mathrm{Im}\,\overline{\Pi}-\eta)^2} + i\,\frac{\mathrm{Im}\,\overline{\Pi}-\eta}{(k^2-m^2-\mathrm{Re}\,\overline{\Pi})^2+(\mathrm{Im}\,\overline{\Pi}-\eta)^2}$$

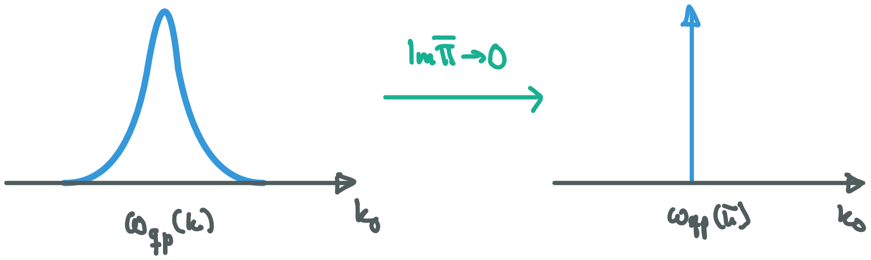
$$= a+ib \quad \Rightarrow D_F = ia-b$$

Similarly $D_F^* = -ia-b$. Then, using (D2) and (Π1) on p.24 we can write

$$D_{ab} = i\sigma_3 \frac{\overbrace{k^2-m^2-\mathrm{Re}\,\overline{\Pi}}^{\text{Principal part}}}{(k^2-m^2-\mathrm{Re}\,\overline{\Pi})^2+(\mathrm{Im}\,\overline{\Pi}-\eta)^2} + \frac{2(\eta-\mathrm{Im}\,\overline{\Pi})}{(k^2-m^2-\mathrm{Re}\,\overline{\Pi})^2+(\mathrm{Im}\,\overline{\Pi}-\eta)^2}\begin{pmatrix} n+\frac{1}{2} & \theta_-+n \\ \theta_++n & n+\frac{1}{2}\end{pmatrix}$$

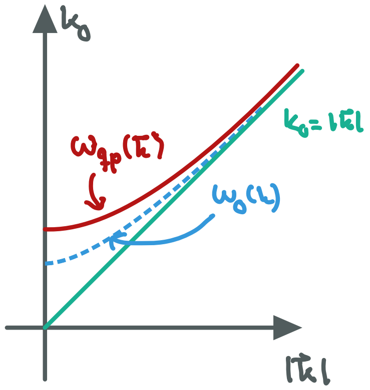
Let us suppose that $\mathrm{Im}\,\overline{\Pi}$ is either small or zero, but $\mathrm{Re}\,\overline{\Pi}$ is non-negligible. This is the physical case often. We might have for example a case where $\mathrm{Re}\,\overline{\Pi}$ arises at one loop, so that $\mathrm{Re}\,\overline{\Pi} \propto g^2$, while $\mathrm{Im}\,\overline{\Pi}$ arises only at two loops, and is a g^4. In this case it makes sense to make approximation

$$\frac{\eta-\mathrm{Im}\,\overline{\Pi}}{(k^2-m^2-\mathrm{Re}\,\overline{\Pi})^2+(\mathrm{Im}\,\overline{\Pi}-\eta)^2} \xrightarrow{\mathrm{Im}\,\overline{\Pi}\to 0_-} \pi\,\delta(k^2-m^2-\mathrm{Re}\,\overline{\Pi})$$



The peak of the distribution defines the quasiparticle dispersion relation

$$\boxed{k_0^2 - m^2 - \mathrm{Re}\,\bar{\Pi}(k_0, |\vec{k}|) = 0 \quad \Rightarrow \quad k_0 = \omega_{qp}(k)}$$



and we can expand $\left(\omega_0(\vec{k}) = \sqrt{k^2 + m^2}\right)$

$$k_0^2 - \omega_{0k}^2 - \mathrm{Re}\,\bar{\Pi}(\omega_{qp}, \vec{k}) - \underbrace{\frac{d}{dk_0}\mathrm{Re}\,\bar{\Pi}(k_0, \vec{k})\Big|_{k_0 = \omega_k}(k_0 - \omega_k)}_{\approx \frac{1}{2\omega_k}\frac{d}{dk_0}\mathrm{Re}\,\bar{\Pi}\,(k_0^2 - \omega_{qp}^2)} + \dots$$

$$= \left(1 - \frac{1}{2\omega_k}\frac{d}{dk_0}\mathrm{Re}\,\bar{\Pi}\Big|_{\omega_k}\right)\left(k_0^2 - \omega_{qp}^2\right) + \mathcal{O}\left((k_0 - \omega_{qp})^2\right) \approx Z_k^{-1}\left(k_0^2 - \omega_{qp}^2\right)$$

$$\Rightarrow \quad \boxed{D_{ab}^{qp} \approx \sum_{\pm} Z_{\pm}\left(\mathbb{P}\left(\frac{i}{k_0^2 - \omega_{qp}^2}\right)\sigma_3 + 2\pi\begin{pmatrix} n + \frac{1}{2} & \theta_- + n \\ \theta_+ + n & n + \frac{1}{2}\end{pmatrix}\delta(k_0 \mp \omega_{qp})\right)}$$

We have managed to extend the RTF - F-rule to effective degrees of freedom in the plasma. Given this propagator, we can compute even interactions between these states.

Sum-rule

Remember that the full spectral function ρ must satisfy the sum rule $\int \frac{dk_0}{2\pi} k_0 \rho = 1$. Here the full spectral function is

$$\rho = D^{>} - D^{<} = \frac{2(\eta - \mathrm{Im}\,\bar{\Pi})}{(k^2 - m^2 - \mathrm{Re}\,\bar{\Pi})^2 + (\mathrm{Im}\,\bar{\Pi} - \eta)^2}\left(-\theta_+ - n + \theta_- + n\right)$$

$$= 2\epsilon(k_0) \frac{\eta - \mathrm{Im}\,\bar{\Pi}}{(k^2 - m^2 - \mathrm{Re}\,\bar{\Pi})^2 + (\mathrm{Im}\,\bar{\Pi} - \eta)^2}$$

$$\xrightarrow{q.p.l} 2\pi\epsilon(k_0)\,\delta(k^2 - m^2 - \mathrm{Re}\,\bar{\Pi}) = \sum_{\pm} \pm Z_{\pm} \frac{\pi}{\omega_{qp,\pm}} \delta(k_0 \mp \omega_{qp,\pm})$$

So, in the quasiparticle limit

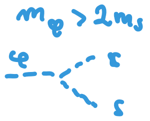
$$\int \frac{dk_0}{2\pi} k_0 \rho(k_0) = \int dk_0 \sum_{\pm} \frac{|k_0|}{2\omega_{qp}} Z_{\pm}\, \delta(k_0 \mp \omega_{qp}) = \sum_{\pm} Z_{\pm}.$$

So, the quasiparticle approximation respects the sum-rule only if $Z_+ + Z_- \overset{<1}{=} 1$, which in general is not the case. The problem is that the the approximation accounts only for a (pole) part of the complex structure of the propagator, or self-energy. What is missing is the "cut"-contribution from the additional complex structure of the propagator. This structure holds information of the inherently multiparticle structure of the propagator that is not encompassed by the quasiparticle picture.

- I should add a discussion of the discontinuity at this point in the next time.

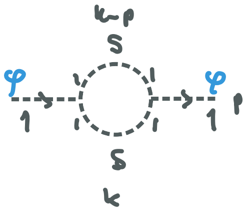
1-loop self-energy with momentum dependence.

Consider two scalars φ and s with interaction: $\mathcal{L} = -\frac{1}{2} g \varphi s s$



1. Real part in RTF-calculation

$$\Pi^h_{11} = i$$



$$= i\left(-\frac{ig}{2}\right)^2 \cdot 2 \int \frac{d^4k}{(2\pi)^4} \left(\overbrace{\mathbb{P}\left(\frac{i}{k^2-m^2}\right) + 2\pi\left(n_k+\frac{1}{2}\right)\delta(k^2-m^2)}^{D_{11}(k)} \right)$$

$$\times \left(\mathbb{P}\left(\frac{i}{(k-p)^2-m^2}\right) + 2\pi\left(n_{k-p}+\frac{1}{2}\right)\delta((k-p)^2-m^2) \right)$$

where $n_k \equiv f(\omega_k)$, and we used $\frac{i}{k^2-m^2+i\eta} = \overset{\text{principal part}}{\mathbb{P}}\left(\frac{i}{k^2-m^2}\right) + \pi\delta(k^2-m^2)$. Let us compute only the real part. This consists of the mixed terms (one $\mathbb{P}$-pr and one δ-function) only:

$$\mathrm{Re}\,\Pi^h_{11} = \frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\, 2\pi\left(n_k+\frac{1}{2}\right)\delta(k^2-m^2)\,\mathbb{P}\left(\frac{1}{(k+p)^2-m^2} + \frac{1}{(k-p)^2-m^2}\right)$$

$$= \frac{g^2}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left(n_k+\frac{1}{2}\right)\sum_{\pm}\left(\frac{1}{p^2+2k_\pm p} + \frac{1}{p^2-2k_\pm p}\right) \qquad k_\pm \equiv (\pm\omega_k, \vec{k})$$

$$= \frac{g^2}{16\pi^2}\int dk\,\frac{k^3}{\omega_k}\left(n_k+\frac{1}{2}\right)\underbrace{\sum_{s,s'=\pm1}\int_{-1}^{1} dz\,\frac{1}{p^2+2s'\omega p_0+2s\,kpz}}$$

$$= \sum_{ss'}\frac{s}{2kp}\log\left(\frac{p^2+2s'\omega p_0+2skp}{p^2+2s'\omega p_0-2skp}\right) = \sum_{s'}\frac{1}{kp}\log\left(\frac{p^2+2s'\omega p_0+2kp}{p^2+2s'\omega p_0-2kp}\right)$$

$$= \frac{1}{kp}\log\left(\frac{(p^2+2kp)^2-4\omega^2p_0^2}{(p^2-2kp)^2-4\omega^2p_0^2}\right)$$

$$= \frac{g^2}{16\pi^2 p}\int_m^\infty d\omega_k\left[n(\omega_k)+\frac{1}{2}\right]\log\left(\frac{(\overset{p_0^2-p^2}{p^2}+2kp)^2-4\omega_k^2p_0^2}{(p^2-2kp)^2-4\omega_k^2p_0^2}\right) = \mathrm{Re}\,\Pi_{11}(p_0,p)$$

2) Imaginary time calculation

Here we will need complex continuation etc..

$$\Pi_{\vec{p},\omega_m} = -\left(-\frac{g}{2}\right)^2 \cdot 2 \int \frac{d^3k}{(2\pi)^3} \sum_n \frac{1}{(\omega_n^2-\omega_{\vec{k}}^2)((\omega_n-\omega_m)^2-\omega_{\vec{k}-\vec{p}}^2)} \quad ; \; \omega_{\vec{k}} \equiv \vec{k}^2+m^2$$

Using the analytic continuation: $\Delta_k(\tau) = \int \frac{dk_0}{2\pi} e^{-ik_0 t} D^>(k_0)\Big|_{t=-i\tau}$

we can write the sum as

$$= \int \frac{dk_0}{2\pi} e^{-k_0\tau} D^>(k_0) = \int \frac{dk_0}{2\pi} e^{-k_0\tau}(1+f(k_0))\rho(k_0)$$

$$\sum_n " = \int_0^\beta d\tau\, d\tau' \, \Delta_{\vec{k}}(\tau)\Delta_{\vec{k}-\vec{p}}(\tau') \sum_n e^{-i\omega_n\tau - i(\omega_n-p_0)\tau'}$$

$$= \int_0^\beta d\tau\, e^{ip_0\tau} \Delta_{\vec{k}}(\tau)\Delta_{\vec{k}-\vec{p}}(\tau)$$

$$= \int \frac{dk_0}{2\pi}\frac{dk_0'}{2\pi} \int_0^\beta d\tau\, e^{-(k_0+k_0'-i\omega_m)\tau} (1+f(k_0))(1+f(k_0'))\rho_{\vec{k}}(k_0)\rho_{\vec{k}-\vec{p}}(k_0')$$

$$= \int \frac{dk_0\,dk_0'}{4\pi^2} \frac{1}{k_0+k_0'-i\omega_m}\left(1-e^{-(k_0+k_0')\beta}\right) e^{\beta(k_0+k_0')} f(k_0)f(k_0')\,\rho_{\vec{k}}(k_0)\rho_{\vec{k}-\vec{p}}(k_0')$$

$= -f(-k_0')$; switch $k_0' \to -k_0'$

$$= \int \frac{dk_0\,dk_0'}{4\pi^2} \frac{1}{k_0+k_0'-i\omega_m}\left(1+f(k_0')+f(k_0)\right) 2\pi\epsilon(k_0)\delta(k_0^2-\omega_k^2)\, 2\pi\epsilon(k_0')\delta(k_0'^2-\omega_{\vec{k}-\vec{p}}^2)$$

$$= \int dk_0\,dk_0' \left(\frac{1}{k_0+k_0'-i\omega_m} f(k_0) + \frac{1}{k_0-k_0'-i\omega_m} f(k_0')\right)\epsilon(k_0)\epsilon(k_0')\delta(k_0^2-\omega_k^2)\delta(k_0'^2-\omega_{\vec{k}-\vec{p}}^2)$$

$$= \int \frac{dk_0}{2\pi} \underbrace{\sum_\pm \frac{1}{2\omega_{\vec{k}-\vec{p}}} \frac{\pm 1}{k_0 \pm \omega_{\vec{k}-\vec{p}} - i\omega_m}} f(k_0)\rho(k_0) + \int \frac{dk_0'}{2\pi} \underbrace{\sum_\pm \frac{1}{2\omega_{\vec{k}}}}\, \frac{\pm 1}{-k_0' \pm \omega_{\vec{k}} - i\omega_m} f(k_0')\rho_{\vec{k}-\vec{p}}(k_0)$$

$$= \frac{1}{2\omega_{\vec{k}-\vec{p}}}\left(\frac{1}{k_0+\omega_{\vec{k}-\vec{p}}-i\omega_m} - \frac{1}{k_0-\omega_{\vec{k}-\vec{p}}-i\omega_m}\right) = \frac{-1}{(k_0-i\omega_m)^2-\omega_{\vec{k}-\vec{p}}^2}$$

first set $k \to \vec{k}+\vec{p}$ [we are under integral]

$\Rightarrow$ use the left result with $\vec{k}-\vec{p} \to \vec{k}+\vec{p}$.

$$= -\int \frac{dk_0}{2\pi} f(k_0)\rho(k_0)\left(\frac{1}{(k_0-i\omega_m)^2-\omega_{\vec{k}-\vec{p}}^2} + \frac{1}{(k_0+i\omega_m)^2-\omega_{\vec{k}+\vec{p}}^2}\right)$$

this can be done only now!

