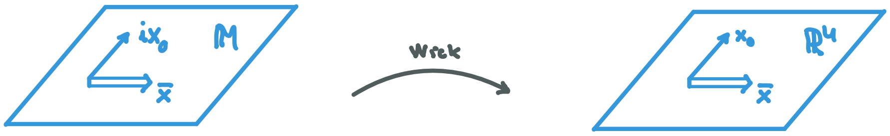
Effective 3d theories

We have already noted that thermal equilibrium FTFT is a theory on $S_1 \otimes \mathbb{R}^3$. Let us now consider the connection between the topology and physics at different length scales.

- T = 0. QFT defined in Minkowski space $\mathbb{M} \propto SO(1,3)$.

  $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Time-ordered Greens functions (Feynman ϵ)

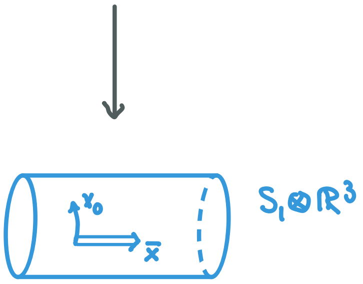
  $\Rightarrow$ Wick $\Rightarrow$ Euclidean $\mathbb{R}^4 \propto SO(4)$ theory $g_{\mu\nu} \to -\delta_{\mu\nu}$.



$$iS_M = i\int d^4x_M \mathcal{L}_M$$

$$S_E = -\int d^4x_E \mathcal{L}_E$$

- T ≠ 0 FTFT was found to be equivalent on a QFT defined on $S_1 \otimes \mathbb{R}^3$, where $x_0 \in [0,\beta]$, and $\phi_a(\beta) = \phi_a(0)$, $\psi_a(\beta) = -\psi_a(0)$.

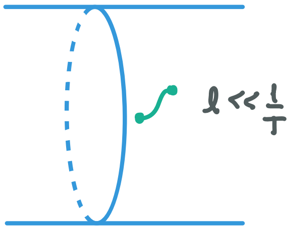


$$S_E^\beta = -\int_0^\beta dt \int d^3\vec{x}\, \mathcal{L}_E$$

When $T \to 0$, $\beta \to \infty$ and $S_1 \longrightarrow \mathbb{R}$. Periodicity loses meaning as Matsubara frequences coalesce and one recovers the continuum $\mathbb{R}^4$-theory.

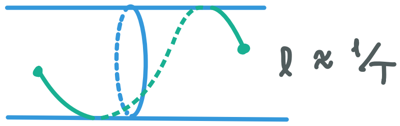
Moreover, for a finite $T \neq 0$, theory looks different at different scales.

i) $l \propto \frac{1}{k_E} \ll \frac{1}{T}$. In these length scales the only the largest Matsubara frequencies contribute. Modes



effectively coalesce. The paths that mainly contribute to the PI are not sensitive to periodicity. Low Matsubara modes not relevant.

ii) $\ell \sim 1/T$ Temperature corrections essential. All Matsubara modes dynamical & relevant.



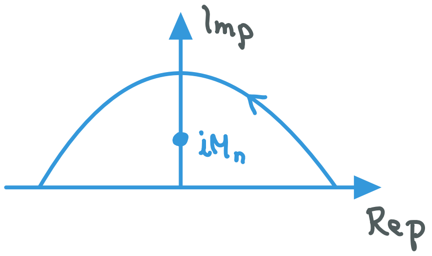
iii) $\ell \gg 1/T$. In these length scales τ-dimension does not show anymore. Only the zero mode contributes to dynamical correlations. Theory has been effectively reduced to three dimensions.

Free theory correlators

$M_n^2 \equiv m_R^2 + \omega_n^2$

$$\langle \phi_n(x)\,\phi_n^*(0)\rangle = \beta\int\frac{d^3p}{(2\pi)^3}\,\frac{e^{i\vec{p}\cdot\vec{x}}}{\omega_n^2+\omega_{\vec{p}}^2} = \frac{\beta}{4\pi^2}\int_0^\infty dp\,\frac{p^2}{p^2+M_n^2}\int_{-1}^{1}dz\,e^{ip|\vec{x}|z}$$

$$= -\frac{i\beta}{4\pi^2|\vec{x}|}\sum_{s=\pm1}\int_0^\infty dp\,\frac{sp}{p^2+M_n^2}\,e^{isp|\vec{x}|}$$



$$= -\frac{i\beta}{4\pi^2|\vec{x}|}\int_{-\infty}^{\infty}dp\,\frac{p}{p^2+M_n^2}\,e^{ip|\vec{x}|}$$

$$= \frac{\beta}{4\pi|\vec{x}|}\,e^{-M_n|\vec{x}|} \qquad M_n = \sqrt{m_R^2+(2\pi nT)^2}$$

Thus all matsubara modes with $n>0$ decouple for $|\vec{x}| \gg 1/T$. Then, for large distances only the zero-mode correlator survives:

$$\langle \phi_0(\vec{x})\, \phi_0^*(0) \rangle = \frac{\beta}{4\pi|\vec{x}|} e^{-m_R|\vec{x}|} \xrightarrow{m_R \to 0} \frac{\beta}{4\pi|\vec{x}|} \qquad (\text{Yukawa} \longrightarrow \text{Coulomb})$$

Interacting theory. For $n \geq 1$ modes the free theory result remains a good approximation. For the zero mode, the leading correction is the thermal mass correction $m_R^2 \to m_D^2(T)$.

$$\langle \phi_0(\vec{x})\, \phi_0^*(0) \rangle \longrightarrow \frac{\beta}{4\pi|\vec{x}|} e^{-m_D(T)|\vec{x}|} \quad ; \quad m_D^2(T) = m_R^2 + \frac{\lambda T^2}{24} .$$

The full 4-D-theory correlator then is

$$\langle \phi(\vec{x})\, \phi(0) \rangle_\beta = T^2 \sum_n e^{-i\omega_n \tau} \langle \phi_n(\vec{x})\, \phi_n(0) \rangle$$

$$= T^2 \sum_n e^{-i\omega_n \tau} \frac{\beta}{4\pi|\vec{x}|} e^{-M_n|\vec{x}|} \xrightarrow{|\vec{x}| \gg \frac{1}{T}} \frac{T}{4\pi|\vec{x}|} e^{-m_D(T)|\vec{x}|}$$

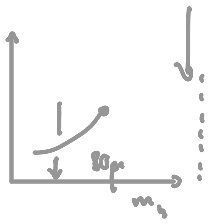
More generally, based on Wick's theorem, also all higher order Greens functions reduce to those of the zero-modes only.

If we are mainly interested in the dynamics of long-wave modes, it would be sensible to derive an effective theory for zero modes only.

* Phase diagram of the SM: ⇒ Electroweak phase transition is cross-over.

⇒ Baryogenesis not possible in the SM.

⇒ must be beyond SM physics.



inflation

How to do this systematically? ≙ How to compute the form of the effective 3d action, and the relation of the 3d-effective parameters to physical parameters in 4d-theory.

- Obvious way: integrate out all heavy modes ⇒ problems with nonlocal terms
- Practical way: dimensional reduction by matching of 4d & 3d greens functions.

1. Trivial reduction.

Here one simply restricts to static modes, neglecting the $n \neq 0$ -modes altogether

zero mode, not $\tilde{\phi}$

$$\int_0^\beta d\tau \int d^3x \, \mathcal{L}_E(\partial_\mu\phi, \phi) \longrightarrow \frac{1}{T}\int d^3x \, \mathcal{L}_E(\nabla\phi_0, \phi_0) \equiv \int d^3x \, \mathcal{L}_{3D}(\nabla\phi_3, \phi_3)$$

In $\lambda\phi^4$-theory then $\mathcal{L}_{3D} = \frac{1}{2}(\nabla\phi_3)^2 + \frac{1}{2}m_3^2\phi_3^2 + \frac{\lambda_3}{4!}\phi_3^4$

$$\Rightarrow \quad \begin{aligned} \phi_3 &= \sqrt{T}\phi_0 \qquad ([\mathcal{L}_3] = L^{-3} = T^3) \\ m_3 &= m_R \\ \lambda_3 &= \lambda_R T \qquad \text{dimensionful coupling.} \end{aligned}$$

2. Integrating out $n \neq 0$ modes.

Writing the partition function in mode basis we can write

ϕ_0-independent part

$$Z = \int \prod_n [\mathcal{D}\phi_n]_\beta \, e^{-S_E[\phi_n]} \equiv Z_{n\neq 0} \int [\mathcal{D}\phi_0]_\beta \, e^{-S_{eff}[\phi_0]}$$

Our goal then is to derive $S_{\text{eff}}[\phi_0]$ by integrating out all $\phi_{n\neq 0}$-modes. To this end we write

$$\phi(\tau;\vec{x}) = T\sum_{n=-\infty}^{\infty} \overset{\text{dimensionless}}{\phi_n(\vec{x})}\, e^{-i\omega_n\tau} \iff \phi_n(\vec{x}) = \int_0^\beta d\tau\, \phi(\tau,\vec{x})\, e^{i\omega_n\tau}$$

Which then gives $\phi_{-n}(\vec{x}) = \phi_n^*(\vec{x})$. Also $\phi_0(\vec{x}) = \int_0^\beta d\tau\, \phi(\tau,\vec{x}) = \beta\overset{\text{average field}}{\bar{\phi}} = \frac{1}{\sqrt{T}}\overset{\text{real}}{\phi_3}$.

In terms of the mode-functions, Lagrangian becomes: (eg. $\phi_3 = \frac{1}{\sqrt{T}}\bar{\phi}$)

$$\mathcal{L}_E[\phi_n] = T\sum_n\Big(\underbrace{\tfrac{1}{2}|\nabla\phi_n|^2 + \tfrac{1}{2}M_n^2|\phi_n|^2}_{\text{diagonal free part}} + \underbrace{\frac{\lambda T^2}{4!}\sum_{k,l}\phi_n\phi_k\phi_l\phi^*_{n+k+l}}_{\text{mode-coupling interactions}}\Big)$$

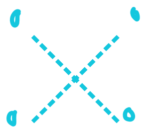
$$= \frac{1}{2}(\nabla\phi_3)^2 + \frac{1}{2}m_R^2\phi_3^2 + \sum_{n\neq 0}\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{Int}}.$$

Interacting part can be divided into 3 pieces depending on how many zero modes are involved:

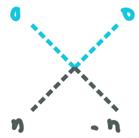
$$\mathcal{L}_{\text{int}} = \mathcal{L}_0[\phi_3] + \mathcal{L}_{\text{mix}}[\phi_3, \phi_{n\neq 0}] + \mathcal{L}_{n\neq 0}[\phi_{n\neq 0}]$$

where

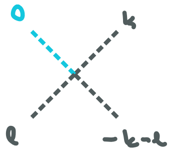
- $$\mathcal{L}_0 = \frac{\lambda T^3}{4!}\phi_0^4 = \frac{\lambda T}{4!}\phi_3^4 = \frac{\lambda_3}{4!}\phi_3^4$$



- $$\mathcal{L}_{\text{mix}} = \frac{\lambda T^3}{4!}\Big(6\phi_0^2\sum_{n\neq 0}|\phi_n|^2 + 4\phi_0\sum_{k,l\neq 0}\phi_k\phi_l\phi^*_{k+l}\Big)$$

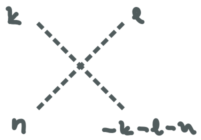


$$= \frac{\lambda_3 T}{4}\phi_3^2\sum_{n\neq 0}|\phi_n|^2 + \frac{\lambda_3 T^{3/2}}{6}\phi_3\sum_{k,l\neq 0}\phi_k\phi_l\phi^*_{k+l}$$



and

- $$\mathcal{L}_{n\neq 0} = \frac{\lambda}{4!}T^3\sum_{k,l,n\neq 0}\phi_n\phi_k\phi_l\phi^*_{n+k+l}$$



With these preparations we can attempt to derive $S_{eff}[\phi_3]$:

$$\int [\mathcal{D}\phi_{n\neq0}]_\beta \exp(-S[\phi_n]) = Z_{n\neq0}\, e^{-S_{eff}(\phi_3)}$$

$$= e^{-S_0[\phi_3]} \int [\mathcal{D}\phi_{n\neq0}]_\beta \exp\left[-S_{free}[\phi_{n\neq0}] - S_{mix} - S_{n\neq0}\right]$$

$$= e^{-S_0[\phi_3]} \int [\mathcal{D}\phi_{n\neq0}]_\beta\, e^{-S_{free}[\phi_{n\neq0}]} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (S_{mix}+S_{n\neq0})^k$$

$$= e^{-S_0[\phi_3]} \left(\prod_{n\neq0} Z_{free}^{n\neq0}\right) \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left\langle (S_{mix}+S_{n\neq0})^k \right\rangle$$

where

$$\langle X \rangle \equiv \frac{1}{\prod_{n\neq0} Z_{free}^{n\neq0}} \int \left[\prod_{n\neq0} \mathcal{D}\phi_n\right]_\beta X\, e^{-S_{free}[\phi_{n\neq0}]}$$

By inspection, we now get

free gas contribution from n≠0-modes

$$\log Z_{n\neq0} = \sum_{n\neq0} \log Z_{free}^{n\neq0} + \log \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \langle S_{n\neq0}^k \rangle .$$

vacuum graphs from n≠0-modes

whereas the effective 3d-action is

$$S_{eff}[\phi_3] = S_0[\phi_3] - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left\langle (S_{mix}+S_{n\neq0})^k - S_{n\neq0}^k \right\rangle \Big|_{connected}$$

Remember that taking log means one gets only connected graphs. Subtracting

Renormalization. Vacuum ct's give rise to

$$\mathcal{L}_{ct} = \frac{T}{2}\sum_n \left(\delta_m + p^2\delta_\phi\right)\phi_n\phi_n^* + \frac{\delta_\lambda}{4!}T^3\sum_{k,l,n}\phi_k\phi_l\phi_n\phi^*_{k+l+n}$$

$$= \frac{1}{2}\left(\delta_m + p^2\delta_\phi\right)\phi_3^2 + \frac{\lambda_3}{4!}\phi_3^4 + \mathcal{L}^{ct}_{mixed} + \mathcal{L}^{ct}_{n\neq 0}$$

Ⓘ Lowest order: $S_{eff}[\phi_3] = S_0[\phi_3] = \int d^3x\, \mathcal{L}_{3d}[\phi_3]$

Ⓘ1-loop order

$$\langle S_{mix}\rangle\Big|_{connected} = \sum_{n\neq 0}\frac{1}{Z_n}\int[\mathcal{D}\phi_n]_\beta\left(\beta\int d^3x\,\frac{\lambda_R T^2}{4}\phi_3^2|\phi_n|^2\right)e^{-S_{n\neq 0}[\phi_n]}$$

$$\underbrace{\Delta_n(0)} = \beta\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_n^2+\omega_p^2}$$

$$= \int d^3x\,\frac{\lambda_R T^2}{4}\phi_3^2(x)\sum_{n\neq 0}\langle\phi_n(x)\phi_n^*(x)\rangle$$

$$= \int d^3x\,\frac{\lambda_R}{4}\left(\frac{2}{\lambda_R}\Pi_{vac} + \frac{T^2}{12}\right)\phi_3^2(x)$$

Combined with the counter term $\int d^3x\,\frac{1}{2}\delta_m\phi_3^2 = -\int d^3x\,\frac{1}{2}\Pi_{vac}\phi_3^2$

we get simply

$$\delta S^{(1)} = \int d^3x\,\frac{\lambda T^2}{24}\phi_3^2$$

and

$$S_{eff}(\phi_3, T) = \int d^3x\left(\frac{1}{2}(\nabla\phi_3)^2 + \frac{1}{2}m_3^2(T)\phi_3^2 + \frac{\lambda_3}{4!}\phi_3^4\right)$$

where still $\underset{= \frac{1}{\sqrt{T}}\bar{\phi}}{\phi_3 = \sqrt{T}\phi_0}$, $\lambda_3 = \lambda_R T$ and $m_3^2(T) = m_R^2 + \frac{\lambda_R T^2}{24} = m_D^2(T)$.