Finally putting this back to original expression and continuing $i\omega_m \to p_0$, we get

even in $k_0 \leftrightarrow -k_0$

$$\mathrm{Re}\,\Pi_{11}(p_0,p) = -\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\underbrace{f(k_0)\rho(k_0)}\left(\frac{1}{(k_0-p_0)^2-\omega^2_{\vec{k}-\vec{p}}}+\frac{1}{(k_0+p_0^2)^2-\omega^2_{\vec{k}+\vec{p}}}\right)$$

$$= \left(n(k_0)+\theta(k_0)\right)2\pi\delta(k^2-m^2)$$

odd in $k_0 \leftrightarrow -k_0$

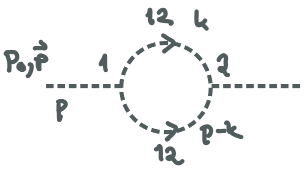
$$= \left(n(k_0)+\frac{1}{2}\right)2\pi\delta(k^2-m^2) + \pi\epsilon\cancel{(k_0)}\delta(k^2+m^2)$$

$$= \frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\,2\pi\left(n_k+\frac{1}{2}\right)\delta(k^2-m^2)\left(\frac{1}{p^2-2k\cdot p}-\frac{1}{p^2+2k\cdot p}\right)$$

Same that we got in RTF, but somewhat longer computation. (Sum could have been done somewhat faster using contour integration, but still.)

$D^>(-k_0) = D^<(k_0)$

2. Imaginary part in RTF-calculation



$$\pi^<(p_0,\vec{p}) = i\left(\frac{-ig}{2}\right)\left(\frac{ig}{2}\right)2\int\frac{d^4k}{(2\pi)^4}\,2\pi\,\epsilon(k_0)f_s(k_0)\,\delta(k^2-m^2)\;2\pi\,\epsilon(p_0-k_0)f_s(p_0-k_0)\,\delta((k-p)^2-m^2)$$

$$= i\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\,2\pi\,\epsilon(k_0)f_s(k_0)\,\delta(k^2-m^2)\;2\pi\,\epsilon(k_0')f_s(k_0')\,\delta(k'^2-m^2)\,(2\pi)^4\delta^4(p_0-k_0-k_0')$$

Consider $p_0>0$ ⇒ $k_0+k_0'>0$. Kinematic constraints now imply $k_0>0$ & $k_0'>0$

$$\Rightarrow\quad \pi^<(\omega_p,\vec{p}) = i\frac{g^2}{2}\int\frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\,f_s(\omega_k)f_s(\omega_{k'})\,(2\pi)^4\delta^4(p-k-k')$$

"backward scattering"

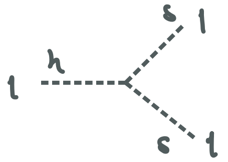
Similarly, using $\pi^>(p_0,\vec{p}) = e^{\beta p_0}\pi^<(p_0,\vec{p})$ $\quad\left(e^{\beta p_0}f(\omega_k)f(\omega_{k'}) = e^{\beta(\omega_k+\omega_{k'})}f(\omega_k)f(\omega_{k'})\right)$

$$\pi^>(\omega_p,\vec{p}) = i\,\frac{g^2}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\,\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\,\underbrace{(1+f_s(\omega_k))(1+f_s(\omega_{k'}))}_{\text{"forward scattering"}}\,(2\pi)^4\,\delta^4(p-k-k')$$

and then

$$\operatorname{Im}\bar{\Pi} = \frac{i}{2}(\Pi^>-\Pi^<) = \overset{\text{sign!}}{-}\,\frac{g^2}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\,\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\,\underset{\text{"thermal decay rate"}}{(1+2f(\omega_k))\,(2\pi)^4\,\delta^4(p_0-k_0-k_0')}$$

- One cannot help noting the connection of of $\pi^<$ and $\pi^>$ with the usual backward and forward scattering terms in the Boltzmann equation for h, describing the processes $ss \to h$ and $h \to ss$, respectively. Likewise $\operatorname{Im}\bar{\Pi}$ is just the thermally corrected decay rate. (or damping rate). Indeed



; $\mathcal{M} = -\frac{ig}{2}\cdot 2 \Rightarrow |\mathcal{M}|^2 = g^2$; $S = \frac{1}{2!} = \frac{1}{2}$

Then, accounting for the symmetry factor $\frac{1}{2!}$ in the final state, the usual BE for f_h is

$$\frac{df_h}{dt} = -\frac{g^2}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\,\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\,(2\pi)^4\,\delta^4(p_0-k_0-k_0')$$

$$\times\left[f_h(\omega_p)(1+f_s(\omega_k))(1+f_s(\omega_{k'})) - (1-f(\omega_p))\,f_s(\omega_k)\,f_s(\omega_{k'})\right]$$

$$= f_h(\omega_p)\,i\pi^>(\omega_p,\vec{p}) - (1-f(\omega_p))\,i\pi^<(\omega_p,\vec{p})$$

Looks like we are onto something interesting? Yes, in RTF one can formulate out-of-eq. thermal field theory in the on-shell limit, by simply continuing f_{eq} into an out-of-equilibrium distributions in the F-rules. More on that on part. 4!

Fermion field in RTF

Fermion fields are introduced analogously to bosons, just moving to anti-commutation relations. First we write the Mills representation:

$$
\begin{aligned}
S_c(t) &= \theta_c(t)\langle \psi(t,\vec{x})\,\bar{\psi}(0)\rangle_\beta - \theta_c(-t)\langle \bar{\psi}(0)\,\psi(t,\vec{x})\rangle_\beta \\
&= \theta_c(t)\,S^>(t) - \theta_c(-t)\,S^<(t) \\
&= -S^<(t) + \theta_c(t)\left[S^>(t) + S^<(t)\right] \\
&= -S^<(t) + \theta_c(t)\,\rho_F(t)
\end{aligned}
$$

eg: $S^> \equiv \langle \psi(t,\vec{x})\,\bar{\psi}(0)\rangle_\beta$

$S^< \equiv +\langle \bar{\psi}(0)\,\psi(t,\vec{x})\rangle_\beta$

the sign definition varies in literature

For a free field

$$
\begin{aligned}
\rho_{F0} &= \int d^4x\, e^{-ik\cdot x}\langle\{\psi(t,\vec{x}),\bar{\psi}(0)\}\rangle_\beta \\
&= \int d^4x\, e^{-ik\cdot x}\int_{q,q'}\sum_{s,s'}\Big\langle\Big\{a^s_{\vec{q}}\,u_s(\vec{q})\,e^{-iq\cdot x} + b^{s\dagger}_{\vec{q}}\,v_s(\vec{q})\,e^{iq\cdot x}, \\
&\qquad\qquad a^{s\dagger}_{\vec{q}'}\,\bar{u}_s(\vec{q}') + b^{s'}_{\vec{q}'}\,\bar{v}_s(\vec{q}')\Big\}\Big\rangle_\beta \\
&= \int d^4x\, e^{-ik\cdot x}\int_{\vec{q},\vec{q}'}\sum_{s,s'}\Big(\langle\{a^s_{\vec{q}},a^{s'\dagger}_{\vec{q}'}\}\rangle_\beta\, u_s(\vec{q})\,\bar{u}_{s'}(\vec{q}')\,e^{-iq\cdot x} \\
&\qquad\qquad + \langle\underbrace{\{b^{s\dagger}_{\vec{q}},b^{s'}_{\vec{q}'}\}}_{(2\pi)^3\, 2\omega_q\,\delta^3(\vec{q}-\vec{q}')}\rangle_\beta\, v_s(\vec{q})\,\bar{v}_{s'}(\vec{q}')\,e^{iq\cdot x}\Big) \\
&= \int d^4x\, e^{-ik\cdot x}\int\frac{d^4q}{(2\pi)^4}\,2\pi\,\theta(q_0)\,\delta(q^2-m^2)\sum_s\Big(\underbrace{u_s(\vec{q})\,\bar{u}_s(\vec{q})}_{\not{q}+m}\,e^{-iq\cdot x} + \underbrace{v_s(q)\,\bar{v}_s(\vec{q})}_{\not{q}-m}\,e^{iq\cdot x}\Big) \\
&= 2\pi\,\epsilon(k_0)\,(\not{k}+m)\,\delta(k^2-m^2).
\end{aligned}
$$

KMS-condition:

$$S^>(t) = \frac{1}{\mathrm{Tr}\hat\rho}\,\mathrm{Tr}\left[e^{-\beta(\hat H-\mu\hat Q)}\,\hat\psi(t)\,\hat{\bar\psi}(0)\right]$$

$$= \frac{1}{\mathrm{Tr}\hat\rho}\,\mathrm{Tr}\left[e^{-\beta(\hat H-\mu\hat Q)}\,\hat{\bar\psi}(0)\,\underbrace{e^{\beta\mu\hat Q}\,\hat\psi(t+i\beta)\,e^{-\beta\mu\hat Q}}_{=\,e^{-\beta\mu}\,\hat\psi(t+i\beta)}\right]$$

$$= e^{-\beta\mu}\,S^<(t+i\beta)$$

where I again used $\{\psi_\alpha(x), i\psi^\dagger_\beta(y)\} = i\delta^3(\bar x-\bar y)\,\delta_{\alpha\beta}$ and $\hat Q = \int d^3x\,\hat\psi^\dagger\hat\psi$ whereby $[\hat Q,\hat\psi] = -\hat\psi$. In the momentum space

$$S^>(k_0) = e^{\beta(k_0-\mu)}\,S^<(k_0)$$

$$\Rightarrow\quad S^> + S^< = \left(1+e^{\beta(k_0-\mu)}\right)S^<(k_0) = \rho_F(k_0) \;\Rightarrow\; \begin{cases} S^<(k_0) = f_F(k_0-\mu)\,\rho_F(k_0) \\ S^>(k_0) = \left(1-f_F(k_0-\mu)\right)\rho_F(k_0) \end{cases}$$

with $f_F(k_0-\mu) = \dfrac{1}{e^{\beta(k_0-\mu)}+1}$

Based on the Mills representation

$$S_{11}(k_0) = \int\frac{d^4k}{(2\pi)^4}\,e^{ik\cdot x}\left(\theta(t) - f(k_0-\mu)\right)2\pi\,\epsilon(k_0)\,(\not k-m)\,\delta(k^2-m^2)$$

Now

- $$\int\frac{dk_0}{2\pi}\,e^{ik_0t}\,\theta(t)\,2\pi(\not k-m)\,\frac{1}{2\omega}\left(\delta(k_0-\omega_k)-\delta(k_0+\omega_k)\right)$$

$$= \frac{\theta(t)}{2\omega_k}\left((\not k_+ + m)\,e^{-i\omega_k t} - (\not k_- + m)\,e^{i\omega_k t}\right) \qquad ;\; k^\mu_\pm = (\pm\omega_{\bar k}, \bar k)$$

$$= \frac{i}{2\omega_k}\int\frac{dk_0'}{2\pi}\frac{e^{-ik_0't}}{k_0'+i\eta}\left((\not{k}_+ + m)e^{-i\omega_k t} - (\not{k}_- + m)e^{i\omega_k t}\right)$$

$$= \frac{i}{2\omega_k}\int\frac{dk_0}{2\pi}e^{-ik_0t}\left(\frac{\not{k}_+ + m}{k_0-\omega_k+i\eta} - \frac{\not{k}_- + m}{k_0+\omega_k+i\eta}\right)$$

$$= \frac{i}{2\omega_k}\int\frac{dk_0}{2\pi}e^{-ik_0t}(\not{k}+m)\underbrace{\left(\frac{1}{k_0-\omega_k+i\eta} - \frac{1}{k_0+\omega_k+i\eta}\right)} \quad ; \; k^\mu = (k_0;\vec{k})$$

$$= \frac{2\omega_k}{(k_0+i\eta)^2-\omega_k^2} = \frac{2\omega}{k_0^2-\omega_k^2+i\epsilon(k_0)\eta} = \frac{2\omega}{k^2-m^2+i\eta(1-2\theta(-k_0))}$$

$$= \ldots \int\frac{dk_0}{2\pi}e^{-ik_0t}(\not{k}+m)\left(\frac{i}{k^2-m^2+i\eta} - \theta(-k_0)\,2\pi\,\delta(k^2-\omega^2)\right)$$

$$\Rightarrow S_{11}(k_0) = (\not{k}+m)\left(\frac{i}{k^2-m^2+i\eta} - \underbrace{\left(f(k_0-\mu)+\theta(-k_0)\right)}_{f(k_0-\mu)\,\theta(k_0)\,+\,f(|k_0|+\mu)\,\theta(-k_0)}2\pi\,\delta(k^2-m^2)\right)$$

Similarly, one can derive an expression for S_{22}. The final results are

$$\boxed{\begin{aligned}
S_{11}(k_0) &= (\not{k}+m)\left(\frac{i}{k^2-m^2+i\eta} - \left(f(k_0-\mu)+\theta(-k_0)\right)2\pi\,\delta(k^2-m^2)\right)\\
S_{22}(k_0) &= (\not{k}+m)\left(\frac{-i}{k^2-m^2-i\eta} - \left(f(k_0-\mu)+\theta(-k_0)\right)2\pi\,\delta(k^2-m^2)\right).\\
S_{12}(k_0) &= S^<(k_0) = 2\pi\,\epsilon(k_0)\,f_F(k_0-\mu)\,(\not{k}+m)\,\delta(k^2-m^2)\\
S_{21}(k_0) &= S^>(k_0) = 2\pi\,\epsilon(k_0)\left(1-f_F(k_0-\mu)\right)(\not{k}+m)\,\delta(k^2-m^2)
\end{aligned}}$$

Some alternative forms:

$$f_F(k_0-\mu)+\theta(-k_0) = \theta(k_0)\,f_F(k_0-\mu) + \theta(-k_0)\,f_F(|k_0|+\mu) = f_F(|k_0|-\mu\,\epsilon(k_0)) \equiv n_F(k_0)$$

$$\epsilon(k_0)\,f_F(k_0-\mu) = \theta(k_0)\,f_F(k_0-\mu) - \theta(-k_0)\,f_F(-|k_0|-\mu) = n_F(k_0) - \theta(-k_0)$$

$$\epsilon(k_0)\left(1-f_F(k_0-\mu)\right) = \theta(k_0) - \theta(-k_0) - n_F(k_0) + \theta(-k_0) = \theta(k_0) - n_F(k_0)$$

Quasifermions

Just as for bosons, we can derive general results for fermionic thermal propagator including interactions, using the rotated basis. We can again write

$$S_{ab} = U_F \begin{pmatrix} S_F & 0 \\ 0 & -S_F^* \end{pmatrix} U_F \quad , \quad \text{where} \quad S_F^* = \gamma^0 S_F^\dagger \gamma^0$$

This follows from the fact that $S_F(x-y) = -\gamma^0 S_F^\dagger(x-y)\gamma^0$.

Let us actually derive U_F. Start by writing $U_F \equiv \begin{pmatrix} a & c \\ b & a \end{pmatrix}$ and then note

$$S_{ab} = \begin{pmatrix} a & c \\ b & a \end{pmatrix}\begin{pmatrix} S_F & 0 \\ 0 & -S_F^* \end{pmatrix}\begin{pmatrix} a & c \\ b & a \end{pmatrix} = \begin{pmatrix} a & c \\ b & a \end{pmatrix}\begin{pmatrix} S_F a & S_F c \\ -S_F^* b & -S_F^* a \end{pmatrix}$$

$$= \begin{pmatrix} a^2 S_F - bc S_F^* & ac(S_F - S_F^*) \\ ab(S_F - S_F^*) & -a^2 S_F^* + bc S_F \end{pmatrix} = (a^2 - bc)\begin{pmatrix} S_F & 0 \\ 0 & -S_F^* \end{pmatrix} + \begin{pmatrix} bc & ac \\ ab & bc \end{pmatrix}(S_F - S_F^*)$$

In momentum space: $S_F = \dfrac{i}{\not{p} - m + i\eta}$, so that $-S_F^* = \gamma^0 \dfrac{i}{\not{p}^\dagger - m - i\eta}\gamma^0 = \dfrac{i}{\not{p} - m - i\eta}$

$$\Rightarrow S_F - S_F^* = \frac{i}{\not{p} - m + i\eta} - \frac{i}{\not{p} - m - i\eta} = 2\pi(\not{p} + m)\delta(p^2 - m^2)$$

It is then easy to see that

$$\left.\begin{aligned} a^2 - bc &= 1 \\ bc &= -n_F \end{aligned}\right\} \Rightarrow \underline{a = \sqrt{1 - n_F}}$$

$$\left.\begin{aligned} ac &= n_F - \theta_- \\ ab &= \theta_+ - n_F \end{aligned}\right\} \Rightarrow c = \frac{\theta_- - n_F}{\sqrt{1 - n_F}} \qquad \Rightarrow b = \frac{\theta_+ - n_F}{\sqrt{1 - n_F}}$$

Now $n_F = f(|k_0| - \epsilon(k_0)\mu) \Rightarrow 1 - n_F = e^{\beta(|k_0| - \epsilon(k_0)\mu)} n_F \equiv e^x n_F(x)$: $x \equiv \beta(|k_0| - \epsilon(k_0)\mu)$.

and $$\frac{\theta_- - n_F}{\sqrt{1 - n_F}} = \begin{cases} \sqrt{1 - n_F} = \sqrt{n_F}\, e^{x/2} & ; k_0 < 0 \\ -\dfrac{n_F}{\sqrt{1 - n_F}} = -\sqrt{n_F}\, e^{-x/2} & ; k_0 > 0 \end{cases}$$ eg $\underline{c = -\epsilon\sqrt{n_F}\, e^{-\epsilon x/2}}$.

and similarly $b = \frac{\theta_+ - n_F}{\sqrt{1-n_F}} = \begin{cases} \sqrt{1-n_F} = \sqrt{n_F}\, e^{x/2} & ; k_0 > 0 \\ -\frac{n_F}{\sqrt{1-n_F}} = -\sqrt{n_F}\, e^{-x/2} & ; k_0 < 0 \end{cases}$ eg $b = \epsilon \sqrt{n_F}\, e^{\epsilon x/2}$

$$\Rightarrow U_F = \begin{pmatrix} \sqrt{1-n_F} & \frac{\theta_+ - n_F}{\sqrt{1-n_F}} \\ \frac{\theta_- - n_F}{\sqrt{1-n_F}} & \sqrt{1-n_F} \end{pmatrix} = \sqrt{n_F} \begin{pmatrix} e^{x/2} & \epsilon e^{\epsilon x/2} \\ -\epsilon e^{-\epsilon x/2} & e^{x/2} \end{pmatrix}$$

$$\Rightarrow U_F^{-1} = \begin{pmatrix} \sqrt{1-n_F} & \frac{n_F - \theta_-}{\sqrt{1-n_F}} \\ \frac{n_F - \theta_+}{\sqrt{1-n_F}} & \sqrt{1-n_F} \end{pmatrix} = \sqrt{n_F} \begin{pmatrix} e^{x/2} & \epsilon e^{-\epsilon x/2} \\ -\epsilon e^{\epsilon x/2} & e^{x/2} \end{pmatrix}$$

Given U_F and U_F^{-1} we can diagonalize the Dyson equation. In diagonal basis

$$S_F = S_0 + S_0(-i\Sigma)\, S_F \Leftrightarrow (S_0^{-1} + i\Sigma)\, S_F = 1 \Leftrightarrow S = \frac{1}{S_0^{-1} + i\Sigma_F}$$

Then, as with bosons $S = \begin{pmatrix} \frac{i}{\not p - m - \Sigma} & 0 \\ 0 & \frac{i}{\not p - m - \Sigma^*} \end{pmatrix}$, $\Sigma^* \equiv \gamma^0 \Sigma^\dagger \gamma^0$

From Dyson $S^{-1} = S_0^{-1} + i\Sigma$; Σ diagonalized by U_F & inverted back by U_F^{-1}:

$$\Rightarrow \Sigma_{ab} = n_F \begin{pmatrix} e^{x/2} & \epsilon e^{-\epsilon x/2} \\ -\epsilon e^{\epsilon x/2} & e^{x/2} \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma^* \end{pmatrix} \begin{pmatrix} e^{x/2} & \epsilon e^{-\epsilon x/2} \\ -\epsilon e^{\epsilon x/2} & e^{x/2} \end{pmatrix}$$

$$= n_F \begin{pmatrix} e^{x/2} & \epsilon e^{-\epsilon x/2} \\ -\epsilon e^{\epsilon x/2} & e^{x/2} \end{pmatrix} \begin{pmatrix} \Sigma e^{x/2} & \epsilon \Sigma e^{-\epsilon x/2} \\ \epsilon \Sigma^* e^{\epsilon x/2} & -\Sigma^* e^{x/2} \end{pmatrix}$$

$$= n_F \begin{pmatrix} \Sigma e^x + \Sigma^* & \epsilon(\Sigma - \Sigma^*)\, e^{(1-\epsilon)x/2} \\ -\epsilon(\Sigma - \Sigma^*)\, e^{(1+\epsilon)x/2} & -\Sigma^* e^x - \Sigma \end{pmatrix}$$

$$\text{Thus} \begin{cases} \Sigma_{11} = n_F(\Sigma e^x + \Sigma^*) \\ \Sigma_{11}^* = n_F(\Sigma^* e^x + \Sigma) \end{cases} \Rightarrow e^x \Sigma_{11} - \Sigma_{11}^* = (e^{2x} - 1)\, n_F \Sigma = (e^x - 1)\Sigma$$

$$\Rightarrow \Sigma = \frac{1}{e^x - 1}\left(e^x \Sigma_{11} - \gamma^0 \Sigma_{11}^\dagger \gamma^0\right)$$

Let us now define $\bar{\Sigma}_{11} \equiv \gamma^0 \Sigma_{11}$. Then $\bar{\Sigma}^{*} = \Sigma_{11}^{\dagger}\gamma^0 = (\gamma^0\Sigma_{11})^{\dagger}$, and we can write

$$\underline{\bar{\Sigma}} = \frac{1}{e^x-1}\left(e^x\bar{\Sigma}_{11} - \bar{\Sigma}_{11}^{\dagger}\right) = \bar{\bar{\Sigma}}_{11}^{H} + i\coth\frac{x}{2}\,\bar{\Sigma}_{11}^{A}$$

$$= \underline{\bar{\bar{\Sigma}}_{11}^{H} + i\epsilon(k_0)\coth\frac{\beta}{2}(k_0-\mu)\,\bar{\Sigma}_{11}^{A}}$$

where $\Sigma_{11} = \Sigma_{11}^{H} + i\Sigma_{11}^{A}$ where $\bar{\Sigma}_{11}^{H} \equiv \frac{1}{2}(\Sigma_{11}+\Sigma_{11}^{\dagger})$ and $\bar{\Sigma}_{11}^{A} \equiv \frac{i}{2}(\Sigma_{11}-\Sigma_{11}^{\dagger})$ are both Hermitean. So we find the connections

$$\underline{\bar{\Sigma}^{H} = \bar{\Sigma}_{11}^{H}} \quad \text{and} \quad \underline{\bar{\Sigma}^{A} = \epsilon(k_0)\coth\frac{\beta}{2}(k_0-\mu)\,\bar{\Sigma}_{11}^{A}}$$

On the other hand

$$\underline{\bar{\Sigma}^{<}} = (\bar{\Sigma}-\bar{\Sigma}^{\dagger})\,\epsilon n_F \overbrace{e^{(1-\epsilon)\frac{x}{2}}}^{e^{+\theta_- x} = 1+\theta_-(e^x-1) = \theta_- e^x + \theta_+} = 2i(\theta_+ - \theta_- e^x)\,n_F\,\bar{\Sigma}^{A}$$

$$= 2i\left(\theta_+ n_F - \theta_-(1-n_F)\right)\bar{\Sigma}^{A} = 2i(n_F-\theta_-)\,\bar{\Sigma}^{A} = \underline{2i\epsilon(k_0)\,f_F(k_0-\mu)\,\bar{\Sigma}^{A}}$$

$$\underline{\bar{\Sigma}^{>}} = -(\bar{\Sigma}-\bar{\Sigma}^{\dagger})\,\epsilon n_F e^{(1+\epsilon)\frac{x}{2}} = 2i(\theta_- - \theta_+ e^x)\,n_F\,\bar{\Sigma}^{A}$$

$$= -2i\left(\theta_- n_F - \theta_+(1-n_F)\right)\bar{\Sigma}^{A} = 2i(\theta_+ - n_F)\,\bar{\Sigma}^{A} = \underline{2i\epsilon(k_0)\left(1-f_F(k_0-\mu)\right)\bar{\Sigma}^{A}}$$

$$= e^{-\beta(k_0-\mu)}\bar{\Sigma}^{<}$$

That is $\underline{\bar{\Sigma}^{A} = -\frac{i}{2}\epsilon(k_0)(\bar{\Sigma}^{>}+\bar{\Sigma}^{<}) = -\frac{i}{2}\epsilon(k_0)\left(1+e^{\beta(k_0-\mu)}\right)\bar{\Sigma}^{<}}$

Fermion sum-rule Canonical commutation relation $\{\hat{\psi}(x), \hat{\psi}^{\dagger}(y)\} = \delta^3(\vec{x}-\vec{y})\mathbb{1}$.

$$\Rightarrow \quad \underline{\rho_{F,\vec{k}}(t,t)} = \left\langle \{\hat{\psi}_{\vec{k}}(t), \hat{\bar{\psi}}_{\vec{k}}^{\dagger}(t)\} \right\rangle_{\beta} = \underline{\gamma^0} \quad \Leftrightarrow \quad \boxed{\int \frac{dk_0}{2\pi}\,\rho_F(k_0,t) = \gamma^0}$$

Quasifermions Just as we did for the scalar field, we can introduce quasifermions as an effective description in plasma. Let us define $\bar{S} = S\gamma^0$

$$S_F = U_F \begin{pmatrix} S & 0 \\ 0 & -S^* \end{pmatrix} U_F \qquad \text{where} \quad S^{-1} = \not{p} - m - \Sigma_H - i\Sigma_A$$

$$= \begin{pmatrix} S & 0 \\ 0 & -S^* \end{pmatrix} + \begin{pmatrix} m_F & \theta_+ - m_F \\ \theta_- - m_F & m_F \end{pmatrix} (S - S_*)$$

To compute this explicitly we should be able to invert $S = (S^{-1})^{-1}$. For most general Σ_H this is cumbersome, so let us only consider a restricted form,

$$\Sigma_H \equiv -a(p_0, \vec{p})\not{p} - b(p_0, \vec{p})\not{u} + c(p_0, \vec{p}) \qquad (1)$$

Here $\not{u} = \gamma_\mu u^\mu$, where u^μ is the plasma 4-velocity and a, b & c are scalar functions. In the plasma frame $u^\mu = (1, \vec{0})$. The form (1) covers vector-like gauge interactions such as QED & QCD.

Here we will also drop the antihermitean part, eg we will not consider width.

The dispersion relation is given by $\det(\not{p} - m - \Sigma_H) = 0$ with the form (1) this becomes

$$\det\left((1+a)\not{p} + b\not{u} - (m+c)\right) = 0$$

We work in the plasma frame and define

$$\vec{q} \equiv (1+a)\,\vec{p} \equiv r\vec{p}$$

$$q_0 \equiv (1+a)p_0 + b = r p_0 + b \equiv r n p_0$$

$$m_c \equiv m + c.$$

$$\det(\not{q}-m_c)=0 \iff q^2=m_c^2 \iff q_0=\pm\sqrt{\vec{q}^{\,2}+m_c^2}$$

$$\iff rnp_0=\pm\sqrt{(r\vec{p})^2+(m+c)^2}$$

$$\implies p_0=\omega_\pm$$

We will see that we can have $rn \gtrless 0$ for $p_0>0$, so there will be two branches of particle like solutions. Here + refers to usual particles and − to new collective excitations often called holes. For negative frequencies one finds the same solutions (given no $\cancel{CP}$). $p_0 \to s_c\omega_\pm$, where $s_c=+1$ in particle and $s=-1$ in antiparticle sectors. Inverting propagator is simple

$$S^{-1}=\not{q}-m_c \implies S=\frac{\not{q}+m_c}{q^2-m_c^2}=\frac{mrp_0\gamma^0-r\vec{\gamma}\cdot\vec{p}+m_c}{q^2-m_c^2}$$

Near quasiparticle poles

$$q^2-m_c^2 \simeq \frac{\partial(q^2-m_c^2)}{\partial p_0}(p_0-\overset{\substack{\text{particle-antiparticle sign}\\ \downarrow}}{s_c}\,\underset{\substack{\uparrow\\ \text{particles (+) and holes (−)}}}{\omega_\pm}) \equiv (2q_0Z^{-1})_\pm^{s_c}(p_0-s_c\omega_\pm)=2(rnp_0Z^{-1})_\pm^{s_c}(p_0-s_c\omega_\pm)$$

$$\implies \bar{S}=S\gamma^0 \simeq \sum_{s_c,\pm}\frac{Z_\pm^{s_c}}{p_0-s_c\omega_\pm}\frac{1}{2}\left(1+\frac{s_c}{(nr)_\pm}\left(\frac{r_\pm\vec{\alpha}\cdot\vec{p}+m_c\gamma^0}{\omega_\pm}\right)\right)$$

$$\simeq \sum_{s_c,\pm}\frac{Z_\pm^{s_c}}{p_0-s_c\omega_\pm}\frac{1}{2}\left(1+s_c\frac{H_\pm}{\omega_\pm}\right)=\sum_{s_c,\pm}\frac{Z_\pm P_\pm^{s_c}}{p_0-s_c\omega_\pm}$$

where we wrote $\vec{\alpha}=\vec{\gamma}\gamma^0$. This looks complicated but is not. We recognize the matter Hamiltonian $H_\pm=\frac{1}{(nr)_\pm}(r_\pm\vec{\alpha}\cdot\vec{p}+m_c\gamma^0)$

Indeed it is easy to show that

$$H_\pm H_\pm = \frac{1}{(nr)^2_\pm}\left(r_\pm \vec{\alpha}\cdot\vec{p} + m_c \gamma^0\right)^2 = \frac{(r_\pm \vec{p}\,)^2 + m_c^2}{(nr)^2_\pm} = \frac{(nr\omega)^2_\pm}{(nr)^2_\pm} = \omega^2_\pm$$

So, $P_\pm^{s_c}$ are idempotent and obey $P_\pm^{s} P_\pm^{s'} = P_\pm^{s}\,\delta_{ss'}$. So $\bar{S}$ clearly describes quasiparticle-like excitations.