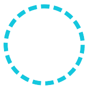
Thus, at 1-loop level the effective 3d-theory is local, and has an effective coupling $\lambda_3 = \lambda_R T$ and effective mass m_D.

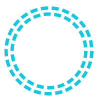
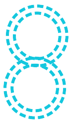
 $\hat{=} \frac{1}{\vec{p}^2 + m_D^2(T)}$  $\hat{=} -\lambda_3$

Ring sum from 3d-perturbation correction

$$\log Z = \log Z_{n\neq 0} + \log \int [D\phi_3]_\beta \, e^{-S_{eff}[\phi_0]}$$

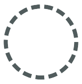
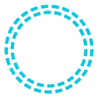
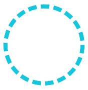
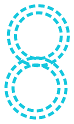
- $\log Z_{n\neq 0} = \sum_{n\neq 0} \log Z_n + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \langle S_{n\neq 0}^k \rangle \Big|_{connected}$

  $= \frac{1}{V} \Big\{$  $-$  $\Big\} + \sum_{n\neq 0}$  $+ \text{ct}'\text{s}$

- $\log \int [D\phi_3]_\beta \, e^{-S_{eff}[\phi_0]} =$  $+$ 

$$= \frac{T}{2} \int \frac{d^3p}{(2\pi)^3} \log\left(\vec{p}^2 + m_D^2\right) + \left(\frac{\lambda_3}{8} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\vec{p}^2 + m_D^2}\right)^2$$

Combined, these give just the ring-improved pressure found earlier

$P = \frac{1}{\beta V} \Big\{$  $+ \Big($  $-$  $\Big) + \sum_{n\neq 0}$  $+$  $+$ ● $\Big\}$

$$= J_T^-(m, T) + \frac{(m_D^3(T) - m_R^3)T}{12\pi} + \frac{1}{2\lambda_R} \pi_T^2 + \frac{\lambda_3}{8} \frac{m_D^2 T^2}{16\pi^2}$$

This is just the ring sum we found earlier resumming the zero modes.

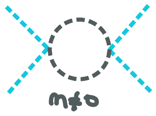
Higher order truncations At order λ^2 we start to get corrections also also to ϕ_3 and λ_3 beyond the trivial mappings found above. The λ^2-correction to λ_3 comes at 1-loop and λ^2-corrections to ϕ_3 & and m_3 at 2 loops.

Coupling constant The lowest order correction is the following

$$S_{eff}^2 \supset -\frac{1}{\prod_{n\neq0} Z_{free}^n} \int \prod_{n\neq0} [\mathcal{D}\phi_n]_\beta \, e^{-S_{free}[\phi_n]} \frac{1}{2!}\left(\frac{\lambda_e T^3}{4}\sum_{k\neq0}\int d^3x\, \phi_0^2 |\phi_k|^2\right)^2 \Bigg|_{connected}$$

$$= -\frac{\lambda_e^2 T^6}{32}\int d^3x\, d^3y\, \phi_0^2(x)\phi_0^2(y) \sum_{k,l\neq0} \langle |\phi_k(x)|^2 |\phi_l(y)|^2\rangle \Big|_{connected}$$

$$= -\frac{\lambda_e^2 T^6}{16}\sum_{k\neq0}\int d^3x\, d^3y\, \phi_0^2(x)\left[\Delta_k(x-y)\right]^2 \phi_0^2(y)$$



This is clearly a non-local term, that cannot be written as an effective local 3d-interaction. However, we can do so approximatively if external momenta in light modes is small $p \lesssim \lambda_R T$. Indeed

$$T^6\sum_{k\neq0}\int d^3x\, d^3y\, \phi_0^2(x)\left[\Delta_k(x-y)\right]^2 \phi_0^2(y)$$

$r = x-y$

$X = \frac{1}{2}(x+y)$

$$= T^3\int d^3X\, d^3r\, \phi_0^2(X+\tfrac{r}{2})\, T^3\sum_{k\neq0}\Delta_k^2(r)\, \phi_0^2(X-\tfrac{r}{2})$$

$$\simeq T^3\int d^3X \int d^3r\, T^3\sum_{k\neq0}\Delta_k^2(r)\left(\phi_0^4(x) - \frac{2}{3}\vec{r}^2\phi_0^2(x)\left(\nabla_X\phi_0(x)\right)^2 + \dots\right)$$

$$= T^3\int d^3X\left(\#_1\,\phi_0^4(x) + \#_2\,\phi_0^2(x)(\nabla\phi)^2\right)$$

First coefficient is

$$\#_1 = T^3 \sum_k \int d^3\vec{r}\, \Delta_k^2(r) \simeq T \sum_k \int_{\vec{p},\vec{q}} \Delta_k(q)\Delta_k(p) \int d^3r\, e^{i(\vec{p}+\vec{t})\cdot\vec{r}}$$

$$= T \sum_{k\neq0} \int_{\vec{p}} \Delta_k(\vec{p})\Delta_k(-\vec{p}) = T \sum_{k\neq0} \int_{\vec{p}} \underbrace{\Delta_k^2(p)}_{\simeq \frac{1}{(\omega_k^2+\vec{p}^2)^2}\left(1-\frac{2m_R^2}{\omega_k^2+\vec{p}^2}+\dots\right)}$$

Now use

$$\oint' \frac{1}{Q^{2p}} \equiv T\mu^{3-d} \sum_{n\neq0} \int \frac{d^dQ}{(2\pi)^d} \frac{1}{(\omega_n^2+Q^2)^p}$$

$$= T \sum_{n\neq0} \frac{\mu^{3-d}}{(4\pi)^{d/2}} \frac{\Gamma(p-\frac{d}{2})}{\Gamma(p)} \left(\frac{1}{\omega_n}\right)^{2p-d} -$$

$$= T^{1-2p+d} \frac{2\pi^{d/2}}{(2\pi)^{2p}} \frac{\Gamma(p-\frac{d}{2})}{\Gamma(p)} \zeta(2p-d)$$

$$\Rightarrow \quad \#_1 = \underbrace{\frac{2\pi^{d/2}}{16\pi^4}\left(\frac{\mu}{T}\right)^{3-d}}_{=\frac{1}{8\pi^{5/2}}\left(1-\frac{\epsilon}{2}\log\frac{\pi T^2}{\mu^2}\right)} \underbrace{\Gamma(2-\tfrac{d}{2})}_{\substack{=\Gamma(\frac{1}{2}+\frac{\epsilon}{2}) \\ \simeq\sqrt{\pi}\left(1-\frac{\epsilon}{2}(\gamma_E+\log 4)\right)}} \underbrace{\zeta(4-d)}_{=\zeta(1+\epsilon)\simeq\frac{1}{\epsilon}+\gamma_E} - 2m_R^2 T^{d-5} \frac{2\pi^{d/2}}{64\pi^6} \frac{\overbrace{\Gamma(3-\tfrac{d}{2})}^{\Gamma(3/2)=\frac{1}{2}\sqrt{\pi}}}{2} \underbrace{\zeta(6-d)}_{=\zeta(3)}$$

$$= \frac{1}{8\pi^2}\left(1-\frac{\epsilon}{2}\left(\gamma_E+\log\frac{4\pi T^2}{\mu^2}\right)\right)\left(\frac{1}{\epsilon}+\gamma_E\right) - \frac{\zeta(3)}{64\pi^4}\left(\frac{m_R}{T}\right)^2$$

$$= \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} + \gamma_E - \log\frac{4\pi T^2}{\mu^2} - \zeta(3)\left(\frac{m_R}{2\pi T}\right)^2\right)$$

The leading term then is

$$T\left\{\frac{\lambda_R}{4!} + \frac{\lambda_R^2}{16}\frac{1}{16\pi^2}\left(\frac{2}{\epsilon}+\gamma_E-\log\frac{4\pi T^2}{\mu^2}-\zeta(3)\left(\frac{m_R}{2\pi T}\right)^2\right)\right\} \underbrace{T^2\phi_0^4(x)}_{\to\phi_3^4(x)}$$

This is divergent as expected. We have a counter-term at our disposal, however; $T\frac{\delta_\lambda}{4!}\phi_0^4$. The counter term depends on scheme of course. In the $m_R\to 0$ - limit renormalization at $p^2=0$ is not possible. We could use some other scale, say $s=t=u=M^2$ to define the coupling λ_M, or we can use just the $\overline{MS}$-scheme, where one just removes the UV- divergence:

$$\delta_\lambda^{\overline{MS}} = -\frac{3}{2}\lambda_R^2\, iB_0(M^2,0,0)_{div} = -\frac{3}{2}\lambda_R^2\frac{1}{16\pi^2}\overbrace{\left(\frac{2}{\epsilon}-\gamma_E+\log 4\pi\right)}^{\frac{2}{\epsilon_{\overline{MS}}}}$$

$$\overline{\lambda}_3 = \lambda_R T + \left\{4!\frac{\lambda_R^2}{16\cdot 16\pi^2}\left(\frac{2}{\epsilon}+\gamma_E-\log\frac{4\pi T^2}{\mu^2}-\zeta(3)\left(\frac{m_R}{2\pi T}\right)^2\right)\right.$$

$$\left.-\frac{3}{2}\frac{\lambda_R^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma_E+\log 4\pi\right)\right\}T$$

$$=\lambda_R T\left\{1+\frac{3\lambda_R}{32\pi^2}\left(2\gamma_E+2\log\frac{\mu}{4\pi T}-\zeta(3)\left(\frac{m_R}{2\pi T}\right)^2\right)\right\}$$

So, we now got a sensible, thermally corrected, local 3d 4-point coupling. However, this was just a first term in the infinite series of operators. We already extracted the second term $\sim \#_2(\nabla\phi_0)^2\phi_0^2$

$$-\frac{2}{3}T^3\sum_k\int d^3\vec{r}\, r^2\Delta_k^2(r) = -\frac{2}{3}T\sum_{k\neq 0}\int_{\vec{p}}\left(\partial_{\vec{p}}\Delta_k\right)^2 = -\frac{8}{3}T\sum_{k\neq 0}\int_p\frac{\vec{p}^2}{(\omega_n^2+\omega_{\vec{p}}^2)^4}$$

$$\simeq -\frac{8}{3}T\sum_{k\neq 0}\int_{\vec{p}}\frac{\omega_n^2+\vec{p}^2-\omega_n^2}{(\omega_n^2+\vec{p}^2)^4} \simeq -\frac{8}{3}\left(\sum\!\!\!\!\!\!\!\int' \frac{1}{Q^6}-\sum\!\!\!\!\!\!\!\int'\frac{\omega_n^2}{Q^8}\right) \quad ;\ Q^2\equiv\omega_n^2+\omega_{\vec{p}}^2.$$

We have a new integral, which we can compute noting that $\sum\!\!\!\!\!\!\!\int'\frac{1}{Q^{2p}}\sim T^{d+1-2p}$ whence

$$T\frac{\partial}{\partial T}\sum\!\!\!\!\!\!\!\int \frac{1}{Q^{2p}} = (1+d-2p)\sum\!\!\!\!\!\!\!\int' \frac{1}{Q^{2p}}$$

$$\frac{\partial}{\partial T}\omega_n^2 = \frac{2}{T}\omega_n^2$$

$$= \sum_{n\neq 0}\int_{\vec{p}} T\frac{\partial}{\partial T}\frac{T}{(\omega_n^2+\omega_p^2)^p} = \sum\!\!\!\!\!\!\!\int' \frac{1}{Q^{2p}} - 2p\sum\!\!\!\!\!\!\!\int' \frac{\omega_n^2}{Q^{2p+2}}$$

$$\Rightarrow \sum\!\!\!\!\!\!\!\int' \frac{\omega_n^2}{Q^{2p+2}} = \left(\frac{2p-d}{2p}\right)\sum\!\!\!\!\!\!\!\int' \frac{1}{Q^{2p}}$$

$$\Rightarrow \#_2 = -\frac{8}{3}\left(1-\frac{6-d}{6}\right)T^{1-6+d}\frac{2\pi^{d/2}}{(2\pi)^6}\frac{\overbrace{\Gamma(3-\tfrac{d}{2})}^{\sqrt{\pi}/2}}{\Gamma(3)}\zeta(6-d)$$

$$= -\frac{4}{3}\frac{1}{T^2}\frac{1}{2^7\pi^4}\zeta(3) = -\frac{\zeta(3)}{32\pi^4T^2}$$

So the second term is

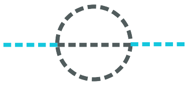
$$\frac{\lambda_R^2T^3}{16}\frac{\zeta(3)}{32\pi^4T^2}\phi_0^2(\nabla\phi_0)^2 = \lambda_R^2T\left(\frac{\zeta(3)}{2(16\pi^2)^2}\right)\frac{1}{T^2}T^2\phi_0^2(\nabla\phi_0)^2 \propto \#\lambda_R^2T\left(\frac{p}{T}\right)^2\phi_3^4$$

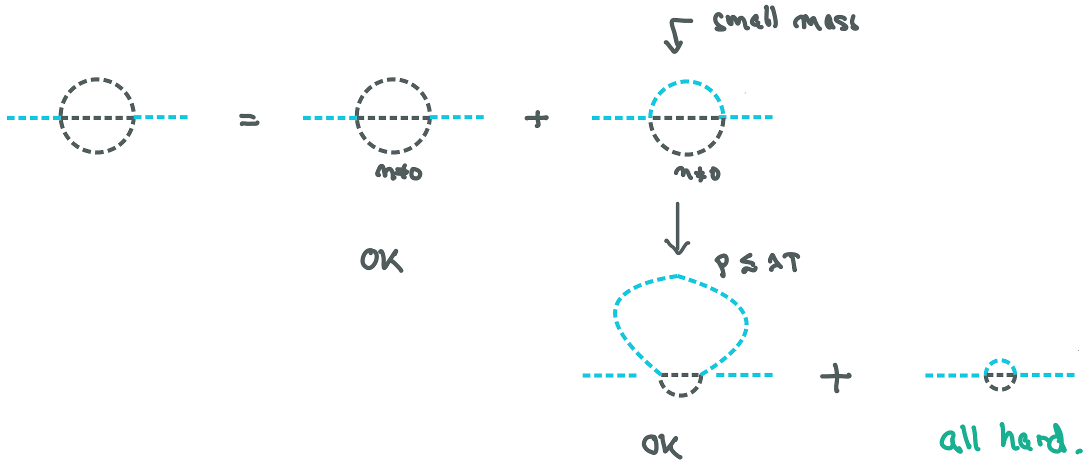
In the dim. reduced theory we assume

higher order.

$$p\lesssim m_D\sim\lambda_R T \Rightarrow \#_2T^3(\nabla\phi)^2\phi_0^2 \lesssim \lambda_R^4T\phi_3^4$$

So the error we make neglecting this term is higher order in coupling, in the regime where we are working.

Things get even more messy with 2-loop correction to self energy. First, all is well with  etc, as we have seen with the ring expansion. However, the graph  is more complicated:



all hard. Not captured by effective theory in our recipe.

This means that to get consistent effective theory, we need to integrate over hard contributions of soft modes as well. How to do this consistently?

Dimensional reduction by matching of greens functions

There are three tasks: define the truncation of the 3d theory, consistent with the symmetries and then define the parameters of the 3D-theory in terms of the full 4d-theory parameters (defined from observables). Finally, one must define the validity-range of the theory.

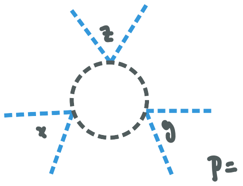
In $\lambda\phi^4$-theory we aim for the super-renormalizable 3d-theory

$$\mathcal{L}_{3D} = \underset{\uparrow}{\frac{1}{2}(\nabla\phi_3)^2} + \underset{\uparrow}{\frac{1}{2}m_3^2\phi_3^2} + \underset{\uparrow}{\frac{\lambda_3}{4!}\phi_3^4} \qquad (+ \ldots \text{ must be small })$$

$= ?\,(\phi_R, m_e, \lambda_R)$

Lowest order additional operators

$$\sim \#\phi_3^6 + \#(\nabla\phi_3)^2\phi_3^2 + \ldots$$



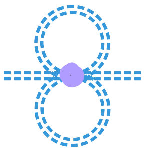
$$\sim \lambda_R^3 T^9 \sum_{k\neq 0} \int d^3x\, d^3y\, d^3z\; \phi_0^2(x)\, \hat{\Delta}_k(x-y) \times$$

$$\times\, \phi_0^2(y)\, \hat{\Delta}_k(y-z)\, \phi_0^2(z)\, \hat{\Delta}_k(z-x)$$

$$\sim \int d^3x\; \lambda_R^3 T^3 \phi_0^6(x)\;\; \underbrace{T^6 \sum_{k\neq 0} \int d^3r_1\, d^3r_2\; \Delta_k(r_1)\, \Delta_k(r_2)\, \Delta_k(r_1-r_2)}_{\sim\, T^3 \sum\!\!\!\!\!\!\int' \frac{1}{(\omega_k^2+\omega_p)^6}}$$

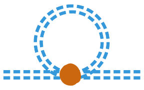
$$\sim \int d^3x\; \overbrace{\#\, \lambda_R^3}^{\equiv\, \lambda_6}\, \phi_3^6$$

This operator would give a contribution to two-point function



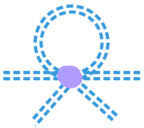
$$\sim \lambda_6\, m_D^2 \quad \sim \lambda_R^4 T^2$$

Similarly, the operator $\#_2 T^3 (\nabla\phi_0)^2 \phi_0^2 \sim \lambda_R^2 \left(\frac{P}{T}\right)^2 T\, \phi_3^4 \overset{P\lesssim\lambda_R T}{\lesssim} \lambda_R^4 T\, \phi_3^4$



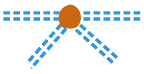
$$\sim \lambda_R^4\, T\, m_D \sim \lambda_R^{\overset{7/2}{9/2}} T^2$$

eg these are shifts $\frac{\Delta m_3}{m_3} \propto \lambda_R^3$ and $\lambda_R^{9/2}$, respectively. The two-loop self energy matching can thus be computed to order λ_R^3 (3 loops), ignoring higher order terms in effective expansion. Similarly



$$\sim \lambda_6\, m_D \sim \lambda_R^{7/2}\, T$$

ok up to 2 loops



$$\sim \lambda_R^2 \left(\frac{P}{T}\right)^2 \sim \lambda_R^4 T\, \phi_3^4$$

~ to 2-loop contribution

So, if we need to trust the theory to $p \sim \lambda_R T$ we can use the effective 3d-theory 2-point function to 3 loops and 4-point function to 2 loops. If we needed to trust the theory to higher momenta, its perturbative validity region would get smaller.

2-point function

The <u>renormalized</u> 2-point function of the zero-mode, computed from 4d-theory can always be written as

this is here only formally. Not computed from 4d th.

$$\Delta_4^{-1}(k^2) \sim \vec{k}^2 + m_R^2 + \overline{\Pi}(k^2) + \Pi_3(k^2) \qquad ; \overline{\Pi}(k^2) = \Pi(k^2, \phi_R, \lambda_R, m_R)$$

where $\Pi_3(k^2)$ comes from the zero-mode and $\overline{\Pi}(k^2)$ contains $n \neq 0$ & mixed $n \neq 0$ and $n=0$ contributions. The 2-point function generated by the effective 3d-theory (we now think that ϕ_3, m_3 & λ_3 are some yet undetermined parameters)

$$\Delta_3^{-1}(k^2) \sim \vec{k}^2 + m_3^2 + \Pi_3(k^2), \qquad ; \Pi_3(k^2) = \Pi_3(k^2, m_3, \lambda_3)$$

which we think is valid for $p \lesssim \lambda_R T$. The function $\overline{\Pi}(k^2)$ is IR-safe, and can be expanded as

$$\overline{\Pi}(k^2) = \overline{\Pi}_T(0) + \overbrace{\overline{\Pi}'_T(0) k^2}^{\mathcal{O}(\lambda^3 T^2)} + \underbrace{\mathcal{O}\left(\lambda^2 \frac{k^4}{T^2}\right)}_{\mathcal{O}(\lambda_R^6)} \qquad \left(\mathcal{O}(\lambda_R^4) \text{ for } p \lesssim \sqrt{\lambda_R} T\right)$$

for max. accuracy. compute to 3 loops

max accuracy 2-loops

$$\vec{k}^2 + m_R^2 + \overline{\Pi}(k^2) \simeq \left(1+\overline{\Pi}'(0)\right)\left(\vec{k}^2 + \frac{m_R^2+\overline{\Pi}(0)}{1+\overline{\Pi}'(0)}\right)$$

Note that we have assumed that renormalization was carried out. This means that $\overline{\Pi}(0)$ and $\overline{\Pi}'(0)$ are finite, purely thermal corrections. Since $\Delta \sim \langle \phi \phi \rangle$, we have $\Delta^1 \propto \phi^{-1}$ & we can absorb $(1+\overline{\Pi}'(0))$ into ϕ_3:

$$\Rightarrow \begin{cases} \phi_3 = \left(\frac{T}{1+\overline{\Pi}'(0)}\right)^{1/2} \phi_0 \simeq \sqrt{T}\left(1-\frac{1}{2}\overline{\Pi}'(0)\right)\phi_0 = \frac{1}{\sqrt{T}}\left(1-\frac{1}{2}\overline{\Pi}'(0)\right)\phi_{4d} \\ m_3^2 = \frac{m_R^2+\overline{\Pi}_T(0)}{1+\overline{\Pi}'(0)} \simeq \left(m_R^2+\overline{\Pi}(0)\right)\left(1-\Pi'(0)\right) = \ldots \end{cases}$$

Similarly for the 4-point coupling:

$$\langle \phi_4 \phi_4 \phi_4 \phi_4 \rangle \sim \frac{\lambda_R T^3}{4!}\left(1 + \sum_{i=1}^{2} \lambda_R^i \#_i\right)\langle \phi_0^4 \rangle$$

$$\sim \underbrace{\frac{\lambda_R T}{4!}\left(1 + \sum_{i=1}^{2} \lambda_R^i \#_i\right)\left(1+\Pi'(0)\right)^2}_{= \lambda_3/4!} \langle \phi_3^4 \rangle \equiv \frac{\lambda_3}{4!}\langle \phi_3^4 \rangle$$

Tämä tietenkin redusoituu 1-luuppitasolla jo aiemmin laskettuun.

Jos halutaan lisätä perturbatiivista tarkkuutta, ei siis riitä mennä korkeampaan kertalukuun, vaan täytyy myös lisätä uusia operaattoreita.

Muistutus. Tämä oli vain lelumalli. DR on hyödyllisimmillään kun tutkitaan teorioita joiden IR-alue on ei-perturbatiivinen.

Esim:

$$\mathcal{L}_{4D-SM}(H, W_{i\mu}, \{\psi\})$$

$$\downarrow \quad \text{Integrate out } n \neq 0 \text{ Matsubara modes (by matching)}$$

$$\mathcal{L}_{3D-SM}(H, W_{iL}^{n=0}, W_{iT}^{n=0})$$

$$\downarrow \quad \text{Integrate out } n=0 \text{-modes for } W_{iL}\text{, which get } m_D \propto gT \neq 0$$

$$\mathcal{L}_{3D-SM}(H, W_{iT}^{n=0}) \qquad (m_T \sim g^2 T \text{, nonperturbative})$$

Reduced 3d-theories are often universal, Eg. the same 3d-theory represents a large class of 4d-theories, They only differ by the perturbative DR-steps, which define the mapping from 4d pm-space to 3d theory.

Examples:
- Yukawa theory
- Scalar electrodynamics with SSB
- QCD
- SM
- BSM-theories MSSM, 2HDM, SSM

More complicated theories may contain more light scalar fields for example. (different universality classes)

EFFECTIVE ACTION

What is the ground state of an interacting theory? Or more generally, what is the classical configuration that is the extremal solution in interacting theory? How do these depend on T?

- Effective action, effective potential

1PI - Generating function.

Let us remind us about generating functions in QFT.

- $Z[J]$ (all graphs, $\xrightarrow{\text{Eucl.}}$ partition function)
- $W[J] = -i\log Z[J]$ (connected, $\longrightarrow -\frac{1}{\beta}\log Z = \Omega$)
- $\Gamma_{1PI}[\phi_c] = W[J] - \int d^4x\, J\phi_c$ (1PI-graphs)

where $\phi_c = \langle\phi\rangle = \frac{\delta W}{\delta J}\Big|_{J=0}$; $J = -\frac{\delta\Gamma_{1PI}}{\delta\phi_c}$

$\uparrow$ expectation value of the quantum field

$\uparrow$ ϕ_c is determined by minimum of Γ_{1PI} when $J=0$.

Effective action is generalization of classical action, that accounts for the effects of quantum fluctuations on classical field dynamics.

- Lowest order: $\phi = \phi_{cl} + \phi_q \longrightarrow \phi_{cl}$; $J = 0$

  $Z[J] \longrightarrow \exp\{iS[\phi_{cl}]\} \Rightarrow \Gamma_{1PI}[\phi_{cl}] = S_{cl}[\phi_{cl}]$.

Γ_{1PI} can be expressed in different ways:

$$\Gamma[\phi_{cl}] = \int d^4x \left\{ -V_{eff}[\phi_{cl}] + \frac{1}{2}(\partial_\mu \phi_{cl})^2 Z[\phi_{cl}] + \dots \right\}$$

shift

$$= -i \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, \Gamma^{(n)}_{v}(x_1, \dots, x_n) \, [\phi(x_1) - v] \cdots [\phi(x_n) - v]$$

Coefficients of a functional Taylor expansion

= 1PI n-point function

The second expression reveals:

$$\left. \frac{\delta^n \Gamma[\phi_{cl}]}{\delta\phi_{cl}(x_1) \dots \delta\phi_{cl}(x_n)} \right|_{\phi_{cl}=v} = \Gamma^{(n)}_{v}(x_1 \dots x_n)$$

$$= \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \dots k_n) \, e^{i(k_1x_1 + \dots + k_nx_n)} \, \tilde{\Gamma}_v(k_1, \dots, k_n)$$

transl. inv.

Making gradient expansion:

$$\tilde{\Gamma}^{(n)}_{v}(k_1, \dots, k_n) = \sum_m \frac{1}{m!} \left(k_1 \cdot \nabla_{k_1} + \dots + k_n \cdot \nabla_{k_n} \right)^m \tilde{\Gamma}^{(n)}_{v}(k_1, \dots, k_n) \Big|_{k_i = 0}$$

$$= \tilde{\Gamma}^{(n)}_{v}(0, \dots, 0) + \dots$$

we get

$$\Gamma[\phi_{cl}] = -i \sum_n \int d^4x_1 \dots d^4x_n \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \int d^4x \, e^{i(k_1\cdot(x_1 - x) + \dots + k_n\cdot(x_n - x))} \times$$

$$\times \tilde{\Gamma}^{(n)}_{v}(0, \dots, 0) \, [\phi_c(x_1) - v] \cdots [\phi(x_n) - v]$$

$$= -i \int d^4x \left\{ \sum_n \frac{1}{n!} \tilde{\Gamma}^{(n)}_{v}(0, \dots, 0) (\phi_c(x) - v)^n + \dots \right\} = \int d^3x \left\{ -V[\phi_{cl}] + \dots \right\}$$

Then in particular

$$\frac{\delta}{\delta\phi_{cl}}\int d^3x\, V[\phi_{cl}]\Big|_{\phi_{cl}=v} = i\tilde{\Gamma}^{(1)}_v(0)$$

For the case of a homogeneous field more simply:

$$\frac{dV_{eff}}{dv} = i\tilde{\Gamma}^{(1)}_v = i$$  $$V = \int^{\phi_{cl}} du\, \frac{dV}{du}$$

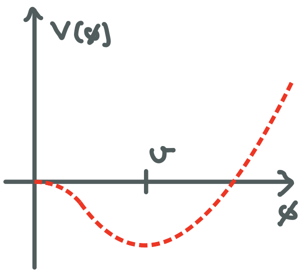
Here we derived an important result: the derivative of the quantum corrected effective potential can be computed as the 1-point function (tadpole) of the shifted theory. Of course, for a homogeneous field $(v=0)$

$$V_{eff}(\phi) = \frac{1}{V_4}\Gamma(\phi) = \sum_n \frac{1}{n!}\frac{1}{V}\Gamma^{(n)}_v(0)\,\phi_{cl}^4$$

$$= \frac{1}{V}\Big\{ \frac{1}{2!}\Big($$  $$\Big)\phi_{cl}^2 + \frac{1}{4!}\Big($$  $$\Big)\phi_{cl}^4 + \dots \Big\}$$

That is $V_{eff}(\phi)$ is the sum of all n-point functions (even here, due to symmetry) in the original unshifted theory.