The thermal wave function renormalization factors $Z_\pm$ are given by

$$Z_\pm^{s_c} = \frac{1}{2(nrp_0)_\pm}\frac{\partial}{\partial p_0}\left(rnp_0 + \sqrt{}\right)\left(rnp_0 - \sqrt{}\right)\Big|_{rnp_0 = \pm\sqrt{}}$$

$$= \frac{\partial}{\partial p_0}\left(rnp_0 \mp \sqrt{}\right)\Big|_{p_0=\omega_\pm} = r\frac{\partial}{\partial p_0}\left(np_0 \mp \sqrt{\vec{p}^{\,2} + \left(\frac{m_c}{r}\right)^2}\right)$$

(Actually indep. of s_c.)

We dropped the antihermitean part, so we have to put the Feynman prescription convergence-factor back, so that

$$\bar{S} = \sum_{s_c,\pm}\frac{Z_\pm P_\pm^{s_c}}{p_0 - s_c\omega_\pm + is_c\eta}$$

$$\Rightarrow \quad \bar{S}^\dagger = \sum_{s_c,\pm}\frac{Z_\pm P_\pm^{s_c}}{p_0 - s_c\omega_\pm - is_c\eta}$$

$$p^2 - m^2 + i\eta = p_0^2 - \omega_p^2 + i\eta$$
$$= (p_0 - s_c\omega)(p_0 + s_c\omega) + i\eta$$
$$= 2s_c\omega\,(p_0 - s_c\omega) + i\eta$$
$$= 2s_c\omega\,(p_0 - s_c\omega + is_c\eta)$$

because $P_\pm^{s_c\,\dagger} = P_\pm^{s_c}$. Thus $\tilde{S} = \sum_{s_c,\pm} Z_\pm P_\pm^{s_c}\left(\mathcal{P}\frac{1}{p_0 - s_c\omega_\pm} + i\pi\,\delta(p_0 - s_c\omega_\pm)\right)$

and

$$\bar{S} - \bar{S}^\dagger = \sum_{s_c\pm} 2\pi i\, Z_\pm P_\pm^{s_c}\,\delta(p_0 - s_c\omega_\pm) = i\rho_{SF}(k_0,k)$$

we then get

$$\bar{S}_F = \sum_{s,\pm} \left\{ Z_\pm P_\pm^{s_c} \left(\sigma_3 \mathbb{P} \frac{1}{p_0 - s_c \omega_\pm} + 2\pi i \begin{pmatrix} n_F + \frac{1}{2} & \theta_+ - n_F \\ \theta_- - n_F & n_F - \frac{1}{2} \end{pmatrix} \delta(p_0 - s_c \omega_\pm) \right\}$$

where $n_F(k_0) = f_F(|p_0| + \epsilon(p_0)\mu) = f_F(|p_0| + s_c\mu)$. This form itself is a result of a resummation. We have solved the propagator including some forward scattering corrections to the dispersion relation.

Gauge fields in RTF

In the ITF the gauge-field propagators followed straightforwardly from the scalar field structures. Following the Faddeev-Popov gauge fixing, also ghosts and Goldstone modes got thermalized. We shall follow similar path in the RTF-formulation. Let us point out however, that in the RTF one could choose differently, and thermalize only the physical excitations. This is due to the fact that for Gauge-fields the partition function is formally a sum just over the physical states:

$$\hat{\rho} \equiv \frac{1}{Z} \hat{\mathbb{P}} e^{-\beta\hat{H}} \quad ; \quad Z = \mathrm{Tr}[\hat{\mathbb{P}} e^{-\beta\hat{H}}]$$

where $\hat{\mathbb{P}}$ is a projection operator onto physical states $\hat{\mathbb{P}} \equiv \sum |i, \mathrm{phys}\rangle\langle i, \mathrm{phys}|$. Whatever one decides to do with unphysical states, then should not matter. This line of argument has indeed been followed by eg. Landshoff and Rebhan. These rules lead to significantly simplified calculations in some cases, but then, to intricacies in others. Here we will only follow the traditional path of FP-gauge fixing in the PI.

R_ξ - gauge propagator.

Let us start from R_ξ-gauge-fixed quadratic Lagrangian for a massive field

$$\mathcal{L}_g = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \frac{1}{2} M^2_{ab} A^a_\mu A^{b,\mu} + \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 \qquad (1)$$

Where indices a,b are SO(N)-indices. The mass matrix M^2_{ab} can always be diagonalized by a rotation of gauge-fields, so we can restrict ourselves to a U(1)-like situation. The massless photon propagator will be obtained in the limit $M \to 0$ at the end. The propagator eom corresponding to the quadratic Lagrangian (1) is

$$\left[(\Box + M^2) g^{\mu\nu} + \left(\frac{1}{\xi} - 1\right)\partial^\mu \partial^\nu\right] D_{\nu\alpha}(x - x') = g^\mu{}_\alpha \, \delta_c(t-t')\, \delta^3(\vec{x} - \vec{x}') \qquad (2)$$

As usual, without the $\frac{1}{\xi}$-factor, this equation does not have a solution. (could not be inverted). However, given a finite $1/\xi$, we can make a guess $D_{\mu\nu} \equiv g_{\mu\nu} D(\vec{x},t) + \partial_\mu \partial_\nu B(\vec{x},t)$. Inserting this to (2) gives

$$g^\mu{}_\alpha (\Box + M^2) D + \partial^\mu \partial_\alpha \left(\underbrace{\left(\frac{1}{\xi} - 1\right) D + \frac{1}{\xi}(\Box + \xi M^2) B}_{=0} \right) = g^\mu{}_\alpha \, \delta_c(t-t')\, \delta^3(\vec{x} - \vec{x}')$$

Let us start by choosing

$$(\Box + M^2) D(\vec{x}, t, M^2) = \delta_c(t-t')\, \delta^3(\vec{x} - \vec{x}')$$

and

$$B = a D(\vec{x}, t; M^2) + b D(\vec{x}, t; \xi M^2) \equiv a D + b D_\xi$$

then $\left(\frac{1}{\xi}-1\right)D + \frac{1}{\xi}\left(\Box+\xi M^2\right)\left(aD + bD_\xi\right)$

$$= \left(\frac{1}{\xi}-1\right)D + \frac{a}{\xi}\left(\Box+M^2+(\xi-1)M^2\right)D + \frac{b}{\xi}\left(\Box+\xi M^2\right)D_\xi$$

$$= \left(\frac{1}{\xi}-1\right)\left(1-aM^2\right)D + \frac{a+b}{\xi}\,\delta_c(t-t')\,\delta^3(\vec{x}-\vec{x}') \equiv 0 \quad \Rightarrow \quad \underline{a=-b=\frac{1}{M^2}}.$$

So, the direct space Gauge propagator is:

$$\boxed{D_{\mu\nu} = g_{\mu\nu}D(\vec{x},t;M^2) + \frac{\partial_\mu\partial_\nu}{M^2}\left[D(\vec{x},t;M^2) - D(\vec{x},t,\xi M^2)\right] \qquad (3)}$$

D is of course our familiar scalar field propagator, so we can proceed directly as before. We may go directly to momentum representation

$$D_{\mu\nu} = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)U(k)\begin{pmatrix} D_F^0(k^2;M^2) & 0 \\ 0 & (D_F^0(k^2;M^2))^* \end{pmatrix}U(k)$$

$$+ \frac{k_\mu k_\nu}{M^2}\,U(k)\begin{pmatrix} D_F^0(k^2;\xi M^2) & 0 \\ 0 & (D_F^0(k^2;\xi M^2))^* \end{pmatrix}U(k)$$

where $D_F^0(k^2,M^2) \equiv \dfrac{i}{k^2-M^2+i\eta}$ and $U(k) = \sqrt{n}\begin{pmatrix} e^{\beta|k_0|/2} & e^{-\beta k_0/2} \\ e^{\beta k_0/2} & e^{\beta|k_0|/2} \end{pmatrix}$; $n = f(|k_0|)$

$$= \begin{pmatrix} \sqrt{1+n} & (\theta_- + n)/\sqrt{} \\ (\theta_+ + n)/\sqrt{} & \sqrt{1+n} \end{pmatrix}$$

In the distribution representation then (Ex)

- $$D_{\mu\nu,11} = -\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2-\xi M^2}\right)\frac{i}{k^2-M^2+i\eta}$$

$$-\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{M^2}\right)\underbrace{2\pi f(|k_0|)\,\delta(k^2-M^2)}_{\text{physical thermal modes}} - \underbrace{\frac{k_\mu k_\nu}{M^2}\,2\pi f(|k_0|)\,\delta(k^2-\xi M^2)}_{\text{unphysical thermal modes}}$$

as usual ● $D_{\mu\nu,22} = (D_{\mu\nu,11})^*$ and in the off-diagonal we find

$$● \; D_{\mu\nu,12} = D^<_{\mu\nu} = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right) 2\pi\epsilon(k_0) f(k_0)\,\delta(k^2-M^2) - \frac{k_\mu k_\nu}{M^2}\, 2\pi\epsilon(k_0) f(k_0)\,\delta(k^2-\xi M^2)$$

and $D_{\mu\nu,21} = D^>_{\mu\nu} = e^{-\beta k_0} D^<_{\mu\nu} = D^<_{\mu\nu}[f \to 1-f]$.

Finally then

$$● \; \underline{\rho_{\mu\nu} = D^>_{\mu\nu} - D^<_{\mu\nu} = 2\pi\epsilon(k_0)\left\{-\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)\delta(k^2-M^2) - \frac{k_\mu k_\nu}{M^2}(k^2-\xi M^2)\right\}}$$

So, each gauge-field propagator contains unphysical states. In the unitary gauge they vanish of course, but in general no. This means that in $\xi \neq \infty$-gauges we need to introduce also the thermal ghosts (and Goldstone modes in SSB-theories).

Ghosts we can now just write the result based on the fact that

$$\mathcal{L}_{ghost} = \bar{c}_a \gamma^\mu \left[\partial_\mu \delta^{ab} + g f^{abc} A^c_\mu\right] c_b$$

Ghosts are then like massless bosonic fields (they obey bosonic KMS-boundary conditions just as do the gauge fields), so we get:

$$D^{ab}_{ghost} = \begin{pmatrix} \frac{i}{p^2+i\eta} & 0 \\ 0 & \frac{-i}{p^2-i\eta} \end{pmatrix} \delta^{ab} + 2\pi\epsilon(p_0)\delta^{ab} \begin{pmatrix} n & \theta_+ + n \\ \theta_- + n & n \end{pmatrix} \delta(p^2).$$

Gauge fields $m \to 0$ limit (photon & gluon propagators)

Our master equation (3) on p.49 is singular when $M \to 0$, but we can get the limit using l'Hopital's rule: $\lim_{x\to x_0} \frac{f}{g} = \lim_{x \to x_0} \frac{f'}{g'}$. if $\lim_{x\to x_0} f = \lim_{x\to x_0} g = 0$ and $\lim_{x\to x_0} \frac{f}{g}$ exists.

(Here $x \to M^2$, $x_0 \to 0$)

$$\lim_{M\to 0} D_{\mu\nu} = g_{\mu\nu} D(\vec{x},t;0) + \partial_\mu \partial_\nu \lim_{M \to 0} \frac{\partial}{\partial M^2}\left(D(\vec{x},t;M^2) - D(\vec{x},t,\xi M^2)\right)$$

$$= g_{\mu\nu} D(\vec{x},t;0) + (1-\xi)\partial_\mu\partial_\nu \frac{\partial}{\partial M^2} D(\vec{x},t;M^2)\Big|_{M^2=0}$$

In the momentum space this becomes

$$D_{\mu\nu} = \left(-g_{\mu\nu} + (1-\xi)k_\mu k_\nu\right) \frac{\partial}{\partial M^2} U(k_0) \begin{pmatrix} \frac{i}{k_0^2 - \vec{k}^2 - M^2 + i\eta} & 0 \\ 0 & \frac{-i}{k_0^2-\vec{k}^2-M^2-i\eta}\end{pmatrix} U(k_0)\Big|_{M=0}$$

$$= U(k_0)\begin{pmatrix} \frac{i}{(k_0^2-\vec{k}^2+i\eta)^2} & 0 \\ 0 & \frac{-i}{(k_0^2-\vec{k}^2-i\eta)^2}\end{pmatrix} U(k_0)$$

We can rewrite $\frac{\pm i}{(k_0^2 - \vec{k}^2 \pm i\eta)^2} = \frac{\partial}{\partial \vec{k}^2} \frac{\pm i}{k_0^2-\vec{k}^2 \pm i\eta}$

$$\simeq \frac{1}{k^2}\frac{\pm i}{k_0^2 - \vec{k}^2 \pm i\eta}$$ well defined for $k^2 \neq 0$ really.

So we can write

$$D^{ab}_{\mu\nu} = \left(-g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \overbrace{U(k_0)\begin{pmatrix}\frac{i}{k^2+i\eta} & 0 \\ 0 & \frac{-i}{k^2-i\eta}\end{pmatrix}U(k_0)}^{\begin{pmatrix}\frac{i}{k^2+i\eta} & 0 \\ 0 & \frac{-i}{k^2-i\eta}\end{pmatrix}\delta^{ab} + 2\pi\epsilon(k_0)\delta^{ab}\begin{pmatrix} n & \theta_{+}+n \\ \theta_{-}+n & n\end{pmatrix}\delta(k^2)}$$

$\hookrightarrow$ $k_\mu k_\nu \frac{\partial}{\partial \vec{k}^2}$ in $\eta \to 0$ limit

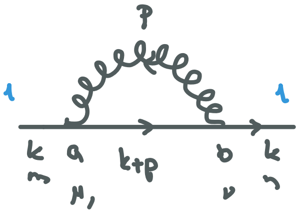
QED and QCD-plasma

To finish this course we perform two detailed computations of fermionic and gauge field excitations in the QED and QCD plasma.