Spontaneously broken $\lambda\phi^4$-theory



do not mix with dim. reg μ.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \underbrace{\frac{1}{2}\bar{\mu}^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{= -V_{tree}(\phi)}$$

- $$\frac{dV_{tree}}{d\phi}\Big|_{\phi=v} = -\phi\left(\bar{\mu}^2 - \frac{\lambda}{6}\phi^2\right) = 0 \Rightarrow \underset{\text{local maximum}}{\phi = 0} \vee \underset{\text{minimum}}{\phi^2 = \frac{6\bar{\mu}^2}{\lambda}} = v^2$$

- $$\frac{d^2V_{tree}}{d\phi^2}\Big|_{\phi=v} = -\bar{\mu}^2 + \frac{\lambda}{2}v^2 = 2\bar{\mu}^2 = \frac{1}{3}\lambda v^2 \equiv m_v^2 > 0$$

Shifted theory: $\eta = \text{const}$

$$\mathcal{L}(\phi+\eta) = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\bar{\mu}^2}{2}(\phi+\eta)^2 - \frac{\lambda}{4!}(\phi+\eta)^4 + \frac{\delta_\phi}{2}(\partial_\mu\phi)^2 - \frac{\delta_m}{2}(\phi+\eta)^2 - \frac{\delta_\lambda}{4!}(\phi+\eta)^4$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\underbrace{\left(-\bar{\mu}^2 + \frac{\lambda}{2}\eta^2\right)}_{m_\eta^2}\phi^2 - \frac{\lambda}{6}\eta\phi^3 - \frac{\lambda_R}{4!}\phi^4 + \underbrace{\left[(\bar{\mu}^2 - \delta_m)\eta - \frac{\lambda+\delta_\lambda}{6}\eta^3\right]}_{= -\frac{dV}{d\phi}\big|_{\varphi=\eta}}\phi$$

$$+ \frac{\delta_\phi}{2}(\partial_\mu\phi^2) - \frac{1}{2}\left(\delta_m + \frac{\delta_\lambda}{2}\eta^2\right)\phi^2 - \frac{\delta_\lambda}{6}\eta\phi^3 - \frac{\delta_\lambda}{4!}\phi^4 + V(\eta)$$

All Feynman rules can be directly read from this lagrangian.

① Tree level:

$$i\tilde{\Gamma}^{(1)}_{tree} = i$$

$$= i(i)\left(\bar{\mu}^2\eta - \frac{\lambda}{6}\eta^3\right) = -\bar{\mu}^2\eta + \frac{\lambda}{6}\eta^3 = \frac{dV}{d\phi}\Big|_{\phi=\eta} \quad ✓.$$

$$\Rightarrow V_{eff}(\phi) = \int^{\phi_{cl}} d\eta\, i\tilde{\Gamma}^{(1)}_\eta = -\frac{1}{2}\bar{\mu}^2\phi_{cl}^2 + \frac{\lambda}{4!}\phi_{cl}^4 = V_{tree}(\phi_{cl}) \quad ✓$$

② 1-loop level:

$$i\tilde{\Gamma}^{(1)}_{1-loop} = i$$

$$+ i$$

$$= i\left(-\frac{i\lambda_R\eta}{6}\right)\cdot 3\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M_\eta^2} + i\left(-i\delta_m - i\frac{\delta_\lambda}{6}\eta^2\right)\eta$$

$$= \frac{\lambda_R}{2}\eta\,\underbrace{iA_0(M_\eta^2)}_{= -\frac{m_\eta^2}{16\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}} + 1 - \log\frac{m_\eta^2}{\mu^2}\right)} + \delta_m\eta + \frac{\delta_\lambda}{6}\eta^3 = \frac{dV_{1-loop}}{d\phi}\Big|_{\phi=\eta}$$

To fix counter terms we need to define renormalization conditions. We require

- $$\left.\frac{dV}{d\phi}\right|_{\phi=v_{tree}} \equiv 0$$ and
- $$\left.\frac{d^2V}{d\phi}\right|_{\phi=v} \equiv m_v^2 = \frac{1}{3}\lambda v_{tree}^2$$

This is equivalent to $\left.\frac{dV_{1\text{-loop}}}{d\phi}\right|_{\phi=v} = \frac{d^2V_{1\text{-loop}}}{d\phi^2} = 0.$ Now:

$$\frac{d}{d\eta} iA_0(m_\eta^2) = \frac{dm_\eta^2}{d\eta}\frac{d}{dm_\eta^2} iA_0(m_\eta^2) = \overbrace{\frac{dm_\eta^2}{d\eta}}^{=\lambda\eta} iB_0(0,m_\eta^2,m_\eta^2) = -\lambda\eta\left(\frac{1}{16\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}} - \log\frac{m_\eta^2}{\mu^2}\right)\right)$$

Thus we have:

$$\frac{\lambda_R}{2} iA_0(m_v^2) + \delta_m^{(1)} + \frac{\delta_\lambda^{(1)}}{6}v^2 = 0 \quad \&$$

$$\frac{\lambda_R}{2} iA_0(m_v^2) + \frac{\lambda_R^2 v^2}{2} iB_0(0,m_v^2,m_v^2) + \delta_m^{(1)} + \frac{\delta_\lambda^{(1)}}{2}v = 0$$

$$\Rightarrow \delta_\lambda^{(1)} = -\frac{3\lambda_R^2}{2} iB_0(0,m_v^2,m_v^2) \qquad \Rightarrow \delta_m^{(1)} = -\frac{\lambda_R}{2} iA_0(m_v^2) + \frac{\lambda_R^2 v}{4} iB_0(0,m_v^2,m_v^2)$$

$$\Rightarrow \left.\frac{dV_{1\text{-loop}}}{d\phi}\right|_{\phi=\eta} = \frac{\lambda_R}{2}\eta\left(iA_0(m_\eta^2) - iA_0(m_v^2)\right) + \frac{\lambda_R^2\eta}{4}(v^2-\eta^2)\, iB_0(0,m_v^2,m_v^2)$$

$$= -\frac{\lambda_R}{32\pi^2}\left\{\left(-\bar{\mu}^2+\frac{\lambda_R}{2}\eta^2\right)\eta\left(\frac{2}{\epsilon_{\overline{MS}}}+1-\log\frac{m_\eta^2}{\mu^2}\right) - \left(\bar{\mu}^2+\frac{\lambda_R}{2}v^2\right)\eta\left(\frac{2}{\epsilon_{\overline{MS}}}+1-\log\frac{m_v^2}{\mu^2}\right)\right.$$

$$\left. + \frac{\lambda_R\eta}{2}(v^2-\eta^2)\left(\frac{2}{\epsilon_{\overline{MS}}} - \log\frac{m_v^2}{\mu^2}\right)\right\}$$

$$= \frac{\lambda_R}{32\pi^2}\left\{-\bar{\mu}^2\eta\log\frac{m_\eta^2}{m_v^2} - \frac{\lambda_R}{2}\eta^3\left(1-\log\frac{m_\eta^2}{m_v^2}\right) + \frac{\lambda_R\eta v^2}{2}\right\}$$

$$= \frac{\lambda_R}{32\pi^2}\left\{\eta m_\eta^2\log\frac{m_\eta^2}{m_v^2} - \frac{\lambda_R}{2}\eta\left(\eta^2-v^2\right)\right\}$$

$$= \frac{\lambda_R\eta}{32\pi^2}\left\{m_\eta^2\log\frac{m_\eta^2}{m_v^2} + m_v^2 - m_\eta^2\right\}$$ finite & goes to zero at $\eta = v$.

This can be easily integrated:

$$V_{\text{eff}} = V_{\text{tree}}(\phi_a) + \int_v^{\phi} d\eta \; \underbrace{\frac{\lambda_R \eta}{32\pi^2}}_{\frac{dm_\eta^2}{d\eta}} \left\{ m_\eta^2 \log \frac{m_\eta^2}{m_v^2} + m_v^2 - m_\eta^2 \right\}$$

$$= V_{\text{tree}} + \frac{1}{32\pi^2} \int_{m^2(v)}^{m^2(\phi)} dx \left(x \log \frac{x}{m_v^2} + m_v^2 - x \right)$$

$$= V_{\text{tree}} + \frac{1}{64\pi^2} \left\{ m^4(\varphi) \left(\log\left(\frac{m^2(\varphi)}{m_v^2} \right) - \frac{3}{2} \right) + 2 m_v^2 m^2(\varphi) \right\}$$

What was our renormalization scheme exactly?

① We know that $\left.\frac{d^2 V_{\text{eff}}}{d\phi^2}\right|_{\phi=v} = \Gamma_v^{(2)} = \frac{1}{W^{(2)}(0)} = \left.(p^2 - m_R^2 + \overset{\text{incl. ct's}}{\Pi})\right|_{p^2=0} = m_R^2 + \underbrace{\Pi(0)}_{\equiv 0}$

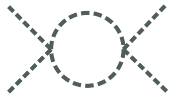
So setting $m^2 \equiv V''|_{\phi=v}$ corresponds to setting m^2 to $p^2=0$ - mass. Indeed

$\Pi(0) =$ i  + i  + i 

$$= \frac{\lambda v}{2} i A_0(m_v^2) + \frac{6 \cdot 3 \cdot 2}{2!} \left(-\frac{i \lambda v}{6} \right)^2 i^2 \, i B_0(0, m_v^2, m_v^2) + \delta_m + \frac{\delta_\lambda}{2} v^2 \equiv 0$$

$$= \frac{\lambda v}{2} i A_0(m_v^2) + \frac{\lambda_R^2 v^2}{2} i B_0(0, m_v^2, m_v^2) + \delta_m + \frac{\delta_\lambda}{2} v^2 \equiv 0 \quad \blacksquare$$

② Consider defining $\lambda_R \equiv \Gamma^{(4)}(0,0,0)$

$\Rightarrow$ ( $+ \; t + u \Big) + \delta_\lambda = 0 \quad \Rightarrow \quad \delta_\lambda = -\frac{3}{2} \lambda_R^2 \, i B_0(0, m_R^2, m_R^2)$

Thus our scheme corresponds to $\lambda_R \equiv \Gamma^{(4)}(0,0,0)$ and m_R^2 is $p^2=0$ mass at broken minimum.

Changing scheme? Suppose we want to define $m =$ pole-mass
and $\lambda = \Gamma^{(4)}(s_*, t_*, u_*)$?

It is easiest to continue to use our current form for the potential, and just define mapping between schemes.

$$\begin{aligned} \bar{\mu}_0^2 &= \bar{\mu}_R^2 + \delta\bar{\mu}_R^2 \\ \lambda_0 &= \lambda_R + \delta\lambda_R \\ \phi_0 &= Z_R^{1/2}\phi_R \end{aligned} \quad \Rightarrow \quad \begin{aligned} \bar{\mu}_{R'}^2 &= \bar{\mu}_R^2 + \delta\mu_R^2 - \delta\mu_{R'}^2 \\ \lambda_{R'} &= \lambda_R + \delta\lambda_R - \delta\lambda_{R'} \\ \phi_{R'} &= (Z_R / Z_{R'})\phi_R \end{aligned}$$

The differences in ct's are finite. From $-\delta_m^R = Z_\phi^R(\bar{\mu}_R^2 + \delta\bar{\mu}_R^2) - \bar{\mu}_R^2 = \delta_\phi\bar{\mu}_R^2 + (1+\delta_\phi)\delta\bar{\mu}_R^2$.
$\delta_\lambda = Z_\phi^2(\lambda_R + \delta\lambda_R) - \lambda_R = (Z_\phi^2 - 1)\lambda_R + Z_\phi^2\delta\lambda_R$ one can solve

$$\delta\bar{\mu}_R^2 = -\frac{\delta_m^R - \delta_\phi^R\bar{\mu}_R^2}{1+\delta_\phi^R} \simeq -\delta_m^R + \delta_\phi^R\bar{\mu}_R^2 + \ldots$$

$$\delta\lambda_R = \frac{\delta_\lambda^R - [(\delta_\phi^R)^2 + 2\delta_\phi^R]\lambda_R}{(1+\delta_\phi^R)^2} \simeq -\delta_\lambda^R - 2\delta_\phi^R\lambda_R + \ldots$$

At 1-loop level, we did not need to set wfr-factors. However, the definition $m^2 = \frac{d^2V}{d\phi^2}$ and identifying m^2 as the $p^2=0$-mass corresponds to setting $\delta_\phi = -\Pi'(0)$. order by order.

Effective potential at finite T

One-loop result is straightforward. Just note that in Euclidean space $iS \to -S_E$; $i\Gamma \to -\Gamma_E$, eg

$$\frac{d\delta V_{1\text{-loop}}}{d\eta} = -\Gamma_E^{(1)} = -\Big(\qquad + \qquad\Big)$$

$$= +\lambda\eta\,\frac{1}{2}\,\SumInt \frac{1}{\omega_n^2+\omega_p^2} + \delta_m\eta + \frac{\delta_\lambda}{6}\eta^3$$

$$= \frac{\lambda\eta}{2}\,\overset{iA_0(m_R^2)}{I_0^-(m_R)} + \delta_m\eta + \frac{\delta_\lambda}{6}\eta^3 + \frac{\lambda\eta}{2}\,I_T^-(m_R)$$

$$= \frac{dV^{\text{vac}}_{1\text{-loop}}}{d\eta} + \frac{1}{2}\frac{dm_\eta^2}{d\eta}\,\overbrace{I_T^-(m_R,T)}^{2\frac{d}{dm^2}J_T} = \frac{dV^{\text{vac}}_{1\text{-loop}}}{d\eta} + \frac{d}{d\eta}J_T^-(m_R)$$

$$\Rightarrow \boxed{V(\phi_{cl},T) = V_{\text{tree}}(\phi_{cl}) + V^{\text{vac}}_{1\text{-loop}}(\phi_{cl}) + J_T^-(m(\phi_{cl}))}$$

So the thermal correction to $V(\phi_{cl},T)$ is entirely given by the J_T^--integral.

To remind: (I changed my notation to follow L&V also for bosonic J_T^-)

$$J_T^\pm(m) = T\int\frac{d^3p}{(2\pi)^3}\log\left(1\pm e^{-\beta\omega_p}\right)$$

where $\omega_p = \sqrt{\vec{p}^2+m^2(\varphi)}$ is now φ-dependent. In particular for high T:

$$J_T^- \simeq -\frac{\pi^4}{90}T^4 + \frac{m_\varphi^2T^2}{24} - \frac{m_\varphi^3T}{12\pi} - \frac{m_\varphi^4}{64\pi^2}\left[2\gamma_E - 2\log 4\pi + \log\frac{m_\varphi^2}{T^2} - \frac{3}{2}\right]$$

Combining this with the vacuum term, we find that for $T \gg m$:

$$\delta V_{1\text{-loop}} = \frac{m_\phi^2 T^2}{24} - \frac{m_\phi^3 T}{12\pi} - \frac{m_\phi^4}{64\pi^2}\Big[\log \frac{m_v^2}{T^2} + \underbrace{2\gamma_E - 2\log 4\pi}_{= C_B \simeq -3.9076} - 2 m_v^2 m_\phi^2\Big] + \varphi\text{-indep. pieces}$$

const.

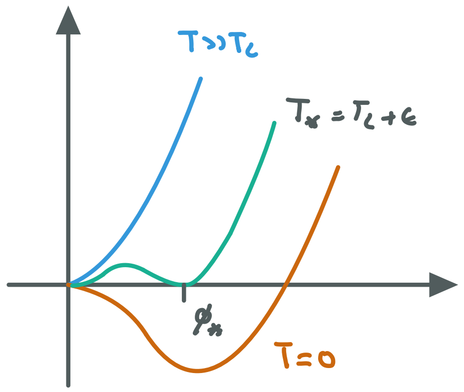
$$= a(T) + b(T)\phi^2 + c(T)\, m_\phi^3 + d(T)\phi^4$$

Apart from m_ϕ^3 $\delta V_{1\text{-loop}}$ becomes again a simple polynominal at high T. The most important corrections are the first two terms, giving

$$V = V_{\text{tree}} + \delta V_{1\text{-loop}}$$

$$\simeq \underbrace{\frac{1}{2}\left(-\bar{\mu}^2 + \frac{\lambda T^2}{24}\right)}_{m_D^2(T)}\phi^2 - \overset{\sim \phi^3 T}{\frac{m_\phi^3 T}{12\pi}} + \frac{\lambda\phi^4}{4!} + \dots$$

↑ can cause the bump



This example qualitatively displays the main physical interest on $V_{\text{eff}}(\phi, T)$: it reveals the possibility of phase transition. ϕ is the order parameter of the system that changes from $\phi = 0$ (high T-phase) to $\phi \neq 0$ (low-T-phase) somewhere around the critical temperature

$$-\bar{\mu}^2 + \frac{\lambda T_c^2}{24} = 0 \Rightarrow T_c = \sqrt{\frac{24\bar{\mu}^2}{\lambda}} \quad \hat{=} \text{ function of model par.}$$

Important questions:

- what is the order of the transition? 1st order 2nd order
- thermodynamics of the transition
- dynamics of the transition.
- Observables of transition. BAU? GW-signal?

Contributions from other fields

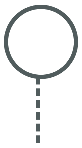
All fields that couple to ϕ contribute to V_{eff}. How? Consider next the

Yukawa interactions

$$\mathcal{L}_Y = -\frac{y_f}{\sqrt{2}}\bar{\psi}_f \phi \psi_f \xrightarrow[\text{theory}]{\text{shifted}} -\frac{y_f}{\sqrt{2}}\bar{\psi}_f(\phi_y + \eta)\psi_f$$

η- dependent mass for ψ $\quad m_{f\eta} = \frac{y_f}{\sqrt{2}}\eta$

$$\Rightarrow \frac{d\,\delta V_f}{d\eta} = -$$



$+\,ct's$

$$= (-1)(-1)\left(-\frac{y_f}{\sqrt{2}}\right) \SumInt_F \mathrm{Tr}\left(\frac{1}{\tilde{\not{p}} + m_f}\right) \quad ; \tilde{\not{p}} = p_\mu \tilde{\gamma}^\mu$$

$$\tilde{\gamma}^\mu = (i\gamma^0, \vec{\gamma})$$

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = -2\delta^{\mu\nu}$$

$$= -\underbrace{\overbrace{\frac{y_f}{\sqrt{2}}}^{=\frac{dm_\eta}{d\eta}} 4m_f}_{=2\frac{\partial m_f^2}{d\eta}} \SumInt_F \frac{1}{\omega_n^2 + \omega^2}$$

$$= -4\frac{d}{d\eta}\frac{1}{2}\SumInt_F \log(\omega_n^2 + \omega_f^2)$$

$$\Rightarrow \delta V_f = -4\,\frac{1}{2}\SumInt_F \log(\omega_n^2 + \omega_f^2) = -4\,J^+(m_f) = -4\left(J_0 + J_T^+(m)\right)$$

Of course $J_0(m_\psi^2)$ is combined with the counter-term. We can add new pieces to existing ct.'s just setting:

$$\delta_m = \delta_m^\phi + \delta_m^\psi + \ldots \qquad \text{cte.}$$

where δ_i^ψ are again set by $\left.\frac{d\,\delta V^\psi_{T=0}}{d\eta}\right|_{\eta=v} = \left.\frac{d^2\delta V_\psi}{d^2\eta}\right|_{\eta=v} = 0$. Because $J_0(m_f)$ is exactly of the same form as the earlier scalar correction, the calculation is analogous. One then finds.

$$\delta V_f^{ren} = (-4)\left\{ \frac{m_f^4(\phi)}{64\pi^2}\left(\log\left(\frac{m_f^2(\varphi)}{m_f^2(v)}\right) - \frac{3}{2}\right) + 2 m_f^2(\varphi)\, m_f^2(v) \right\} - 4\, J_T^+(m_f)$$

At high T- limit $J_T^+(m_f)$ becomes:

$$J_T^+(m_f^2) = \frac{7}{8}\,\frac{\pi^4}{90}\,T^4 - \frac{m_f^2 T^2}{48} - \frac{m_f^4}{64\pi^2}\left(\log \frac{m_f^2(\varphi)}{T^2} + 2\gamma_E - 2\log\pi - \frac{3}{2}\right)$$

So that for $T \gg m_f$

$$\delta V_f \simeq -4\left\{ \frac{m_f^2 T^2}{48} - \frac{1}{64\pi^2}\left[m_f^4(\varphi)\overbrace{\left(\log \frac{m_f^2(v)}{T^2} +\right.}^{\text{const}(T)} \underbrace{2\gamma_E - 2\log\pi}_{= E_n \sim .-1\,13} \Big) - 2 m_f^2(\varphi)\, m_f^2(v) \right] \right\}$$

Complex scalar field

$$\mathcal{L} = |\partial_\mu \Phi|^2 + \bar{\mu}^2 |\Phi|^2 - \lambda |\Phi|^4 \qquad ; \quad \Phi \to \frac{1}{\sqrt{2}}(\varphi + i\chi)$$

$$= \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}(\partial_\mu\chi)^2 + \underbrace{\frac{1}{2}\bar{\mu}^2\varphi^2 + \frac{1}{2}\bar{\mu}^2\chi^2 - \frac{\lambda}{4}(\varphi^2+\chi^2)^2}_{-V_{tree}(\varphi,\chi)}$$

Tree level. Minimum at direction $\chi = 0$

- $$\left.\frac{dV_{tree}}{d\varphi}\right|_{\substack{\chi=0\\ \varphi=v}} = (-\bar{\mu}^2 + \lambda\varphi^2)\varphi \equiv 0 \Rightarrow \varphi = 0 \vee \varphi = \frac{\bar{\mu}^2}{\lambda}$$

$$\left.\frac{\partial^2 V_{tree}}{\partial\varphi^2}\right|_{\substack{\chi=0\\ \varphi=v}} = -\mu^2 + 3\lambda v^2 = 2\lambda v^2 = 2\bar{\mu}^2 = m_v^2$$ (this needs to be refined, in fact)

and $$\left.\frac{\partial^2 V}{\partial\chi^2}\right|_{\substack{\chi=0\\ \varphi=v}} = -\bar{\mu}^2 + \lambda v^2 = 0$$ Goldstone boson.

Shifted theory:

shifted theory

$\varphi \to h+\eta$

$\varphi^2 \to h^2+2h\eta+\eta^2$

$\varphi^4 \to h^4+6h^2\eta^2$
$+4h\eta^3$
$+4\eta h^3$
$+\eta^4$

$$\to \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}\overbrace{(-\bar{\mu}^2+3\lambda\eta^2)}^{m_h^2}h^2 + \frac{1}{2}(\partial_\mu \chi^2) - \frac{1}{2}\overbrace{(-\mu^2+\lambda\eta^2)}^{m_\chi^2}\chi^2$$

$$-\frac{\lambda}{4}(h^4+\chi^4+2h^2\chi^2) - \lambda\eta h^3 - \lambda\eta h\chi^2 + h(\bar{\mu}^2-\lambda\eta^2)$$

$+$ ct.'s (same structure with $\lambda \to \delta_\lambda$, $\bar{\mu}^2 \to -\delta_m$ & $(\partial_\mu h)^2 \to \delta_\varphi(\partial_\mu h)^2$ & $(\partial_\mu \chi)^2 \to \delta_\varphi(\partial_\mu\chi)^2$.)

One-loop calculation:

$\frac{d\delta V^{\text{vac.}}_{\text{1-loop}}}{d\eta} = i$ 

$+\ i$ 

$+$ ct. $= 3\lambda\eta\, iA_0(\overset{-\mu^2+3\lambda\eta^2}{m_h^2}) + \lambda\eta\, iA_0(\underset{-\mu^2+\lambda\eta^2}{m_\chi^2}) + (\delta_m+\delta_\lambda\eta^2)\eta$

After renormalization and adding the thermal parts we again get (Ex.)

$$\delta V_{1\text{-loop}} = \sum_{i=1}^{2}\left\{\frac{1}{64\pi^2}\left[m_i^4(\varphi)\left(\log\left(\frac{m_i^2(\varphi)}{m_i^2(v)}\right)-\frac{3}{2}\right)+2m_i^2(\varphi)\,m_i^2(v)\right]+J_T^-(m_i^2(\varphi))\right\}$$

where $m_1^2(\varphi) \equiv m_h^2(\varphi) = -\bar{\mu}^2+3\lambda\varphi^2$ and $m_2^2(\varphi) \equiv m_\chi^2(\varphi) = -\bar{\mu}^2+\lambda\varphi^2$.

This calculation went formally through without a problem. However, note that since $m_\chi^2(v) = 0$, this result is actually ill-defined. What is the problem. The renormalization scheme! The $p^2=0$-mass is not well defined in this context. The problem is the IR-singularity due to Goldstone boson.

Mass renormalization with Goldstone modes more carefully.

$$\underbrace{}_{\Pi_A} \qquad \underbrace{}_{= \Pi_B(p^2)}$$

$$= 3\lambda_R iA_0(m_h^2) + \lambda_R iA_0(m_\chi^2) + 18\lambda_R^2 h^2 iB_0(p^2, m_h^2, m_h^2) + 2\lambda_R h^2 iB_0(p^2, m_\chi^2, m_\chi^2)$$
$$+ \delta_m + 3h^2\delta_\lambda + p^2\delta_\phi$$

The self energy is well behaved for $p^2 \neq 0$ or $m_i^2 \neq 0$. However, if $p^2 = 0$ & one tries to take $m_\chi^2 \to 0$, the  - term is IR-divergent! This means that the $p^2=0$ - mass is IR-divergent. We can write it in terms of the pole mass & a divergent part, by moving from $p^2=0$ scheme to on-shell scheme:

- **$p^2=0$ -scheme:**

$$\delta_m + \delta_\lambda v^2 \equiv -\Pi_A$$
$$\delta_m + 3\delta_\lambda v^2 \equiv -\Pi_A - \Pi_B(0) - \Pi_B'(0)p^2 - \delta_\phi p^2$$

$$\Rightarrow \delta_\phi = -\Pi_B'(0), \quad \delta_\lambda = -\frac{1}{2v^2}\Pi_B(0) \quad \text{and} \quad \delta_m = -\Pi_A + \frac{1}{2}\Pi_B(0)$$

- **$p^2 = m_p^2$ -scheme** (note that $\bar v \neq v$!)

$$\bar\delta_m + \bar\delta_\lambda \bar v^2 \equiv -\Pi_A$$
$$\bar\delta_m + 3\bar\delta_\lambda \bar v^2 \equiv -\Pi_A \overbrace{- \Pi_B(m_p^2) - (p^2 - m_p^2)\Pi_B'(m_p^2)}^{= -\Pi_B(p^2) + \tilde{\Pi}_B(p^2)} + \overset{\swarrow\, -\Pi_B'(m_p^2)}{\bar\delta_\phi} p^2$$

$$\Rightarrow \bar\delta_\phi = -\Pi'(m_p^2), \quad \bar\delta_\lambda = -\frac{1}{2\bar v^2}\left(\Pi_B(m_p^2) - m_p^2\Pi_B'(m_p^2)\right) \quad \text{and}$$
$$\bar\delta_m = -\Pi_A + \frac{1}{2}\left(\Pi_B(m_p^2) - m_p^2\Pi_B'(m_p^2)\right)$$

Let us first note that

$$\delta_m^R = -Z_\phi^R\,\delta\bar\mu_R^2 - (Z_\phi^R-1)\,\bar\mu_R^2 \quad\Rightarrow\quad \delta\bar\mu_R^2 = -\frac{\delta_m^R + \delta_\phi^R\,\bar\mu_R^2}{1+\delta_\phi^R} \simeq -\delta_m^R - \delta_\phi^R\,\bar\mu_R^2$$

Using $m_{v_0}^2 = 2\bar\mu_0^2 = 2(\mu_R^2 + \delta\mu_R^2)$ for the bare mass, we can write the $p^2=0$-scheme mass m_v^2 in terms of the pole mass as

$$\begin{aligned}
\underline{m_v^2} = 2\bar\mu^2 &= m_p^2 + \overset{\text{pole scheme}}{2\bar\delta\bar\mu^2} - \overset{p^2=0\text{ scheme}}{2\,\delta\bar\mu^2} \\
&\simeq m_p^2 - 2(\bar\delta_m - \delta_m) + 2(\bar\delta_\phi - \delta_\phi)\overset{\simeq\, m_p^2/2}{\bar\mu^2} \\
&\simeq m_p^2 - \left(\Pi_B(m_p^2) - \Pi_B(0) - m_p^2\,\cancel{\Pi'}(m_p^2)\right) + m_p^2\left(-\cancel{\Pi'}(m_p^2) - \Pi_B'(m_p^2)\right) \\
&= m_p^2 - \Big[\underbrace{\Pi_B(m_p^2)}_{\text{UV-finite, but IR-divergent.}} - \Pi_B(0) - m_p^2\,\Pi_B'(0)\Big] \equiv \underline{m_p^2 - \Delta\Pi}
\end{aligned}$$

The problem is that $\Pi_B(0)$ is not well defined at $p^2=0$! We can keep the Goldstone mass as a formally nonzero regulator for a while. Now observe (all $\Delta\delta_i$ are finite)

$$\begin{aligned}
\Delta\delta_\phi &\equiv \bar\delta_\phi - \delta_\phi = -\Pi_B'(m_p^2) + \Pi_B'(0) \\
\Delta\delta_m &\equiv \bar\delta_m - \delta_m = \frac{1}{2}\left(\Pi_B(m_p^2) - m_p^2\,\Pi'(m_p^2) - \Pi_B(0)\right) \\
&= \frac{1}{2}\left(\Delta\Pi - m_p^2\,\Delta\delta_\phi\right) \equiv \frac{1}{2}\Delta\Sigma. \\
\Delta\delta_\lambda &\equiv \bar\delta_\lambda - \delta_\lambda \simeq -\frac{1}{v^2}\Delta\delta_m = -\frac{1}{2v^2}\Delta\Sigma.
\end{aligned}$$

We can now obtain the effective potential corresponding to renormalization conditions

$$\left.\frac{d\bar V}{d\bar\varphi}\right|_{\bar\varphi=\bar v} = 0\,, \qquad \overset{\text{obs.}}{\left.\frac{d^2\bar V}{d\bar\varphi^2}\right|_{\bar\varphi=\bar v}} = m_p^2$$

by adding the difference of ct-lagrangians with old ct's

$$\bar{V}_{\text{eff}} = \bar{V}_{\text{tree}} + V^{\text{earlier}}_{\text{1-loop}} - \underbrace{\frac{1}{2}\Delta\delta_m \varphi^2 - \frac{1}{4}\Delta\delta_\lambda \varphi^4}$$

different from V_{tree}

$$= -\frac{1}{4v^2}\left(2\Delta\Sigma\, v^2\varphi^2 - \Delta\Sigma\varphi^4\right) \qquad m_\chi^2 = -\mu^2 + \lambda_\chi \varphi^2$$

$$= -\frac{\Delta\Sigma}{8v^2}(\varphi^2 - v^2)^2 + \text{const} = -\frac{m_\chi^4(\varphi)}{8\lambda_\chi^2 v^2}\Delta\Sigma + \text{const}$$

$$\frac{-\mu^2+\lambda\varphi^2}{\lambda} - \frac{\mu^2-\lambda v^2}{\lambda} = \frac{1}{\lambda}\left(m_\chi^2(\varphi) - m_\chi^2(v)\right) = \frac{1}{\lambda} m_\chi^2(\varphi)$$