1) Plasmino dispersion relation; particles and holes

The 1-loop quark self-energy correction in the QCD-plasma is:

$$-i\Sigma_{11} =$$



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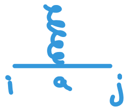


$$\hat{=} \; i g_s T^a_{ij}$$

$$D^{11}_{\mu\nu} = -g_{\mu\nu}\left(\frac{i}{p^2+i\eta} - 2\pi n_B(p_0)\,\delta(p^2)\right)$$

$$S_{11} = (\not{p}+m)\left(\frac{i}{p^2-m^2+i\eta} + 2\pi n_F(p)\,\delta(p^2-m^2)\right)$$

$$\Sigma_{11} = i(-i)^2 g_s^2 (T^aT^a)_{mn} \int \frac{d^4p}{(2\pi)^4} D^{11}_{\mu\nu}(p)\,\gamma^\mu S^{11}(p+k)\gamma^\nu$$

we only want the Hermitean part of $\Sigma_H = \frac{1}{2}(\Sigma_{11}+\Sigma_{11}^\dagger)\gamma^0$. This comes solely from the cross terms of the propagators $((\not{p}\gamma^0)^\dagger = \not{p}\gamma^0)$.

$$\Sigma_H = -i g_s^2 \underset{\substack{\uparrow \\ \text{Casimir}}}{C(N)} \int \frac{d^4p}{(2\pi)^4} \underbrace{\gamma^\mu(\not{p}+\not{k}+m)\gamma_\mu}_{=4m-2(\not{p}+\not{k})} \left(\frac{i}{p^2}\, 2\pi\, n_F(p_0+k_0)\,\delta((p+k)^2-m^2) - \frac{i}{(p+k)^2-m^2}\, 2\pi\, n_B(p_0)\,\delta(p^2)\right)$$

$p+k \to p$

$$= -2 g_s^2 C(N) \int \frac{d^4p}{(2\pi)^3} \left(\frac{\not{p}-2m}{\underbrace{(p-k)^2}_{p^2-2k\cdot p+k^2 \,=\, k^2-2k\cdot p+m^2}}\, n_F(p_0)\,\delta(p^2-m^2) - \frac{\not{p}+\not{k}-2m}{\underbrace{(p+k)^2-m^2}_{=p^2+2k\cdot p+k^2-m^2 \,=\, k^2-m^2+2k\cdot p}}\, n_B(p_0)\,\delta(p^2) \right)$$

$$\equiv -a\not{k} - b\not{u} + c$$

Now use the following relations:

$$\left\{ \begin{aligned} -\tfrac{1}{4}\mathrm{Tr}\,\Sigma_H &= c \\ -\tfrac{1}{4}\mathrm{Tr}\,\not{k}\Sigma_H &= ak^2 + bk_0 \\ -\tfrac{1}{4}\mathrm{Tr}\,\not{u}\Sigma_H &= ak_0 + b \end{aligned} \right\}$$

$$\Rightarrow \begin{cases} a = \frac{1}{|\vec{k}|^2}\left(\frac{1}{4}\mathrm{Tr}\,\not{k}\Sigma_H - \frac{k_0}{4}\mathrm{Tr}\,\not{u}\Sigma_H\right) \\ b = \frac{-1}{|\vec{k}|^2}\left(\frac{k_0}{4}\mathrm{Tr}\,\not{k}\Sigma_H - \frac{k^2}{4}\mathrm{Tr}\,\not{u}\Sigma_H\right) \end{cases} \qquad (1)$$

So we need to compute three different traces $\mathrm{Tr}\,\Sigma_H$, $\mathrm{Tr}\,\not{k}\Sigma_H$ and $\mathrm{Tr}\,\not{u}\Sigma_H$.

- $$\frac{1}{4}\mathrm{Tr}\,\Sigma_H = 2g_s^2 C(N)\, 2m \int \overbrace{\frac{d^4p}{(2\pi)^3}}^{=\frac{1}{(2\pi)^2}\int dp_0\int dp\,p^2\int_{-1}^{1}dz}\left(\frac{n_F(p_0)}{k^2+m^2-2k\cdot p}\,\overbrace{\delta(p^2-m^2)}^{=\frac{1}{2\omega_p}(\delta(p_0-\omega_p)+\delta(p_0+\omega_p))} - \frac{n_B(p_0)}{k^2-m^2+2k\cdot p}\,\overbrace{\delta(p^2)}^{=\frac{1}{2p}(\delta(p_0-p)+\delta(p_0+p))}\right)$$

- $$k^2\pm m^2\mp 2k\cdot p = k^2\pm m^2\mp 2k_0p_0 \pm 2kpz \quad ; \quad z=\cos\theta$$
$$= C_\pm(p_0)\pm 2kpz \qquad\qquad C_\pm = k^2\pm m^2\mp 2k_0p_0$$

- $$\int_{-1}^{1}dz\,\frac{1}{C_\pm(p_0)\pm 2kpz} = \pm\frac{1}{2kp}\log\frac{C_\pm(p_0)\pm 2pk}{C_\pm(p_0)\mp 2pk} = \frac{1}{2kp}\log\frac{C_\pm(p_0)+2pk}{C_\pm(p_0)-2pk}$$
$$= \frac{1}{2kp}\log\frac{k^2\pm m^2\mp 2k_0p_0+2pk}{k^2\pm m^2\mp 2k_0p_0-2pk} \equiv \frac{1}{2kp}\,\ell_\pm(p_0)$$

$$\Rightarrow \frac{1}{4}\mathrm{Tr}\,\Sigma_H = \overbrace{2C(N)m g_s^2\frac{1}{8\pi^2 k}}^{\frac{g_s^2}{4\pi^2}C(N)\frac{m}{k}}\int_0^\infty dp\left[\frac{p}{\omega_p}n_F(\omega_p)\overbrace{\left(\ell_+(\omega_p)+\ell_+(-\omega_p)\right)}^{\equiv L_+(\omega_p)} - n_B(p)\overbrace{\left(\ell_-(p)+\ell_-(-p)\right)}^{\equiv L_-(p)}\right]$$

- $$\frac{1}{4}\mathrm{Tr}\,\not{k}\Sigma_H = -2g_s^2 C(N)\int\frac{d^4p}{(2\pi)^3}\left(\frac{k\cdot p}{k^2-2k\cdot p+m^2}\,n_F(p_0)\,\delta(p^2-m^2) - \frac{k\cdot p+k^2}{k^2+2k\cdot p-m^2}\,n_B(p_0)\,\delta(p^2)\right)$$

Now

- $$\int_{-1}^{1}dz\,\frac{k_0p_0-kpz}{C_+(p_0)+2pkz} = \int_{-1}^{1}dz\,\frac{k_0p_0+\frac{1}{2}C_+-\frac{1}{2}(C_++2kpz)}{C_++2pkz}$$
$$= -1+\frac{C_++2k_0p_0}{4kp}\,\ell_+(p_0) = -1+\frac{k^2+m^2}{4kp}\,\ell_+(p_0)$$

$$\int_{-1}^{1} dz \, \frac{k_0 p_0 + k^2 - kpz}{C_-(p_0) - 2kpz} = \int_{-1}^{1} dz \, \frac{k_0 p_0 + k^2 - \frac{1}{2} C_- + \frac{1}{2}(C_- - 2kpz)}{C_- - 2kpz}$$

$$= 1 - \frac{C_- - 2k_0 p_0 - 2k^2}{4kp} \, \ell_-(p_0) = 1 + \frac{k^2 + m^2}{4kp} \, \ell_-(p_0)$$

$$\Rightarrow \frac{1}{4} \mathrm{Tr} \not{K} \Sigma_\mu = \frac{g_s^2}{4\pi^2} C(N) \int_0^\infty dp \, p \left[\left(\frac{p}{\omega_p} - \frac{k^2 + m^2}{8k\omega_p} L_+(\omega_p) \right) n_F(\omega_p) + \left(1 + \frac{k^2 + m^2}{8kp} L_-(p) \right) n_B(p) \right]$$

$$\frac{1}{4} \mathrm{Tr} \not{u} \Sigma_\mu = -2C(N) g_s^2 \int \frac{d^4 p}{(2\pi)^3} \left(\frac{p_0}{k^2 - 2k \cdot p + m^2} n_F(p_0) \, \delta(p^2 - m^2) - \frac{p_0 + k_0}{k^2 + 2k \cdot p - m^2} n_B(p_0) \, \delta(p^2) \right)$$

$$= -2C(N) g_s^2 \frac{1}{16\pi^2 k} \int_0^\infty dp \left[p \, n_F(\omega_p) \left(\ell_+(\omega_p) - \ell_+(-\omega_p) \right) - n_b(p) \left((k_0 + p) \, \ell_-(p) + (k_0 - p) \, \ell_-(-p) \right) \right]$$

This is how these expressions back into (1) on p. 51 we get a & b ($c = \frac{1}{4} \mathrm{Tr} \Sigma_\mu$). These 1d integrals are easily computed numerically. However, we are here interested only on the hard thermal loop limit (HTL), defined as $T \gg m$ and $T \gg k_0$, which corresponds to soft modes in high-temperature plasma. Here we can simplify:

HTL - limit

$$\omega^2 - k^2 = (k \pm \mathrm{sgn}(p_0)\omega)(k \overbrace{\mp \mathrm{sgn}(p_0)}^{=s} \omega) = (\omega + sk)(\omega - sk)$$

$$\ell_\pm(p_0) = \log \frac{k^2 \pm m^2 \mp 2k_0 p_0 + 2pk}{k^2 \pm m^2 \mp 2k_0 p_0 - 2pk} \simeq \log \frac{k^2 + 2p(k \mp \mathrm{sgn}(p_0)\omega)}{k^2 - 2p(k \pm \mathrm{sgn}(p_0)\omega)}$$

$$= \log \frac{\omega^2 - k^2 + 2sp(\omega - sk)}{\omega^2 - k^2 + 2sp(\omega + sk)} = \log \frac{\omega_{-s}}{\omega_{+s}} + \log \frac{\omega_s + sp}{\omega_{-s} + sp} \quad ; \quad s = \pm \mathrm{sgn}(p_0)$$

$$\Rightarrow L_\pm = \ell_\pm(p_0) + \ell_\pm(-p_0) \longrightarrow \ell_\pm(p) + \ell_\pm(-p) = \log \left(\frac{\omega_\pm \pm p}{\omega_\mp \pm p} \right) + \log \left(\frac{\omega_\mp \mp p}{\omega_\pm \mp p} \right)$$

$$= \log \left(\frac{\omega_+ - p}{\omega_- - p} \right) - \log \left(\frac{\omega_+ + p}{\omega_- + p} \right) \simeq -\frac{\omega_+ - \omega_-}{p} - \frac{\omega_+ - \omega_-}{p} \simeq -\frac{2k}{p}$$

And $\ell_\pm(p_0) - \ell_\mp(p_0) \longrightarrow 2\log\frac{\omega_\mp}{\omega_\pm} + \log\left(\frac{\omega_\pm \pm p}{\omega_\mp \pm p}\right) - \log\left(\frac{\omega_\mp \mp p}{\omega_\pm \mp p}\right)$

$$= \mp 2\log\frac{\omega_+}{\omega_-} + \log\left(\frac{\omega_+ + p}{\omega_- + p}\right) + \log\left(\frac{\omega_+ - p}{\omega_- - p}\right)$$

$$\approx \mp 2\log\frac{\omega_+}{\omega_-} + \cancel{\frac{\omega_+ - \omega_-}{p}} + \frac{1}{2}\left(\frac{\omega_+^2 - \omega_-^2}{p^2}\right) - \cancel{\frac{\omega_+ - \omega_-}{p}} + \frac{1}{2}\frac{(\omega_+^2 - \omega_-^2)}{p^2}$$

$$\simeq \underline{\mp 2\log\frac{\omega_+}{\omega_-} + \frac{\omega k}{p^2}}$$

Implementing these approximations on traces we get:

- $$\underline{\frac{1}{4}\mathrm{Tr}\,\Sigma_H} \approx 4C(N)\, m_0 g_s^2 \frac{1}{8\pi^2 k}\int_\Lambda^\infty dp\, \big(n_F(p) - n_B(p)\big)\left(-\frac{2k}{p}\right) \lesssim \log T$$ Small! neglect

Similarly drop L-terms in others:

- $$\underline{\frac{1}{4}\mathrm{Tr}\,\not{k}\Sigma_H} = 2C(N)\frac{g_s^2}{4\pi^2}\int_0^\infty dp\, p\, \big(n_F(p) + n_B(p)\big) = \frac{g_s^2}{2\pi^2}C(N)T^2\left(1+\frac{1}{2}\right)\overset{\to \frac{\pi^2}{6}}{\zeta(2)} = \underline{\overbrace{\frac{C(N)}{8}g_s^2 T^2}^{= m_T^2}}$$

- $$\underline{\frac{1}{4}\mathrm{Tr}\,\not{u}\Sigma_H} = -2C(N)\frac{g_s^2}{16\pi^2 k}\int_0^\infty dp\, p\, \Big(n_F(p)\big(\ell_+(\omega_p) - \ell_+(-\omega_p)\big) - n_b(p)\big(\ell_-(\omega_p) - \ell_-(-\omega_p)\big)\Big)$$

$$\approx C(N)\frac{g_s^2}{8\pi^2 k}\, 2\log\left(\frac{\omega_+}{\omega_-}\right)\int_0^\infty dp\, p\,\big(n_F(p) + n_B(p)\big) = \underline{\frac{g_s^2 T^2}{16k}C(N)\log\frac{\omega_+}{\omega_-}}$$

Now we finally get

$a = \frac{1}{|k|^2}\left(\frac{1}{4}\mathrm{Tr}\,\not{k}\Sigma_H - \frac{k_0}{4}\mathrm{Tr}\,\not{u}\Sigma_H\right)$

$b = \frac{-1}{|k|^2}\left(\frac{k_0}{4}\mathrm{Tr}\,\not{k}\Sigma_H - \frac{k^2}{4}\mathrm{Tr}\,\not{u}\Sigma_H\right)$

limit $k \to 0$

$$\begin{cases} a(\omega,k) = \dfrac{M_T^2}{k^2}\left(1 - \dfrac{\omega}{2k}\log\left(\dfrac{\omega+k}{\omega-k}\right)\right) \simeq 2\dfrac{M_T^2}{\omega^2} \\ b(\omega,k) = -\dfrac{M_T^2}{k^2}\left(\omega - \dfrac{\omega^2-k^2}{2k}\log\left(\dfrac{\omega+k}{\omega-k}\right)\right) \simeq -4\dfrac{M_T^2}{\omega} \end{cases}$$

Where we finally defined the thermal mass

$$M_T^2 \equiv \frac{g_s^2}{8} C(N) T^2 \longrightarrow \begin{cases} \frac{g_s^2 T^2}{6} & \text{in QCD, where } C(3) = \frac{4}{3} \\ \frac{e^2 T^2}{8} & \text{in QED, where } C(1) = 1. \end{cases}$$

These are the famous HTL self-energy expressions derived by Weldon in 80's.