This term can be combined with the χ-term in the vacuum effective potential:

- $$\frac{1}{64\pi^2} m_\chi^4(\varphi)\left(\log\left(\frac{m_\chi^2(\varphi)}{m_\chi^2(v)}\right) - \frac{3}{2}\right) - \frac{1}{8} m_\chi^4(\varphi)\Big(\underbrace{\widehat{\Pi}_B(m_p^2) - \widehat{\Pi}_B(0) - m_p^2\,\widehat{\Pi}'_B(m_p^2)}_{= \Delta\Sigma/\lambda_\chi^2 v^2}\Big)$$

where $\widehat{\Pi}_B(p^2) = 18\, iB_0(p^2; m_h^2, m_h^2) + 2iB_0(p^2; m_\chi^2, m_\chi^2)$. The only divergent part in $\Delta\Sigma$ is the χ-part in $\widetilde{\Pi}_B(0)$. Using the result:

$$iB_0(p^2, m^2, m^2) = -\frac{1}{16\pi^2}\Big(\underbrace{\frac{2}{\epsilon_{\overline{MS}}} + \log\mu^2}_{\text{cancel in } \Delta\Sigma} - \int_0^1 dx\, \log\left(x(1-x)p^2 - m^2 - i\epsilon\right)\Big)$$

- $$\widehat{\Pi}_{B,\chi}(m_p^2) - \widehat{\Pi}_{B,\chi}(0) - m_p^2\,\widehat{\Pi}'_{B\chi}(m_p^2) = \frac{2}{16\pi^2}\left(\int_0^1 dx\, \log(x(1-x)m_p^2) - \log(-m_\chi^2(v)) - 1\right)$$

  can put $m_\chi^2(v) = 0$ here

  $$= \frac{2}{16\pi^2}\left(\log\left(\frac{m_p^2}{m_\chi^2(v)}\right) - 3\right) \qquad \int_0^1 dx \log x(1-x) = 2\int_0^1 dx \log x = -2$$

- $$\widehat{\Pi}_{B,h}(m_p^2) - \widehat{\Pi}_{B,h}(0) - m_p^2\,\widehat{\Pi}'_{Bh}(m_p^2)$$

$$= \frac{18}{16\pi^2}\int_0^1 dx\left(\log\left([x(1-x)-1]m_p^2 + i\epsilon\right) - \log(-m_p^2 + i\epsilon) - \frac{m_p^2}{x(1-x)m_p^2 - m_p^2}\right)$$

$$= \frac{18}{16\pi^2}\int_0^1 dx\left(\log(1 - x(1-x)) + \frac{1}{1-x(1-x)}\right) = \frac{18}{16\pi^2}\left(-2 + \frac{\pi}{\sqrt{3}} + \frac{2\pi}{3\sqrt{3}}\right)$$

Combining the logs, the χ-term in the new vacuum term becomes

- $$= \underbrace{\frac{1}{64\pi^2} m_\chi^4(\varphi) \left\{ \log\left(\frac{m_\chi^2(\varphi)}{m_p^2}\right) - \frac{3}{2}\right.}_{\text{The old term with } m_\chi^2 \to m_p^2 \text{ in the log}} + \overbrace{3 + 9\left(2 - \frac{5\pi}{3\sqrt{3}}\right)}^{\simeq 12.21 \;\; : \text{This is usually neglected.}} \left.\vphantom{\frac{1}{1}}\right\}$$

So, our full pole-mass renormalized 1-loop correction becomes:

$$\delta\bar{V}_{1\text{-loop}} = \sum_{i=1}^{2} \left\{ \frac{1}{64\pi^2}\left[m_i^4(\phi)\left(\log\left(\frac{m_i^2(\varphi)}{m_{p_i}^2}\right) - \frac{3}{2}\right) + 2 m_i^2(\varphi)\, m_i^2(v)\right] + J_T^-\left(m_i^2(\varphi)\right) \right\}$$

$\uparrow$ for χ-field $m_{p_1}^2 = m_p^2$

$$+ \frac{12.21}{64\pi^2} m_\chi^4(\varphi)$$

Check:

- $$\left.\frac{d}{d\bar{\varphi}}(\delta\bar{V} - \delta V)\right|_{\varphi=v} = \left.\frac{d}{d\varphi}\left(\Delta\delta_m \varphi^2 + \frac{1}{4}\Delta\delta_\lambda \varphi^4\right)\right|_{\varphi=v} = \varphi\left(\Delta\delta_m - v^2\Delta\delta_\lambda\right) = 0.$$

- $$\frac{d^2\bar{V}}{d\bar{\varphi}^2} = \left.\frac{d^2}{d\bar{\varphi}^2}(V + \Delta V)\right|_{\bar{\varphi}=\bar{v}} \simeq \left.\left(\frac{Z_\phi}{\bar{Z}_\phi}\frac{d^2V}{d\varphi^2} - \Delta\delta_m - 3\varphi^2\Delta\delta_v\right)\right|_{\varphi=v}$$

$$\simeq m_v^2\left(1 - \bar{\delta}_\phi + \delta_\phi\right) + 2\left(\bar{\delta}_m - \delta_m\right)$$

$$\simeq m_v^2 + \left[2\left(\bar{\delta}_m - \delta_m\right) - m_p^2\left(\bar{\delta}_\varphi - \delta_\phi\right)\right]$$

$$= m_v^2 + \Delta\Pi = m_p^2 \qquad \checkmark.$$

Show that $\delta v_R = -\frac{v_R}{2m_p^2}\Pi_A$, so that $v_R = v_{R'}$ at least to 1 loop order.

Gauge fields. Here we encounter another problem: gauge dependence. This is a very profound problem that starts from the fact that for the complex (or more general) field the selection of the SSB vacuum state breaks the symmetry: eg the 1-point function $\varphi_a = \langle\varphi\rangle$ is not GI.

In practice, one works in Landau-gauge, which has been found to give results that are usually consistent with GI-invariant lattice simulations.

Scalar-QED

$$\mathcal{L} = |D_\mu\phi|^2 + \mu^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This is like our previous example, except that now we get new gauge interactions.

$$|D_\mu\phi|^2 = [(\partial_\mu + ieA_\mu)\phi][(\partial_\mu - ieA_\mu)\phi^*] \qquad \phi \equiv \frac{1}{2}(h+\eta+i\chi)$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(\partial_\mu\chi^2) - eA_\mu(\chi\partial^\mu h - h\partial^\mu\chi) + \frac{e^2}{2}(h^2+\chi^2)A_\mu A^\mu$$

$$+ \underbrace{e^2\eta h A_\mu A^\mu}_{\text{coupling to scalar} \Rightarrow \text{tadpole}} + \underbrace{\frac{1}{2}e^2\eta^2 A_\mu^2}_{\text{gives } m_\eta = e\eta \text{ for gauge field}} + \underbrace{e\eta A_\mu(\partial^\mu\chi)}_{= -e\eta(\cancel{\partial^\mu A_\mu})\chi}$$

removes Goldstone -photon mixing.

R$_\xi$-gauge-fixing:

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\overbrace{\partial_\mu A^\mu - \xi e\eta\chi}^{G})^2 = \underbrace{-\frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{\text{photon propagator}} + e\eta\chi(\cancel{\partial^\mu A_\mu}) - \underbrace{\frac{1}{2}\xi e^2\eta^2}_{\xi\text{-dep mass for the Goldstone mode}}\chi^2$$

In this gauge the full quadratic Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}(-\mu^2 + 3\lambda\eta^2)h^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}(-\mu^2 + \overbrace{\lambda\eta^2 + \xi(e\eta)^2}^{(\lambda+\xi e^2)\eta^2})\chi^2$$

$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(e\eta)^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$

In a general gauge $m_\chi^2(v) = \xi e^2 v^2 \neq 0$, but in the Landau gauge $\xi = 0$ $m_\chi^2(v) = 0$.

General gauge propagator:

$$D^{-1}_{\mu\nu} = i\left[(q^2 - m_{A\eta}^2)g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)q_\mu q_\nu\right] \qquad ; \; m_{A\eta}^2 \equiv (e\eta)^2$$

$$\Rightarrow D_{\mu\nu} = \frac{-i}{q^2 - m_{A\eta}^2}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2 - \xi m_{A\eta}^2}\right) \xrightarrow[\substack{\text{Landau} \\ \xi = 0}]{\text{Eucl.}} \frac{1}{q^2 + m_\eta^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$

Finally, there is a Ghost: $G = \partial_\mu A^\mu - \xi e\eta\chi = 0$

$$\begin{cases} \phi \to (1 - i\alpha)\phi \\ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha \end{cases} \Rightarrow \delta\chi = -(h+\eta)\alpha \;\; ; \; \delta h = \alpha\chi \qquad \delta A_\mu = \frac{1}{e}\partial_\mu \alpha$$

$$\Rightarrow \delta G_\alpha = \frac{1}{e}\partial^\mu\partial_\mu \alpha + \xi e\eta(h+\eta)\alpha \quad \Big| \; \alpha \to e\alpha$$

$$\Rightarrow \frac{\delta G}{\delta \alpha} = \partial^2 + \xi(e\eta)^2 + \xi e^2 \eta h$$

$$\Rightarrow \mathcal{L}_{ghost} = \bar{c}\left[\partial^2 + \underbrace{\xi(e\eta)^2 + \xi e^2\eta h}\right]c$$

decouple when $\xi = 0$

e.g. we can forget ghosts in L-gauge

(we are only interested in field dependent corrections)

So, to one-loop, the only relevant new interaction is $+e^2\eta h A_\mu A^\mu$

$$\Rightarrow \frac{d\delta V^{\text{tree}}_{1\text{-loop}}}{dh} = i\Big(\quad \Big)$$

$$= 3\lambda\eta\, iA_0(m_h^2) + \lambda\eta\, iA_0(m_\chi^2) + 3e^2\eta\, iA_0(m_A^2) + (\delta_m + \delta_\lambda v^2)\eta$$

In Landau gauge:

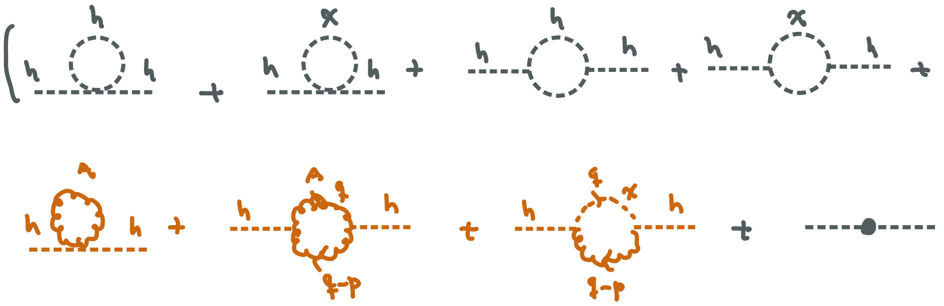
$$m_h^2 = -\mu^2 + 3\lambda\eta^2 \qquad \frac{1}{2}\frac{dm_h^2}{d\eta} = 3\lambda\eta$$

$$m_\chi^2 = -\mu^2 + \lambda\eta^2 \qquad \frac{1}{2}\frac{dm_\chi^2}{d\eta} = \lambda\eta$$

$$m_A^2 = e^2\eta^2 \qquad \frac{1}{2}\frac{dm_A^2}{d\eta} = e^2\eta$$

$$\Rightarrow \frac{\partial V}{\partial h} = \sum_i \frac{g_i}{2}\frac{\partial m_i^2}{\partial\eta} iA_0(m_i^2) + ct's$$

$$\Pi = i\Big(\quad \Big)$$



$- eA_\mu(\chi\partial^\mu h - h\partial^\mu \chi)$

$\simeq ie(p_h^\mu - p_\chi^\mu)$

$$= \overbrace{3\lambda_R iA_0(m_h^2) + \lambda_R iA_0(m_\chi^2)}^{\Pi_A} + \overbrace{18\lambda_R^2 h^2 B_0(p^2, m_h^2, m_h^2) + 2\lambda_R h^2 B_0(p^2, m_\chi^2, m_\chi^2)}^{\equiv \Pi_B(p^2)}$$

$$+ 3ie^2A_0(m_A^2) + \underbrace{2e^4\eta^2 i\int\frac{d^dq}{(2\pi)^d}\frac{2 + \frac{q\cdot(q-p)\,q\cdot(q-p)}{q^2(q-p)^2}}{(q^2-m_A^2)((q-p)^2-m_A^2)} + ie^2\int\frac{d^dq}{(2\pi)^d}\frac{(p+q)^2 - \frac{(p^2-q^2)^2}{(p-q)^2}}{(q^2-m_\chi^2)((q-p)^2-m_\chi^2)}}_{\equiv \Pi_C(p^2)}$$

$\downarrow$ $3iB_0(0, m_A^2, m_A^2)$; $p \to 0$ $\qquad$ $\downarrow$ 0 $\quad p \to 0$

$$+ \delta_m + 3h^2\delta_\lambda + p^2\delta_\phi = \Pi_A + \Pi_B(p^2) + \Pi_C(p^2)$$

$\delta\Sigma_h, \delta\Sigma_\chi$
$\delta\Sigma_\varphi$
$\delta\Sigma_A$

Here the gauge contribution $\Pi_C(p^2)$ is technically slightly more demanding to compute, but otherwise the analysis is exactly the same as with the complex scalar field.

$$\delta \bar{V}_{1\text{-loop}} = \sum_{i=1}^{2} g_i \left\{ \frac{1}{64\pi^2} \left[m_i^4(\varphi) \left(\log\left(\frac{m_i^2(\varphi)}{m_{pi}^2} \right) - \frac{3}{2} \right) + 2 m_i^2(\varphi)\, m_i^2(v) \right] + J_T^-(m_i^2(\varphi)) \right\}$$

$$- \frac{1}{8\lambda_R^2 v_R^2} m_\chi^4(\varphi) \left(\frac{\lambda_R^2 v_R^2}{8\pi^2} 12.21 + \Sigma_C \right)$$

where $m_h^2(\varphi) = -\mu^2 + 3\lambda_R \varphi^2$, $m_\chi^2(\varphi) = -\mu^2 + \lambda_R \varphi^2$, $m_A^2(\varphi) = e^2\varphi^2$; $g_h = g_\chi = 1$, $g_A = 3$ and $m_{ph} = m_{\chi h} \equiv m_p$ and $m_{pA} \equiv ev$. Finally

$$\Sigma_C = \Pi_C(m_A^2(v)) - \Pi_C(0) - m_A^2 \Pi_C'(m_A^2) = \Sigma_C(m_A^2, m_h^2) = \#$$

General expression for the correction term on the second line is

$$- \frac{1}{8\lambda_R^2 v_R^2} m_\chi^4(\varphi) \left\{ \Sigma_1 - \frac{\lambda_R^2 v_R^2}{8\pi^2} \log \frac{m_p^2}{m_\chi^2(v)} \right\}$$

where $\Sigma_1 = \Pi(m_p^2) - \Pi(0) - m_p^2 \Pi'(m_p^2)$ where Π is the full self-energy function in the model.