Using the above expressions for a and b, and dropping m, one can now write the dispersion relation as:

$$(1+a)\,\omega + b = \pm (1+a)\, k \iff (1+a)(\omega \mp k) = -b$$

$$\iff \omega \mp k = -(\omega \mp k) \frac{m_T^2}{k^2} \left(1 - \frac{\omega}{2k} \log\left(\frac{\omega+k}{\omega-k}\right) \right) + \frac{m_T^2}{k^2} \left(\omega - \frac{\omega^2 - k^2}{2k} \log\left(\frac{\omega+k}{\omega-k}\right) \right)$$

$$\iff \underline{\omega = \pm k + \frac{m_T^2}{k} \left(\pm 1 + \frac{k \mp \omega}{2k} \log\left(\frac{\omega+k}{\omega-k}\right) \right)}$$

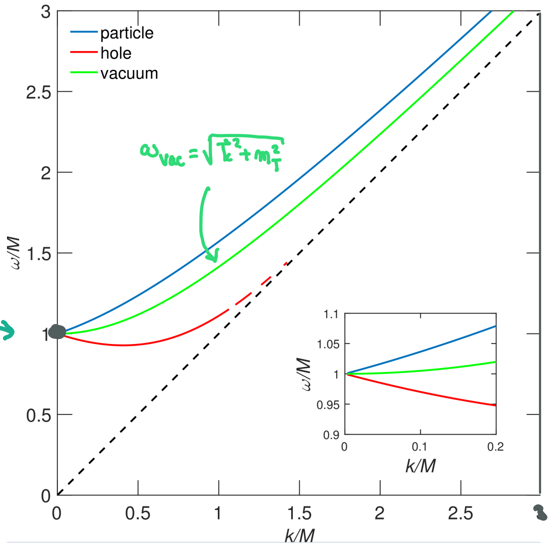
We indeed have two physical branches!

In the limit: $k \ll \omega$ one finds

$$\omega_\pm \simeq \pm k + \frac{m_T^2}{k} \left(\pm 1 + \left(\frac{k}{\omega} \mp 1\right) \left(1 - \frac{2}{3}\left(\frac{k}{\omega}\right)^2\right) + \dots \right)$$

$$= \frac{m_T^2}{\omega} \pm \frac{2}{3} \frac{k}{\omega}$$

$$\Rightarrow \boxed{\omega_\pm \simeq m_T \pm \frac{1}{3} k}$$

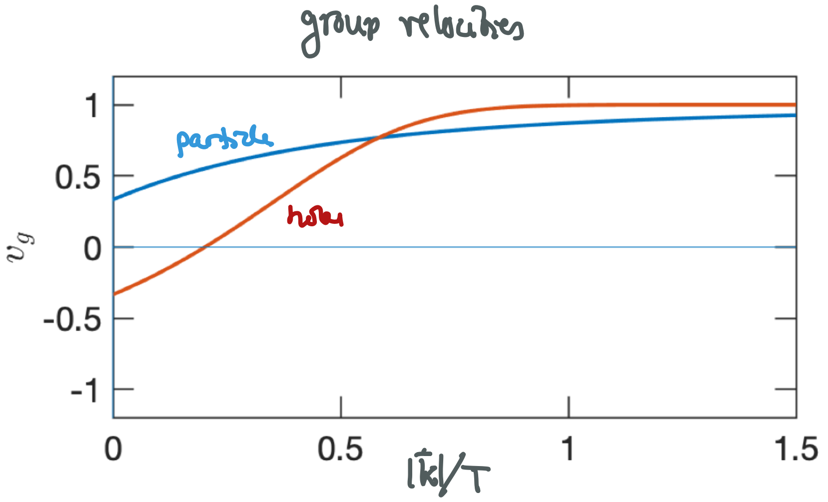


plasmon frequency

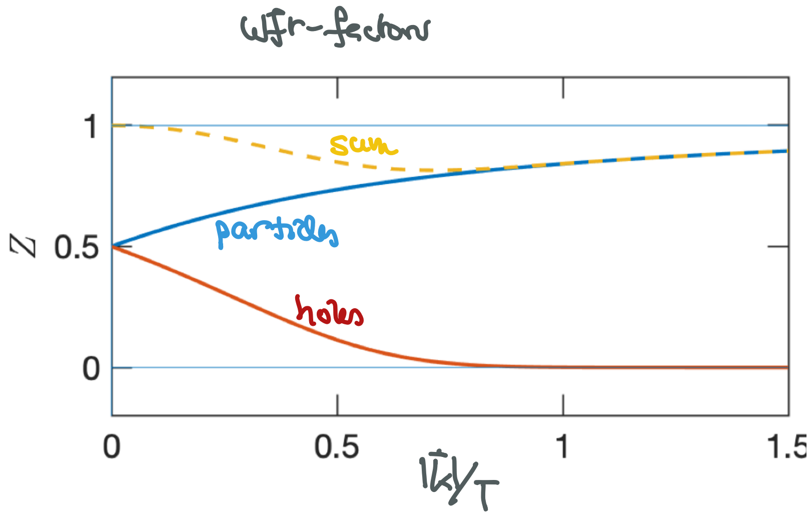
The group velocity $v_g = \frac{\partial \omega_\pm}{\partial k} \longrightarrow \pm \frac{1}{3}$ for $k \ll \omega$!

So, the hole solutions have a negative group velocity, eg. the current of particles is opposite to the momentum.

Group velocities are shown in the figure right as a function of $|\vec{k}|/T$. Note how the hole v_g indeed becomes negative for $k/T \lesssim 0{,}6$



Remember finally the wfr-factors for fermions. In the massless limit



$$Z_\pm^{s_c} = r\frac{\partial}{\partial p_0}\left(m p_0 \mp |\vec{p}|\right)$$

Again the figure to right shows the wfr. factors for particles and holes as a function of $|\vec{k}|/T$. Note that holes exist only for small $|\vec{k}|/T$. Here we used the QCD coupling $g_s = 1{,}216$, which corresponds to $m_T^2 \approx 0{,}2T^2$, or $m_T \sim 0{,}43T$.

$\uparrow$ large!

Sum-rule:

$$\int \frac{dk_0}{2\pi}\bar{\rho}_F(k_0,k) = \int \frac{dk_0}{\cancel{2\pi}} \sum_{s_c\pm} \cancel{2\pi}\, Z_\pm P_\pm^{s_c}\, \delta(p_0 - s_c\omega_\pm) = (Z_+ + Z_-)\cdot 1$$

So, indeed for $k \sim m_T$ the sum-rule fails. It is satisfied however for both $|k| \to 0$ and $|k| \gg T$.

7 QED-plasma

In medium particles become 'dressed' by interactions.
1-particle states → collective modes, or quasiparticles.

Collective modes are characterized by dispersion relation $\omega(\vec{k})$ and by damping rate $\gamma(\vec{k})$ (finite life-time).

DR will be found as a pole of the propagator, corrected with the real part of the self-energy. Complex part of the self-energy can be related to $\gamma(\vec{k})$.

Scales: (consider $m \ll g^2T$)

- $\omega \sim T$; E-scale of individ. particles
  $l \sim 1/T$; interparticle distance

- $\omega \sim gT$; E-scale of collective excitations (electric) ω_D
  $l \sim 1/gT$: size of collective configurations

- $\omega \sim g^2T$; magnetic scale. (Nonperturbative in DR)
  $\gamma(\vec{k}) \sim g^2T$ (decay rate)

Quasiparticles well defined if :

$$\frac{1}{T} \ll \frac{1}{gT} \ll \frac{1}{g^2T} \qquad (7.1)$$

Photon propagator (consider $T \gg m_e$)

The generic finite-T photon polarization tensor is a linear combination of symmetric tensors $g_{\mu\nu}$, $K_\mu K_\nu$, $u_\mu u_\nu$ and $u_\mu K_\nu + u_\nu K_\mu$, where k_μ is the $\overset{\text{photon}}{\text{4-}}$momentum & u_μ is the fluid 4-velocity.
Gauge invariance still implies the Ward-identity:

$$K^\mu \Pi_{\mu\nu} = 0 \qquad (7,2)$$

Eq. gives 2 independent conditions, whereby $\Pi_{\mu\nu}$ can depend only on two scalar functions. It is convenient to decompose $\Pi_{\mu\nu}$ as

$$\Pi_{\mu\nu} = \underbrace{\Pi_T(k,\omega)}_{\text{(transverse)}} P_{\mu\nu} + \underbrace{\Pi_L(k,\omega)}_{\text{(longitudinal)}} Q_{\mu\nu} \qquad (7,3)$$

where we noted that $\Pi_{T,L}$ can only depend on two scalar function

$$\omega \equiv K \cdot u \qquad \& \qquad k \equiv \sqrt{(k \cdot u)^2 - K^2} \qquad (7,4)$$

(energy and 3-momentum), and

$$P_{\mu\nu} = \tilde{g}_{\mu\nu} - \frac{\tilde{K}_\mu \tilde{K}_\nu}{k^2} \qquad (7,5)$$

$$Q_{\mu\nu} = -\frac{1}{k^2 K^2}\left(k^2 u_\mu + \omega \tilde{K}_\mu\right)\left(k^2 u_\nu + \omega \tilde{K}_\nu\right) \qquad (7,6),$$

and

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu \qquad (7,7)$$

$$\tilde{K}_\mu \equiv K_\mu - \omega u_\mu \qquad (7,8)$$

Note that $P_{\mu\nu} \rightarrow -(\delta_{ij} - \frac{k_i k_j}{k^2})$ for $\mu,\nu \neq 0$ (& $P_{0\nu} = P_{\mu 0} = 0$) when $u_\mu \rightarrow (\omega, \vec{0})$. This is why $P_{\mu\nu}$ is the covariant form of the transverse projector. Indeed:

$$P^\mu{}_\alpha P^\alpha{}_\nu = P^\mu{}_\nu \; ; \; Q^\mu{}_\alpha Q^\alpha{}_\nu = Q^\mu{}_\nu$$
$$P^\mu{}_\alpha Q^\alpha{}_\nu = Q^\mu{}_\alpha P^\alpha{}_\nu = 0 \qquad (7.9)$$

Moreover

$$P_{\mu\nu} + Q_{\mu\nu} = g_{\mu\nu} - \frac{K_\mu K_\nu}{K^2} \qquad (7.10)$$

so that when $\Pi_T = \Pi_L$ (at $T = \mu = 0$), we recover the usual vacuum polarization tensor structure.

Dressed photon propagator

We can repeat our analysis for the scalar-propagator with the only exception that we have the overall structure

$$D^{\text{free}}_{\mu\nu} = \left(-g_{\mu\nu} + (1-\xi)\frac{K_\mu K_\nu}{K^2}\right)\frac{1}{K^2} \; ; \text{ imaginary time}$$

$$= -(P_{\mu\nu} + Q_{\mu\nu})\frac{1}{K^2} - \xi\frac{K_\mu K_\nu}{K^4} \, . \qquad (7.11)$$

[Oho, on tullut näemmä kirjoitettua englanniksi. Vaihdetaanpa takaisin suomeen!]

Schwinger-Dyson

$$D_{\alpha\beta} = D^{\text{free}}_{\alpha\beta} + D^{\text{free}}_{\alpha\gamma}(-\Pi^{\gamma\delta})D_{\delta\beta}$$

$$= D^{Free}_{\alpha\beta} - \left(D^{Free}_{\alpha\mu}\Pi^{\mu\nu}D^{Free}_{\nu\beta}\right)_{\alpha\beta} + \left(D^{free}(-\Pi D^{free})^2\right)_{\alpha\beta} + \dots$$

Now

$$(\Pi D^{free})^{\mu}{}_{\beta} = -\left(\Pi_T P^{\mu\nu} + \Pi_L Q^{\mu\nu}\right)\left(-(P_{\nu\beta}+Q_{\nu\beta})\frac{1}{k^2} - \cancelto{0}{\xi\frac{k_\nu k_\beta}{k^4}}\right)$$

$$= +\left(\frac{\Pi_T}{k^2}P^{\mu}{}_{\beta} + \frac{\Pi_L}{k^2}Q^{\mu}{}_{\beta}\right)$$

ja

$$\left[(\Pi D^{free})^n\right]^{\mu}{}_{\beta} = +\left[\left(\frac{\Pi_T}{k^2}\right)^n P^{\mu}{}_{\beta} + \left(\frac{\Pi_L}{k^2}\right)^n Q^{\mu}{}_{\beta}\right](-1)^n$$

$$\Rightarrow D_{\alpha\beta} = D^{free}_{\alpha\beta} + \sum_{n=1}^{\infty}\left(-\frac{1}{k^2}(P_{\alpha\mu}+Q_{\alpha\mu}) + \cancelto{0}{\xi\frac{k_\alpha k_\mu}{k^4}}\right)\left(\left(\frac{\Pi_T}{k^2}\right)^n P^{\mu}{}_{\beta} + \left(\frac{\Pi_L}{k^2}\right)^n Q^{\mu}{}_{\beta}\right)$$

$$= -\frac{1}{k^2}(P_{\alpha\beta}+Q_{\alpha\beta}) + \xi\frac{k_\alpha k_\beta}{k^2} - \sum_{n=1}^{\infty}\left[\frac{1}{k^2}\left(\frac{\Pi_L}{k^2}\right)^n P_{\alpha\beta} + \frac{1}{k^2}\left(\frac{\Pi_T}{k^2}\right)^n Q_{\alpha\beta}\right]$$

$$\underline{\underline{= -\frac{P_{\alpha\beta}}{k^2-\Pi_T} - \frac{Q_{\alpha\beta}}{k^2-\Pi_L} - \xi\frac{k_\alpha k_\beta}{k^2}}} \tag{7,12}$$

Siis transversaali- ja longitudinaalikomponentit saavat omat erilliset napansa jotka määräytyvät Π_T:n ja Π_L:n mukaan,

Π_T ja Π_L voidaan yhdistää plasman sähköiseen ja magneettiseen suskeptibiliteettiin

$$\epsilon_{T,L}(k,\omega) = 1 - \frac{\Pi_{T,L}}{K^2} \tag{7,13}$$

ja

$$\frac{1}{\mu(k,\omega)} = 1 + \frac{K^2\Pi_T - \omega^2\Pi_L}{k^2K^2} \qquad (7.14)$$

Vakuumipolarisaatiolle $\Pi_T = \Pi_L \Rightarrow \frac{1}{\mu} = 1 + \frac{\Pi}{K^2} = \epsilon$ niinkuin pitääkin.

Fysiikka kaavojen (7.13) ja (7.14) takana on seuraavanlainen.

Voimme määritellä kovariantit sähkö- ja magneettikentät:

$$\tilde{E}^\mu \equiv u_\lambda F^{\lambda\mu} \quad ; \quad \tilde{B}^\mu = \frac{1}{2}\epsilon^{\alpha\beta\lambda\mu} F_{\alpha\beta} u_\lambda = F^{*\lambda\mu} u_\lambda \qquad (7.15)$$

Voimme osoittaa että vuorovaikuttamattomassa teoriassa

$$S = -\frac{1}{4}\int d^4x\, F^2 = -\int d^4x \left(\vec{E}^2 - \vec{B}^2\right)$$

$$= -\frac{1}{2}\int \frac{d^4K}{(2\pi)^4}\left(\tilde{E}^\alpha \tilde{E}_\alpha - \tilde{B}^\alpha \tilde{B}_\alpha\right) \qquad (7.16)$$

Kun vuorovaikutukset kytketään päälle saadaan lisätermi

$$S_{\text{quantum}} = -\frac{1}{2}\int \frac{d^4K}{(2\pi)^4} A^\alpha(-K)\, \Pi_{\alpha\beta}\, A^\beta(K) \qquad (7.17)$$

missä $\Pi_{\alpha\beta} = P_{\alpha\beta}\Pi_L + Q_{\alpha\beta}\Pi_T$. Tämä termi voidaan hajoittaa $\hat{E}^\alpha \bar{E}_\alpha$ ja $\hat{B}^\alpha \hat{B}_\alpha$ vastaaviksi tensorirakenteiksi. Esim:

$$-\frac{1}{2}\int d^4x\, \tilde{E}^\alpha \tilde{E}_\alpha = -\frac{1}{2}\int d^4x\; g^{\mu\nu} u^\lambda u^\gamma (\partial_\lambda A_\mu - \partial_\mu A_\lambda)(\partial_\gamma A_\nu - \partial_\nu A_\gamma)$$

$$= \dots = -\frac{1}{2}\int\frac{d^4K}{(2\pi)^4} A^\mu(-K)\overbrace{\left(\omega^2 g^{\mu\nu} + k^2 u^\mu u^\nu - \omega(u^\mu K^\nu + u^\nu K^\mu)\right)}^{\tilde{E}\tilde{E}\text{-tensor.}} A_\nu$$

One may similarly compute a "$\tilde{B}\tilde{B}$-tensor" and then re-express $\Pi_{\alpha\beta}$ in terms of these. The result should be

$$S_0 + S_{quantum} = -\frac{1}{2}\int\frac{d^4K}{(2\pi)^4}\left(\epsilon\, \tilde{E}^\alpha \tilde{E}_\alpha - \frac{1}{\mu}\tilde{B}^\alpha\tilde{B}_\alpha\right) \qquad (7,18)$$

where ϵ & μ are given by eqns. (7,13) and (7,14). (Oho, taas meni enklanniksi)
Tulos (7,18) pätee myös ei-abelisille mittakentille.

Perturbatiivinen lasku $\Pi_{\mu\nu}$:ille

Koska $\Pi_{\mu\nu} = \Pi_T P_{\mu\nu} + \Pi_L Q_{\mu\nu}$ missä $u^\mu P_{\mu\nu} = 0$ saamme

$$u^\mu u^\nu \Pi_{\mu\nu} = \Pi_T u^\mu u^\nu Q_{\mu\nu} = -\frac{\Pi_T}{k^2K^2}\left(-k^2 + \overset{0}{\cancel{\tilde{K}\cdot u}}\right)\left(-k^2 + \cancel{\tilde{K}\cdot u}\right) = -\Pi_T\frac{k^2}{K^2}$$

$$\Rightarrow \Pi_T = \frac{-K^2}{k^2}u^\mu u^\nu \Pi_{\mu\nu} \qquad (7,19)$$

Vastaavasti voidaan osoittaa että

$$\Pi_T = -\frac{1}{2}\Pi_L + \frac{1}{2}g^{\mu\nu}\Pi_{\mu\nu} \qquad (7,20)$$

Tarvitaan siis kontraktiot $u^\mu u^\nu \Pi_{\mu\nu}$ ja $g^{\mu\nu}\Pi_{\mu\nu}$.

Haluamme tehdä laskun RTFissä, joten katsotaan miten (7,12) johdetaan RTFissä. Käytetään Feynmanin mittaa, jotta ei tarvitse huolehtia $\partial/\partial\vec{k}^2$ - tekijöistä.

$$D_{\mu\nu} = -g_{\mu\nu}\, U_{k_0} \tilde{D}\, U_{k_0} \quad ; \quad \tilde{D} \equiv \begin{pmatrix} \frac{1}{k^2+i\epsilon} & 0 \\ 0 & \frac{1}{k^2-i\epsilon} \end{pmatrix} \quad ; \quad U_{k_0} \equiv \sqrt{f(|k_0|)} \begin{pmatrix} e^{\beta|k_0|/2} & e^{-\beta k_0/2} \\ e^{\beta k_0/2} & e^{\beta|k_0|/2} \end{pmatrix}$$

$$= \left(-g_{\mu\nu} + \frac{K_\mu K_\nu}{K^2}\right) U_{k_0} \tilde{D}\, U_{k_0} - \frac{K_\mu K_\nu}{K^2} U_{k_0} \tilde{D}\, U_{k_0}$$

$$= -(P_{\mu\nu} + Q_{\mu\nu})\, U_{k_0} \tilde{D}\, U_{k_0} - \frac{k_\mu k_\nu}{K^2} U_k \tilde{D}\, U_{k_0} \qquad (7,21).$$

Tensorirakenne on nyt sama kuin edellä tekemässämme johdossa. Voimme siten kirjoittaa SD-yhtälön

$$U^{-1}\left(D_{\mu\nu} = D^{free}_{\mu\nu} + D^{free}_{\mu\alpha}(-\Pi)^{\alpha\beta} D_{\beta\nu}\right) U^{-1} \qquad \Pi^{\alpha\beta} = U^{-1}\tilde{\Pi}^{\alpha\beta}U^{-1}$$

$$\Leftrightarrow \quad \bar{D}_{\mu\nu} = \bar{D}^{free}_{\mu\nu} + \bar{D}^{free}_{\mu\alpha}(-\tilde{\Pi}^{\alpha\beta})\bar{D}_{\beta\nu} \qquad \tilde{\Pi}^{\alpha\beta} = \begin{pmatrix} \bar{\Pi}^{\alpha\beta} & 0 \\ 0 & -\bar{\Pi}^{*\alpha\beta} \end{pmatrix} \qquad (7,22)$$

Tässä tietenkin $\bar{D}^{free}_{\mu\nu} = -(P_{\mu\nu} + Q_{\mu\nu})\tilde{D} - \frac{k^\mu k^\nu}{k^2}\tilde{D}$. Lasku menee nyt täsmälleen kuten aiemmin $\bar{D}$:ille; mistä lopulta kääntämällä:

$$D_{\mu\nu} = -P_{\mu\nu}\, U \tilde{D}_T U - Q_{\mu\nu}\, U \hat{D}_L U - \frac{k_\mu k_\nu}{k^4} U \tilde{D} U \, . \qquad (7,23)$$

missä

$$\tilde{D}_{T,L} \equiv \begin{pmatrix} \frac{1}{k^2 - \bar{\Pi}_{T,L}} & 0 \\ 0 & -\frac{1}{k^2 - \bar{\Pi}^*_{T,L}} \end{pmatrix} \qquad (7,24)$$

Nyt

$$U\tilde{D}_i U = f(|k_0|) \begin{pmatrix} \bar{D}_i e^{\beta|k_0|} - \bar{D}_i^* & e^{\beta(|k_0|-k_0)/2}(\bar{D}_i - \bar{D}_i^*) \\ e^{\beta(|k_0|+k_0)/2}(\bar{D}_i - \bar{D}_i^*) & \bar{D}_i - e^{\beta|k_0|}\bar{D}_i^* \end{pmatrix} \qquad (7,25)$$

Esimerkiksi:

$$\mathrm{Re}\, D_{11}^{T,L} = \mathrm{Re}\, \overline{D}_{T,L} = \mathrm{Re}\left(\frac{1}{k^2 - \Pi_{T,L}}\right) = \frac{k^2 - \mathrm{Re}\,\overline{\Pi}_{T,L}}{(k^2 - \mathrm{Re}\,\overline{\Pi}_{T,L})^2 + (\mathrm{Im}\,\overline{\Pi}_{T,L})^2} \qquad (7,26)$$

ja

$$\mathrm{Im}\, D_{11}^{T,L} = -\mathrm{Im}\, D_{22}^{T,L} = \underbrace{f(|k_0|)\left(e^{\beta|k_0|}+1\right)}_{1+2f(|k_0|) \;=\; \epsilon(k_0)(1+2f(k_0))} \frac{\mathrm{Im}\,\overline{\overline{\Pi}}^{T,L}}{(k^2 - \mathrm{Re}\,\overline{\overline{\Pi}}^{T,L})^2 + (\mathrm{Im}\,\overline{\overline{\Pi}}^{T,L})^2} \qquad (7,27)$$

Näitä voi vielä sieventää käyttämällä yhteyksiä (ks. kaavat 6,83 a-b)

$$\mathrm{Re}\,\Pi_{11}^{T,L} = \mathrm{Re}\,\Pi_{22}^{T,L} = \mathrm{Re}\,\overline{\overline{\Pi}}^{T,L} \qquad (7,28)$$

$$\mathrm{Im}\,\Pi_{11}^{T,L} = \mathrm{Im}\,\Pi_{22}^{T,L} = \epsilon(k_0)(1+2f(k_0))\,\mathrm{Im}\,\overline{\Pi}_{T,L} \qquad (7,29)$$

Edelleen

$$D_{12}^{T,L} = 2\epsilon(k_0) f(k_0) \frac{\mathrm{Im}\,\overline{\overline{\Pi}}_{T,L}}{(k^2 - \mathrm{Re}\,\overline{\overline{\Pi}}_{T,L})^2 + (\mathrm{Im}\,\overline{\overline{\Pi}}_{T,L})^2}$$

$$D_{21}^{T,L} = 2\epsilon(k_0)(1+f(k_0)) \frac{\mathrm{Im}\,\overline{\overline{\Pi}}^{T,L}}{(k^2 - \mathrm{Re}\,\overline{\overline{\Pi}}_{T,L})^2 + (\mathrm{Im}\,\overline{\overline{\Pi}}_{T,L})^2} \qquad (7,30)$$

Ja vielä vastaavasti:

$$\Pi_{12}^{T,L} = -2\epsilon(k_0)(1+f(k_0))\,\mathrm{Im}\,\overline{\Pi}_{T,L} \qquad (7,31)$$

$$\Pi_{21}^{T,L} = -2\epsilon(k_0) f(k_0)\,\mathrm{Im}\,\overline{\overline{\Pi}}_{T,L}$$

Huomaa raja-arvot: $\mathrm{Im}\,\overline{\Pi}_{T,L} \to 0$

$$\left.\begin{array}{l} \mathrm{Re}\, D_{11} \to \mathbb{P}\mathbb{P}\, \dfrac{1}{K^2} \\[2ex] \mathrm{Im}\, D_{11} \to (1+2f(|k_0|)\,\pi\,\delta(k^2)) \end{array}\right\} \Rightarrow D_{11} \to \frac{1}{k^2+i\epsilon} + 2f(|k_0|)\,\delta(k^2)$$

$\uparrow$

vacuum part, combine to $\mathrm{Re}\, D_{11}$.

ok

jne...

Kvasihiukkasten dispersiorelaatiot määräytyvät siten kaavoista

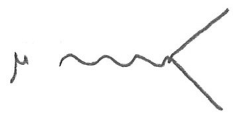
$$k^2 - \mathrm{Re}\,\Pi^{11}_{L,T} = 0 \quad \Rightarrow \quad \omega_{L,T}(\vec{k}) \tag{7,32}$$

ja vaimennuskertoimet kaavoista

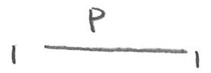
$$\gamma_{L,T}(k) = \tanh \frac{\beta |k_0|}{2} \cdot \frac{1}{2\omega_{L,T}}\, \mathrm{Im}\, \Pi^{L,T}_{11} \tag{7,33}$$

Voimme siis keskittyä laskemaan $(\Pi_{\mu\nu})_{11}$ - tensorin.

Fotonin itseisenergia ; $T \gg k, \omega$; $m_e \to 0$ raja: $\mu \equiv 0$



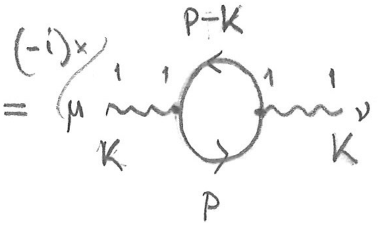
$$\sim -ie\gamma^\mu \qquad (7,34)$$



$$\sim \frac{i}{\not p - m_e + i\epsilon} - 2\pi f(|p_0|)(\not p + m_e)\,\delta(p^2 - m_e^2) \qquad (7,35)$$

$$= iS_F(p)$$

$\Pi^{11}_{\mu\nu} =$



$$= -ie^2 \int \frac{d^dp}{(2\pi)^d} \mathrm{Tr}\left[\gamma_\mu iS_F(p-K)\gamma_\nu iS_F(p)\right]$$

(−1) luupista ; S = 1

$$(7,36)$$

Lasketaan vain termisen $\Pi_{\mu\nu}$in reaaliosa. Tämä sisältää yhden vakuumi- ja yhden termisen propagaattorin kummastakin propagaattorista:

pääarvo! $\mathrm{Re}\left(\frac{1}{\not p - m_e + i\epsilon}\right)$

$$\mathrm{Re}\,\Pi^{11}_{\mu\nu} = -e^2 \int \frac{d^4p}{(2\pi)^3}\Big[P\left(\frac{1}{(p-k)^2 - m_e^2}\right) f(|p_0|)\,\delta(p^2 - m_e^2)$$

$$+ P\left(\frac{1}{p^2 - m_e^2}\right) f(|p_0 - k_0|)\,\delta((p-k)^2 - m_e^2)\Big]$$

$$\times \mathrm{Tr}\left(\gamma_\mu(\not p - \not K + m_e)\gamma_\nu(\not p + m_e)\right) \qquad (7,37)$$

Vaihda $p \to p+k$
jälkimmäisessä
termissä

$$= -e^2 \int \frac{d^4p}{(2\pi)^3} f(|p_0|)\,\delta(p^2 - m_e^2) \times$$

$$\times \Big\{ P\left(\frac{1}{K^2 - 2K\cdot p}\right) \mathrm{Tr}\left(\gamma_\mu(\not p + m_e - \not K)\gamma_\nu(\not p + m_e)\right)$$

$$+ P\left(\frac{1}{K^2 + 2K\cdot p}\right) \mathrm{Tr}\left(\gamma_\nu(\not p + m_e + \not K)\gamma_\mu(\not p + m_e)\right)\Big\} \qquad (7,38)$$

Missä huomioitiin että $\delta(p^2 - m_e^2)$-termin ansiosta $(K \pm p)^2 - m_e^2 = K^2 \pm 2K\cdot p$

Mittainvarianssi:

combines with $\not p + m_e$ to give $p^2-m_e^2=0$

$$K^\mu \,\mathrm{Re}\,\Pi''_{\mu\nu} \propto \int\cdots \frac{1}{K^2-2K\cdot p}\,\mathrm{Tr}\Big(\overbrace{\not K(\not p+m_e-\not K)}^{=(-\not p+m_e)\not K - K^2+2K\cdot p}\gamma_\nu(\not p+m_e)\Big)$$

$$+\frac{1}{K^2+2K\cdot p}\,\mathrm{Tr}\Big(\gamma_\nu\underbrace{(\not p+m_e+\not K)\not K}_{=K^2-2K\cdot p-\not K(\not p-m_e)}(\not p+m_e)\Big)$$