- we did not renormalize e. δe was not needed at this order. At higher loops this will be needed and requires renormalization of gauge sector.
- What is λ_R now? We know $\bar{\delta}_\varphi$ and $\bar{\delta}_\lambda$, so we can relate λ_R to any observable we wish. It is very close to $\Gamma^{(4)}(0,0,0)$, but not quite! ($\delta_\lambda \neq \bar{\delta}_\lambda$ and $\delta_\varphi \neq \bar{\delta}_\varphi$). (Ex.)

- I did not even bother to redo the finite-T-computation with the gauge field. The result is obvious, but let me remind

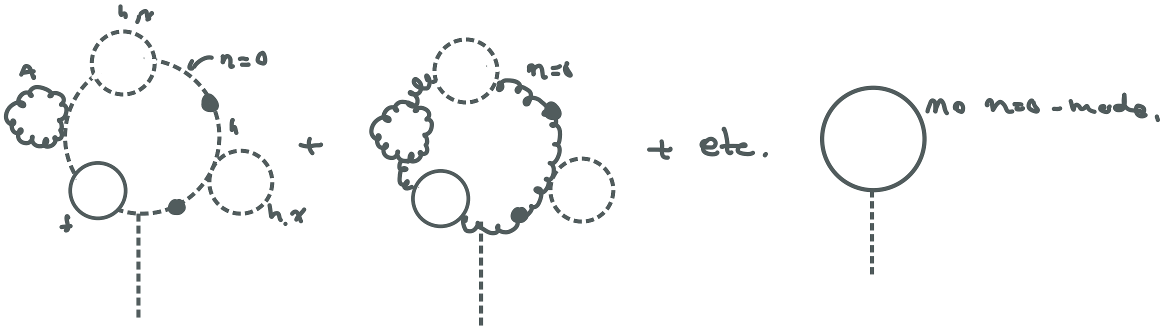
$$\rightarrow 3e^2\eta \sum\!\!\!\!\!\!\int \frac{1}{k^2+\omega_n^2} = 3e^2\eta\left(iA_0(m_A^2(\eta)) + I_T^+(m_A^2(\eta))\right)$$

etc.

Ring resummation

Just as with pressure, we encounter an IR-singularity with massless fields, if we try to go to higher orders. Fields that are particulariy sensitive, are the scalar itself, when the debye mass $m_D^2(T)$ $\left(= -\mu^2 + \frac{\lambda T^2}{24}\right.$ in the singlet model) is small, and the magnetic modes of the gauge fields.

Ring sum resums diagrams like



The sum is needed only for $n=0$-modes. For $n\neq0$-modes thermal mass removes the IR-sensitivity. Again, the resummation could be extended to all modes, to get a ring-improvement consistent with $m(\varphi,T)/T \rightarrow \infty$. (Parwani-scheme), but then, one recovers the same problems encountered earlier when evaluating the pressure.

Here we dress only the $n=0$-mode. (Carrington-Arnold-Espinosa scheme.)

This requires no new effort:

$$n=0: \quad \frac{1}{\vec{p}^2+m_i^2} \longrightarrow \frac{1}{\vec{p}^2+m_{Di}^2(T)} \quad ; \quad m_{Di}^2(T) = m_i^2(\varphi) + \overset{\text{High-T-limit}}{\Pi_{T_i}} . \quad \left(\frac{\lambda\varphi^4}{4!} : \Pi_T = \frac{\lambda}{24} T^2 \right)$$

where Π_{T_i} is the thermal correction to the self-energy of the excitation in the high-T limit. (This corrects only bosons of course.)

$$I_{n=0} = T \int_{\vec{p}} \frac{1}{p^2+m^2} = \frac{mT}{4\pi}$$

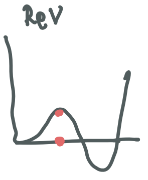
$$\Rightarrow J_{n=0} = -\frac{m^3 T}{12\pi}$$

$$\Rightarrow J_T^-(m_i^2) \longrightarrow J_T^-(m_i^2) - \underbrace{\frac{T}{12\pi}\left(m_{Di}^3(\varphi,T) - m_i^3(\varphi)\right)}_{\text{Ring-correction in CAE-scheme}}$$

So, our final result for the ring-corrected potential is

$$\boxed{\begin{aligned} \delta\bar{V}_{1\text{-loop}} = \sum_{i=1} g_i \Bigg\{ & \frac{1}{64\pi^2}\left[m_i^4(\varphi)\left(\log\left(\frac{m_i^2(\varphi)}{m_{pi}^2}\right) - \frac{3}{2}\right) + 2m_i^2(\varphi)\, m_i^2(v)\right] \\ & + J_T^-(m_i^2(\varphi)) - \frac{T}{12\pi}\left(m_{Di}^3(\varphi,T) - m_i^3(\varphi)\right) \Bigg\} \\ & - \frac{1}{8\lambda_R^2 v_R^2}\, m_\chi^4(\varphi)\left(\frac{\lambda_R^2 v_e^2}{8\pi^2}\left(3\overset{\text{\# of Goldstone modes = 1 in SQED, 3 in MSM.}}{N_g} + 9{,}21\right) + \Sigma_c\right) \end{aligned}}$$

$\Rightarrow \Delta V(T)$



Stronger transition?

where $m_h^2(\varphi) = -\mu^2 + 3\lambda_R\varphi^2$, $m_\chi^2(\varphi) = -\mu^2 + \lambda_R\varphi^2$, $m_A^2(\varphi) = e^2\varphi^2$; $g_h = g_\chi = 1$, $g_A = 3$ and $m_{ph} = m_{\chi h} \equiv m_p$ and $m_{pA} \equiv ev$. Finally, one can include also any number of fermions with $m_f^2(\varphi) = \frac{1}{2}y_f^2\varphi^2$ and $g_f = -4$. In the end

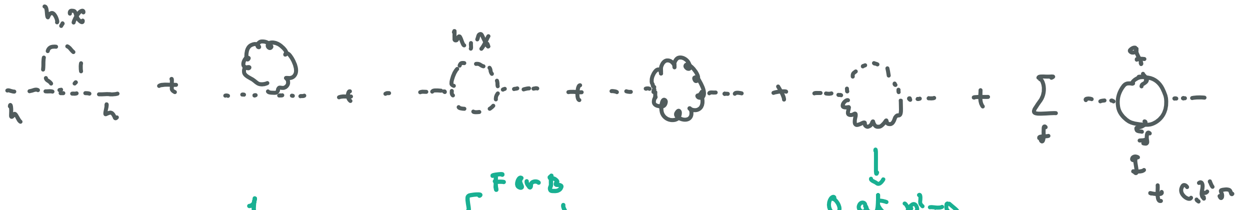
$$\boxed{\Sigma_c = \sum_{i \neq h,\chi}\left(\Pi_i(M_p^2) - \Pi_i(0) - M_p^2\,\Pi_i'(M_p^2)\right) = \sum_{i \neq h,\chi} \Sigma_i}$$

write these down explicitly

This result directly extends to the SM-case. The only difference to the scalar electrodynamics case is the particle-content. Also, in SM there are 3 Goldstone-modes. Otherwise, one just extends the sum over i to include all fields in the SM.

Debye masses required for the SQED:

$-\Pi_h \propto$



Landau gauge: $\left(\Delta_i \equiv \frac{1}{p^2+m_i^2} \; ; \; p^2 = \omega_n^{\overset{F \text{ or } B}{A}} + \vec{p}^{\,2} \right)$

$$\Rightarrow \Pi_h = 3\lambda \underbrace{\sum\!\!\!\!\!\!\int \Delta_h}_{I_0 + I_T^- \simeq I_0 + \frac{T^2}{12}.} + \lambda \sum\!\!\!\!\!\!\int \Delta_\chi + 3e^2 \sum\!\!\!\!\!\!\int \Delta_A + \lambda^2\eta^2 \left(18 \sum\!\!\!\!\!\!\int \Delta_h^2 + 2 \sum\!\!\!\!\!\!\int \Delta_\chi^2 \right)$$

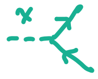
$$+ 6e^4\eta^2 \sum\!\!\!\!\!\!\int \Delta_A^2 + 2\sum_f y_f^2 \underbrace{\sum\!\!\!\!\!\!\int_F \Delta_F}_{I_0 + I_T^+ \simeq I_0 + \frac{T^2}{24}}$$

$$= \Pi^h_{\text{vac}} + \left(4\lambda + 3e^2 + \sum_f y_f^2\right)\frac{T^2}{12} + \mathcal{O}(\lambda^2\eta^2, e^4\eta^2, y_f^4\eta^2)$$

$$\Rightarrow m^2_{D,h}(T) = -\mu^2 + 3\lambda\varphi^2 + \left(4\lambda + 3e^2 + \sum_f y_f^2\right)\frac{T^2}{12} = -\mu^2 + 3\lambda\varphi^2 + c_h T^2$$

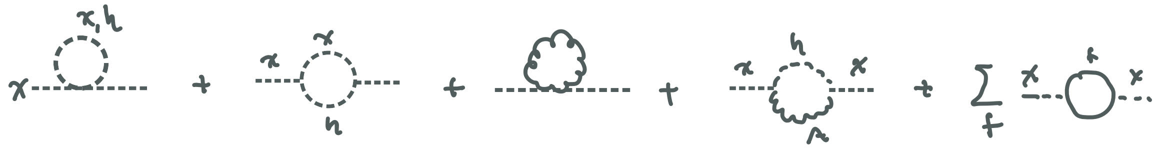
Similarly [Note that $\mathcal{L}_{\text{Yukawa}} = y_f \bar{\Psi}_L \phi \Psi_R + y_f \bar{\Psi}_R \phi^* \Psi_L$

$= \frac{y_f}{\sqrt{2}}(h+\eta)\bar{\Psi}\Psi + \frac{i y_f}{\sqrt{2}} \chi \bar{\Psi}\gamma^5\Psi. \Rightarrow$



$= -\frac{y_f}{\sqrt{2}}\gamma_5$]

$-\Pi_\chi =$

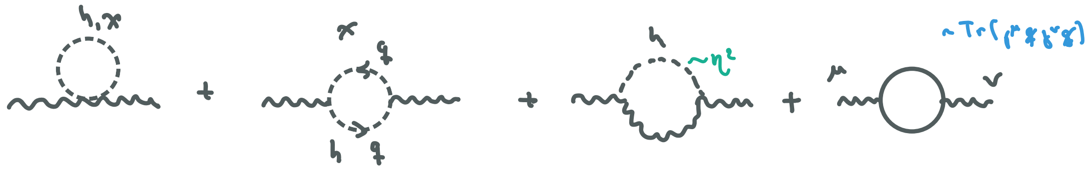


$$\Rightarrow m^2_{D,\chi}(T) = -\mu^2 + \lambda\varphi^2 + c_\chi T^2, \text{ with } \quad c_\chi = 4\lambda + 3e^2 + \sum_f y_f^2 = c_\phi$$

Gauge bosons

$$-eA_\mu(\chi\partial^\mu h - h\partial^\mu\chi) + \frac{e^2}{2}(h^2+\chi^2)A_\mu A^\mu$$
$$+ e^2\eta h A_\mu A^\mu + \frac{1}{2}e^2\eta^2 A_\mu^2 + e\eta A_\mu(\partial^\mu\chi)$$

$-\Pi^A_{\mu\nu} =$



$$\Rightarrow \quad \Pi^A_{\mu\nu} \approx \underbrace{2e^2\delta^{\mu\nu}\sum\!\!\!\!\!\!\!\int_B \frac{1}{q^2} - 4e^2\sum\!\!\!\!\!\!\!\int_B \frac{q^\mu q^\nu}{q^4}}_{= 2e^2\sum\!\!\!\!\!\!\int_B \frac{q^2 - 2q^\mu q^\nu}{q^4}} - 4\sum_f e^2 \sum\!\!\!\!\!\!\!\int_F \frac{2q^\mu q^\nu - q^2\delta^{\mu\nu}}{q^4} + \mathcal{O}(m^2, p^2)$$

- $$\sum\!\!\!\!\!\!\!\int_\pm \frac{q^\mu q^\nu}{q^4} = (\delta_{\mu i}\delta_{\nu i} - \delta_{\mu 0}\delta_{\nu 0})\frac{1}{2}I_T^\pm(0) \quad \Rightarrow \quad \sum\!\!\!\!\!\!\!\int_\pm \frac{2q^\mu q^\nu - q^2\delta^{\mu\nu}}{q^4} = -2\,\delta_{\mu 0}\delta_{\nu 0}\, I_T^\pm(0)$$

$$\Rightarrow \quad \Pi^A_{\mu\nu} = e^2(1+N_f)\frac{T^2}{3}\delta_{\mu 0}\delta_{\nu 0} \equiv C_{A_L} T^2 \delta_{\mu 0}\delta_{\nu 0}$$

$$m_D^{A_L}(\varphi, T) = e^2\varphi^2 + C_{A_L}T^2 \quad ; \quad m_D^{A_T}(\varphi, T) = e^2\varphi^2$$

Todo: give general expressions for Σ_i for $i = \phi, A_\mu, A^a_\mu$

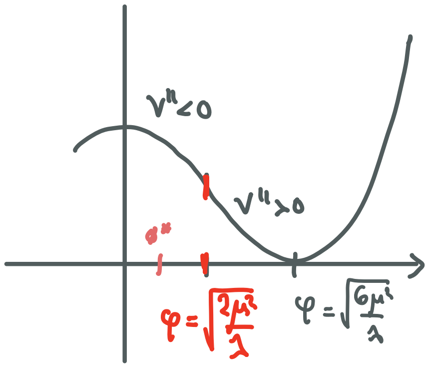
Complex V_{eff}

Effective action can be complex. Indeed, the vacuum-contribution to V contains a part

put back the Feynman $i\epsilon$ $\quad \frac{1}{p^2-m^2+i\epsilon}$

$$\sum_i \frac{m_i^4(\varphi)}{64\pi^2}\log\left(\frac{m_i^2(\varphi)-i\epsilon}{m_{\rho i}^2}\right) = \sum_i \frac{m_i^4(\varphi)}{64\pi^2}\left\{\log\frac{|m_i^2(\varphi)|}{m_{\rho i}^2} - i\pi\,\Theta(-m_i^2(\varphi))\right\}$$

$$\Rightarrow \quad \mathrm{Im}\,V_{eff}^{(vac)} = -\sum_{i=h,\chi}\frac{m_i^4(\varphi)}{64\pi^3}\,\Theta(-m_i^2(\varphi))$$



In the simple $\lambda\varphi^4$-theory with SSB this comes about when

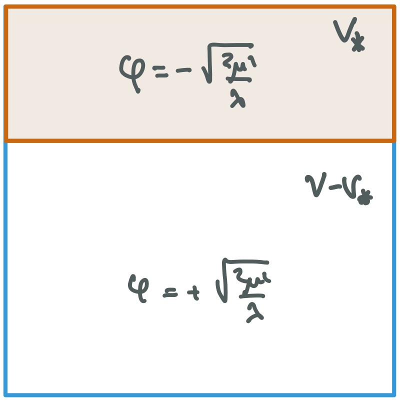
$$V = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4$$

$$m^2(\varphi) = V''(\varphi) = -\mu^2 + \frac{1}{2}\lambda\varphi^2 < 0 \Leftrightarrow |\varphi| < \frac{2\mu^2}{\lambda}.$$

$\phi^* + \delta\varphi_{k<\sqrt{|m^2(\varphi)|}}$

$\langle\phi\rangle = \varphi^*$

In this region, the potential is concave. Negative mass means that the modes with $m^2(\varphi)+k^2<0$ eg $k^2 < -m^2(\varphi)$ are tachyonic (unstable around ϕ = const), and start to grow exponentially, if excited. In the concave region the minimum of V_{eff} is unstable.