$\to p^2-m_e^2=0$

$$\propto \int \quad -\mathrm{Tr}(\gamma_\nu \not p) + \mathrm{Tr}(\gamma_\nu \not p) \quad = \quad 0 \;\square$$

Todetaan vielä:

$$\delta(p^2-m_e^2)\,\mathrm{Tr}\big(\gamma_\mu(\not p+m_e+\not K)\gamma_\nu(\not p+m_e)\big)$$

$$=\delta(p^2-m_e^2)\cdot 4\overbrace{\big(2p_\mu p_\nu-(K_\mu p_\nu+K_\nu p_\mu)+K\cdot p\, g_{\mu\nu}\big)}^{\equiv\, i_{\mu\nu}} \qquad (7,39)$$

Huomataan että integraalissa (7,38) molemmat termit ovat identtiset (HT, selittää myös miksi mittainvarianssi on siltä ok), ja saamme

$$\Rightarrow\ \mathrm{Re}\,\Pi''_{\mu\nu} = -8e^2\int\frac{d^4p}{(2\pi)^3}\,f(|p_0|)\,\delta(p^2-m_e^2)\,P\Big(\frac{1}{K^2-2K\cdot p}\Big)\,i_{\mu\nu} \qquad (7,40)$$

Lasketaan ensin $\mathrm{Re}\,\Pi_L = -\frac{K^2}{k^2}u^\mu u^\nu\,\mathrm{Re}\,\Pi''_{\mu\nu}$. Mennään nyt plasman lepokoordinaatistoon, missä

$$u^\mu u^\nu i_{\mu\nu} = 2p_0^2-2p_0k_0+K\cdot p = 2p_0^2-p_0k_0-\vec p\cdot\vec k \qquad (7,41)$$

$$K^2-2K\cdot p = K^2-2k_0p_0+2\vec k\cdot\vec p \qquad (7,42)$$

Niinpä

$$\mathrm{Re}\,\Pi_L = +8e^2\frac{K^2}{k^2}\sum_{\pm}\int\frac{d^3p}{(2\pi)^3}f(\omega_p)\frac{1}{2\omega_p}\mathcal{P}\left(\frac{2\omega_p^2 \mp \omega_p k_0 - \vec{p}\cdot\vec{k}}{K^2 \mp 2\omega_p k_0 + 2\vec{p}\cdot\vec{k}}\right) \qquad (7{,}43)$$

joka pienen askartelun (HT) jälkeen antaa* $\overset{m\to 0}{\to} (\omega \mp k)(\omega \pm k - 2p)$

$$\mathrm{Re}\,\Pi_L = -\frac{e^2}{\pi^2}\frac{K^2}{k^2}\sum_{\pm}\int d\omega_p\,|\vec{p}|f(\omega_p)\left(1 - \frac{1}{2k|\vec{p}|}\left(\frac{1}{2}K^2 + 2\omega_p(\omega_p \mp k_0)\right)\log\left|\frac{K^2 + 2(kp \mp k_0\omega_p)}{K^2 - 2(kp \pm k_0\omega_p)}\right|\right) \qquad (7{,}44)$$

Tämä tulos on vielä eksakti kun vain $\mu = 0$ (kemiallinen potentiaali). Asetetaan nyt $m_e \equiv 0$, jolloin $\omega_p = |\vec{p}| \equiv p$. Lasketaan nyt myös vain $k_0 \overset{!}{=} \omega$:ille.

Huom

$$\mathrm{Re}\,\Pi_L = \frac{e^2}{\pi^2}\left(1 - \frac{\omega^2}{k^2}\right)\sum_{\pm}\int_0^\infty dp\,\frac{p}{e^{\beta p}+1}\left(1 - \left(\frac{K^2}{4kp} + \frac{1}{k}(p \mp \omega)\right)\log\left|\frac{K^2 + 2p(k \mp \omega)}{K^2 - 2p(k \pm \omega)}\right|\right) \qquad (7{,}45)$$

Huomaa että logaritmeissa esiintyy itseisarvot. Tämä johtuu siitä, että integraali tehdään pääarvomielessä

$$\mathcal{P}\log(p+x) = \frac{1}{2}\left(\log(p+x+i\epsilon) + \log(p+x-i\epsilon)\right) = \log|p+x|$$

Summa $\sum_{\pm}$ voidaan tehdä myös, minkä jälkeen (HT):

$$\mathrm{Re}\,\Pi_L = \frac{2e^2}{\pi^2}\underbrace{\left(1 - \frac{\omega^2}{k^2}\right)}_{= -\frac{K^2}{k^2}}\int_0^\infty dp\,\frac{p}{e^{\beta p}+1}\left(1 - \frac{\omega}{k}\log\left|\frac{\omega-k}{\omega+k}\right| + \frac{\omega}{2k}\log\left|\frac{(\omega+k)^2 - 4p^2}{(\omega-k)^2 - 4p^2}\right| + \frac{1}{2k}\left(\frac{K^2}{4p} + p\right)\log\left|\frac{4p^2 + 4pk - K^2}{4p^2 - 4pk - K^2}\right|\right) \qquad (7{,}46)$$

---

* $$\int_{-1}^{1}dz\left(\frac{a_\mp - pkz}{c_\mp + 2pkz}\right) = \int_{-1}^{1}dz\left(-\frac{1}{2}\frac{c_\mp + 2kpz}{c_\mp + 2kpz} + \frac{a_\mp + \frac{1}{2}c_\mp}{c_\mp + 2kpz}\right)$$

$$= -1 + \frac{1}{2kp}\left(a_\mp + \frac{1}{2}c_\mp\right)\log\left(\frac{c_\mp + 2kp}{c_\mp - 2kp}\right) \quad ; \quad a_\mp + \frac{1}{2}c_\mp = \frac{1}{2}K^2 + 2\omega_p^2 \mp 2\omega_p k_0$$

$$c_\mp = K^2 \mp 2\omega_p k_0$$

$\mathrm{Re}\,\Pi_T$:tä varten tarvitsemme siis vielä suureen $\mathrm{Re}\, g^{\mu\nu}\Pi''_{\mu\nu}$: Voimme käyttää suoraan tulosta (7,40):

$$\mathrm{Re}\, g^{\mu\nu}\Pi''_{\mu\nu} = -8e^2 \int \frac{d^4p}{(2\pi)^3} f(|p_0|)\,\delta(p^2-m_e^2)\,\mathcal{P}\left(\frac{2m_e^2+2K\cdot p}{K^2-2K\cdot p}\right)$$

$$\overset{m_e\to 0}{=} -8e^2 \sum_{\pm} \int \frac{d^3p}{(2\pi)^3} \frac{\cancel{2}}{\cancel{2}p} f(p)\,\mathcal{P}\left(\frac{\pm p\omega - \vec{k}\cdot\vec{p}}{K^2 \mp 2p\omega + 2\vec{k}\cdot\vec{p}}\right) \quad ; \quad \begin{array}{l}\tilde{a}_{\mp} = \pm p\omega \\ \tilde{a}_{\mp} + \frac{1}{2}C_{\mp} = \frac{1}{2}K^2\end{array}$$

$$= -\frac{2e^2}{\pi^2} \sum_{\pm} \int dp \frac{p}{e^{\beta p}+1} \left(-1 - \frac{K^2}{4kp} \log\left|\frac{K^2+2p(k\mp\omega)}{K^2-2p(k\mp\omega)}\right| \right)$$

$$= +\frac{4e^2}{\pi^2} \int dp \frac{p}{e^{\beta p}+1} \left(1 + \frac{K^2}{8kp} \log\left|\frac{4p^2+4pk-K^2}{4p^2-4pk-K^2}\right| \right) \qquad (7,47)$$

$\mathrm{Re}\,\Pi_T$ saadaan nyt kaavoista (7,46) ja (7,47); $\mathrm{Re}\,\Pi_T = -\frac{1}{2}\Pi_L + \frac{1}{2}\mathrm{Re}\, g^{\mu\nu}\Pi''_{\mu\nu}$.

$$\boxed{\begin{aligned}\mathrm{Re}\,\Pi_T = \frac{2e^2}{\pi^2}\int dp \frac{p}{e^{\beta p}+1}\Bigg(& \frac{\omega^2+k^2}{2k^3} - \frac{\omega K^2}{2k^3}\log\left|\frac{\omega-k}{\omega+k}\right| + \frac{\omega K^2}{4k^3}\log\left|\frac{(\omega+k)^2-4p^2}{(\omega-k)^2-4p^2}\right| \\ & + \frac{K^2}{4kp}\left(+p+\frac{\omega^2+k^2}{4k^2}\right)\log\left|\frac{4p^2+4pk-K^2}{4p^2-4pk-K^2}\right| \Bigg)\end{aligned}} \qquad (7,48)$$

Mitä hauskin harjoitustehtävä on osoittaa että rajalla $T \gg k, \omega$

$$\mathrm{Re}\,\Pi_L \approx \frac{e^2T^2}{3}\left(1-\frac{\omega^2}{k^2}\right)\left(1+\frac{\omega}{2k}\log\left(\frac{\omega-k}{\omega+k}\right)\right) \qquad (7,49)$$

$$\mathrm{Re}\,\Pi_T \approx \frac{e^2T^2}{6}\left(\frac{\omega^2}{k^2} - \left(1-\frac{\omega^2}{k^2}\right)\frac{\omega}{2k}\log\left(\frac{\omega-k}{\omega+k}\right)\right) \qquad (7,50)$$

Edelleen, voidaan osoittaa että (HT) kun $k=0$, $T \gg \omega \neq 0$

- $$\mathrm{Re}\,\Pi_L(0,\omega) = \mathrm{Re}\,\Pi_T(0,\omega) = \frac{e^2T^2}{9} \equiv \omega_{pl}^2 \qquad (7{,}51)$$

ja kun $\omega=0$ $(T \gg k \neq 0)$

- $$\mathrm{Re}\,\Pi_L(k,0) = \frac{e^2T^2}{3} \equiv m_D^2(T) \qquad (7{,}52)$$

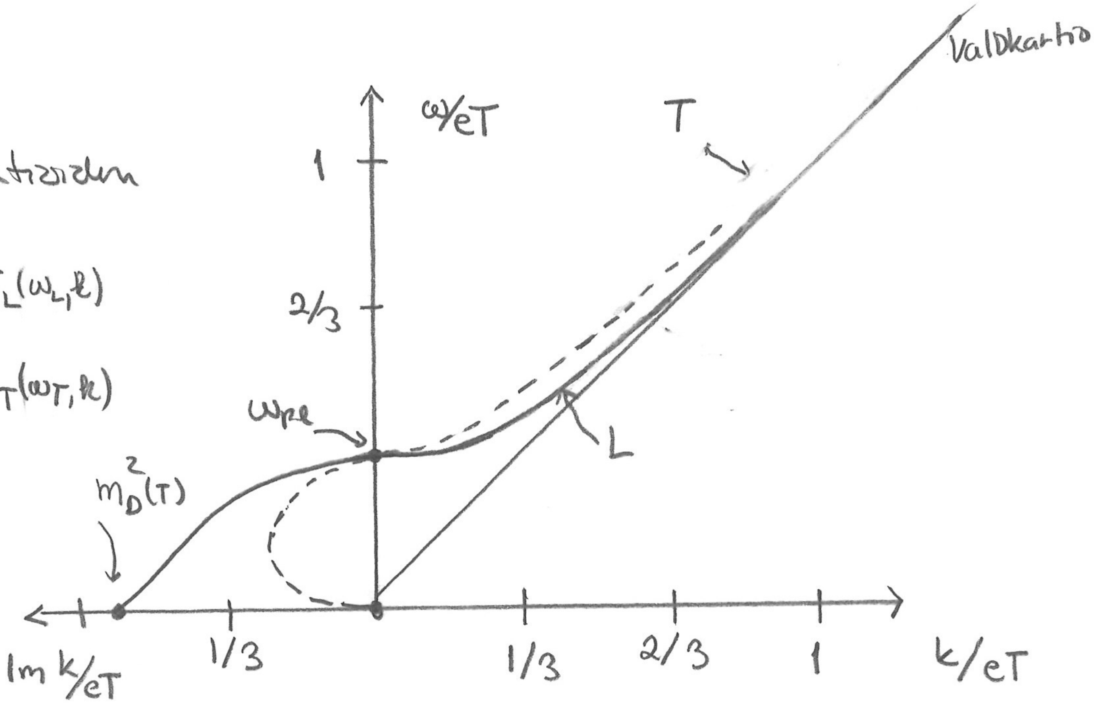
- $$\mathrm{Re}\,\Pi_T(k,0) = 0 \qquad (7{,}53)$$

Dispersiorelaatioiden

$\omega_L^2 = k^2 + \mathrm{Re}\,\Pi_L(\omega_L,k)$

$\omega_T^2 = k^2 + \mathrm{Re}\,\Pi_T(\omega_T,k)$

ratkaisut



Edelleen kun $\omega \gg \omega_{pl}$

$$\omega_L^2 \simeq k^2\left(1 + 4e^{-\frac{2\omega^2}{\omega_{pl}^2}}\right)$$

eli ratkaisu lähestyy eksponentiaalisen nopeasti valokartiota.

Lisäys: tuloksia (7,51 - 7,53) johdettaessa kannattaa johtaa välitulokset

$$\pi_L(\omega,0) = \frac{e^2}{12\omega^2}\int_0^\infty dp\, \frac{p}{e^{\beta p}+1}\left(1+\mathcal{P}\,\frac{2\omega^2}{\omega^2-4p^2}\right) = \pi_T(\omega,0)$$

Kun $q^2>0$ esiintyy siis sekä transversaali- että longitudinaalisia plasmaoskillaatioita.

Imaginääriosa

Voidaan laskea QED:ssä. QCD:ssä jopa 1-luuppi tulos γ:ille) on tavanomaisessa perturbaatioteoriassa usein mittariippuva ja vieläpä negatiivinen kovarianteissa mitoissa. Mittariippumaton tulos esimerkiksi longitudinaalisen plasmonin vaimennukselle QCD:ssä

$$\gamma(\vec{k}=0) = -\frac{1}{2\bar{\omega}_{pl}}\,\mathrm{Im}\,\bar{\Pi}_L(k_0=\omega_{pl}, k=0)$$

$$\simeq 6{,}635\,\frac{gNT}{24\pi} \qquad (\quad)$$

saadaan vasta kun skaalan $\sim gT$ kontribuutiot (HTL) resummataan. Tulos ( ) kertoo että plasmoni on aika marginaalisesti olemassa realistisissa QCD-plasmoissa. Kun $N=3$ on $\gamma \ll T$ kun $g \ll 2{,}2$, kun tyypillisesti $g \gtrsim 1$ $\forall$ T,