$\phi = \varphi_* = \langle\phi\rangle$

$\downarrow$

$\langle\phi\rangle = \varphi_*$

Indeed, consider a very large box. If we only require that $\langle\phi\rangle = \varphi_* < \sqrt{\frac{2\mu^2}{\lambda}}$, we can arrange an inhomogeneous configuration, when in a fraction of the volume

$$\frac{V_*}{V} = 1 - \sqrt{\frac{\lambda}{2\mu^2}}\,\varphi^*$$

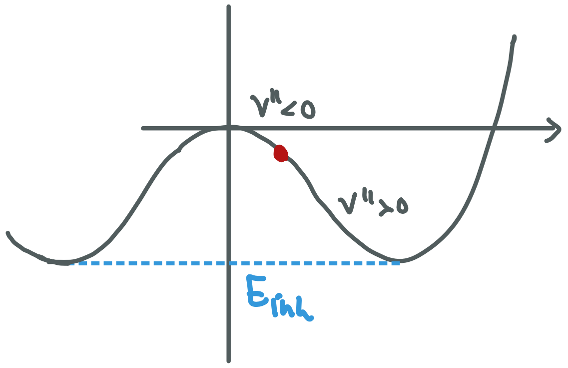
$$\varphi = -\sqrt{\frac{2\mu^2}{\lambda}} \text{ and elsewhere } \varphi = \sqrt{\frac{2\mu^2}{\lambda}} \Rightarrow \langle\phi\rangle = -\frac{V_*}{V}\sqrt{\frac{2\mu^2}{\lambda}} + \frac{V-V_*}{V}\sqrt{\frac{2\mu^2}{\lambda}} \equiv \varphi_*$$

At the infinite volume limit the energy at boundary vanishes

$$\frac{E_{bnd}}{E_{tot}} \propto V^{-1/3} \to 0$$

And we conclude that $E(\varphi^*)_{inh} = 0$.

This is the famous Gibbs construction.



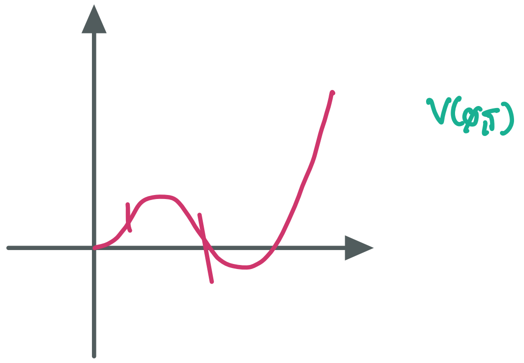
The true minimum of the system with $\langle\varphi\rangle = \varphi^*$ between the two minima is inhomogeneous. If prepared to a state in the concave region, system decays to unstable modes. In cond. matter this is called spinodal instability (spinodal decomposition) and in cosmology tachyonic instability. It can happen eg. at the end of inflation in some models.

- Another complex part can arise from the high-T-expansion term. At high T the vacuum term is canceled, but now instability comes from the cubic term

$$-\frac{m^3(\varphi,T)\,T}{12\pi} \longrightarrow -\frac{(m^2(\varphi,T)-i\epsilon)^{3/2}}{12\pi} \longrightarrow -\overbrace{e^{-\frac{3}{2}i\pi}}^{-i}\,\frac{|m^2(\varphi,T)|^2}{12\pi}$$

$$\Rightarrow \mathrm{Im}\,V_{eff}(\varphi,T)_{HT} = -\frac{|m^2(\varphi,T)|^2}{12\pi}$$

Again we see that the complex part corresponds to the concave area of the V_{eff}.



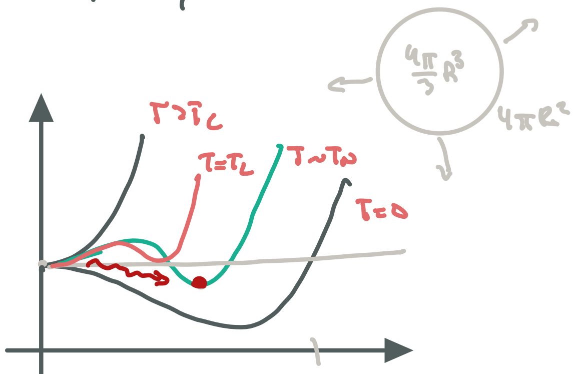
These complex parts are relevant for the dynamics of a transition, where the field may evolve strongly as a function of time, eg. at the end of inflation.

⇒ tachyonic particle production

But neither of these complex parts are relevant for the onset & evolution of phase transitions.

Phase transition (1st order)

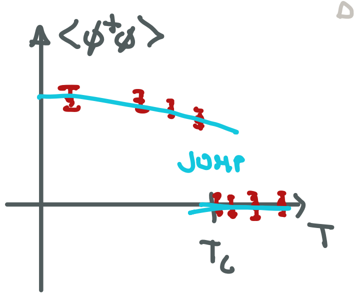
This is a very complicated topic, with several distinct parts, each of which requires different theoretical machinery.



1] Order of transition?

2] Thermodynamics of the transition

- × latent heat, σ, ..
- ● V_{eff}
- ● lattice



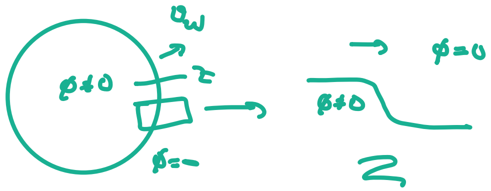
$\Delta E = -\frac{4\pi}{3}R^3\Delta V$
$+ 4\pi R^2\sigma$

$R_c \propto \frac{\sigma}{\Delta V}$

$R_c \searrow T<T_c$

3] Bubble nucleation rate. $\Gamma(T)$

- × Amount of supercooling
- × how does transition complete?



$m_f \propto y_f\phi$

$\rightarrow \delta f \neq 0$

4] Dynamics of transition walls

↳ plasma distortions

- × Boltzmann equations coupling ϕ & $\delta f_i(p,t)$

$\Rightarrow v_w, \underline{\phi(z)}$ (CP-even)

$\phi(t,z)$
$\delta f(t,z)$

5] CP-violating perturbations δf_ϕ $\Rightarrow$ BAU (model, parameters)

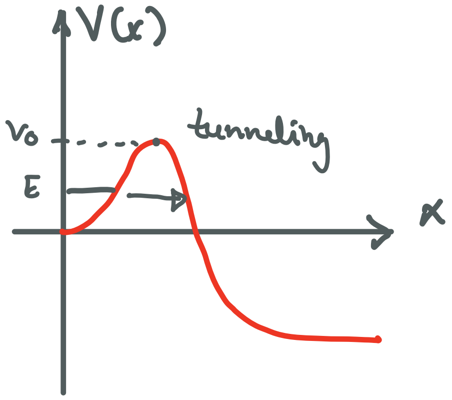
6] GW-signal (model parameters)

Bubble nucleation

This is a complicated problem, because it is nonperturbative. We can not go through the full argument here, but we go through the main points.

First: Consider

$$|\psi(t)\rangle = e^{-iEt}|0\rangle = e^{-i[\mathrm{Re}(E)+i\,\mathrm{Im}(E)]t}|0\rangle$$

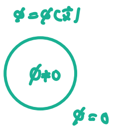


$$\Rightarrow \langle\psi(t)|\psi(t)\rangle = e^{2\mathrm{Im}(E)t}\langle\psi(0)|\psi(0)\rangle$$

$$\equiv e^{-\Gamma(E)t}|\psi(0)|^2 \longrightarrow \text{exponential decay law}$$

$$\Rightarrow \underline{\Gamma(E) = -2\,\mathrm{Im}(E)} \qquad E = E - \frac{i}{2}\Gamma$$

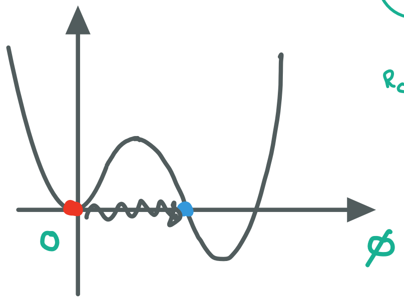
In Field theory one would then expect that



$R_c \sim \frac{\sigma}{\Delta v}$

$$\Gamma \simeq -2\,\mathrm{Im}(F) \qquad (?)$$

↑ Free energy



It turns out that

$$\boxed{\begin{aligned}\Gamma &= -2\mathrm{Im}F \\ \Gamma &= -\frac{\omega}{2\pi T}2\mathrm{Im}F\end{aligned}}$$

when quantum tunneling dominates (Callan & Coleman)

when thermal activation dominates (Langer)

to prove this we should compute both sides of the equation: Γ and F.

Here F is the free energy of the system evaluated by "analytic continuation" of the saddle point contribution to F.

$$F = -T\log\left\{\int \mathcal{D}\phi_{\substack{\phi_i=0\\ \phi_e=0}}\, e^{-S_E(\phi)}\right\} \simeq -T\log\left\{\overset{\substack{\text{fluctuations around } \phi=0\\ \downarrow}}{Z_0} + \underset{\substack{\uparrow\\ \text{what is this?}}}{\mathcal{Z}}\right\}$$

↑ we are interested in instability of configuration φ=0 where $\bar{Z}$ is the contribution from the nontrivial "saddle point". Here $Z_0 \gg \bar{Z}$ and $Z_0 \in \mathbb{R}$, so that

saddle point configuration

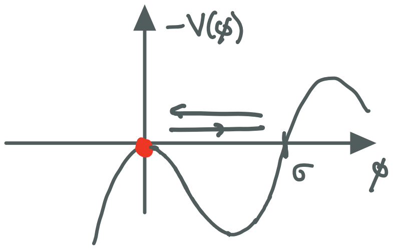
$$\boxed{\operatorname{Im} F \simeq -\frac{T}{Z_0} \operatorname{Im} \bar{Z} \approx T \operatorname{Im} \left\{ e^{-S_E[\overset{\downarrow}{\bar{\phi}}]} \left(\frac{\det \left(\frac{\delta^2 S_E}{\delta \phi^2} \right)_{\phi=\bar{\phi}}}{\det \left(\frac{\delta^2 S_E}{\delta \phi^2} \right)_{\phi=0}} \right)^{-1/2} \right\}}$$

Obvious questions arise:

- what is the "saddle point"?
- Where does the imaginary part come from?
- How to compute the determinants?

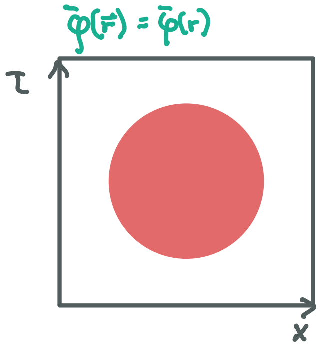
The bouce The saddle point configuration $\bar{\phi}(x,t)$ is a nontrivial solution to the classical equation of motion:

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\bar{\phi}} = -\partial_\mu^2 \bar{\phi} + V(\bar{\phi}) = 0 \,. \quad (1)$$

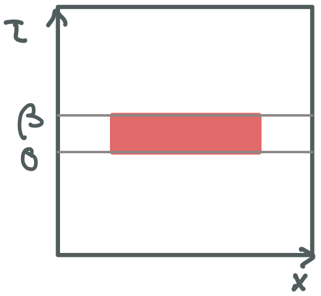


with the boundary condition $\phi(-T) = \phi(+T) = 0$ as $T \to \infty$. Bounce is an example of an instanton. Note that (1) describes motion in a potential $-V$.

?



If $T \ll V''$, then Euclidean time-direction is similar to spatial ones and one expects bounce to have $O(4)$-symmetry. (Vacuum tunneling case)



If $T \gg V''$, then bounce configuration is very large in units $1/T$, the τ-direction gets squeezed and the bounce becomes $O(3)$-symmetric. (thermal activation)

$$S_E = \int d\tau \int d^3x\, \mathcal{L} \rightarrow \beta \int d^3x\, \mathcal{L} = \beta S_{3d}$$

When $V'' \sim T$ the situation is more complicated and both effects are relevant. This is a very narrow region however, and usually one is interested in the case $T \gg V''$. We shall always assume that the bounce is $O(d)$-symmetric.

Because bounce is an extremum, $S[\bar{\phi}]$ must be invariant in particular in the infinitesimal scale transform $x^\mu \to \lambda x^\mu$. To this end define $\phi_\lambda(x) \equiv \bar{\phi}(x/\lambda)$, and

$$S[\phi_\lambda] = \lambda^{d-2}\frac{1}{2}\int d^dx\,(\partial_\mu\phi_\lambda)^2 + \lambda^d\int d^dx\, V \equiv \lambda^{d-2}\langle T\rangle + \lambda^d\langle V\rangle$$

$$\Rightarrow \quad 0 = \frac{\partial S[\phi_\lambda]}{\partial\lambda}\bigg|_{\lambda=1} = (d-2)\langle T\rangle + d\langle V\rangle \quad\Rightarrow\quad \langle V\rangle = \left(\frac{2}{d}-1\right)\langle T\rangle$$

$$\Rightarrow \quad \bullet \quad \bar{S} = \langle T\rangle + \langle V\rangle = \frac{2}{d}\langle T\rangle = \frac{1}{d}\int d^4x\,(\partial\bar{\phi})^2 > 0$$

(Good. probability makes sense.)

$$\bullet \quad \frac{\partial^2 S}{\partial\lambda^2}\bigg|_{\lambda=1} = (d-2)(d-3)\langle T\rangle + d(d-1)\langle V\rangle$$

$$= (d-2)(d-3+1-d)\langle T\rangle = (2-d)\int d^4x\,(\partial\phi)^2 < 0 \quad \text{for } d>2.$$

Because $\bar{S}>0$ $\exp(-\bar{S})$ is small. On the other hand $\frac{\partial^2 S}{\partial\lambda^2}<0$ means that the bounce is not a stable minimum configuration. There must be at least one negative eigenvalue $\bar{\lambda}$ around the bounce. $\Rightarrow$ $\left[\det\left(\frac{\delta^2 S}{\delta\phi^2}\right)\right]^{-1/2}$ becomes complex.

1) At a stable $\overbrace{\text{fixed point}}^{(\phi=0)}$ all eigenmodes of $\frac{\delta^2 S_E}{\delta\phi^2}$ are strictly positive.

$$\Rightarrow \left[\det\left|\frac{\delta^2 S_E}{\delta\phi^2}\right|_{\phi=0}\right]^{-1/2} = \int\mathcal{D}\phi\,\exp\left\{-\int_{x_E}\frac{1}{2}\phi_E\frac{\delta^2 S_E}{\delta\phi^2}\phi_E\right\}$$

is well defined.

Moving to eigen-basis: $\frac{\delta^2 S_E}{\delta\phi^2}\Big|_0 \delta\phi_n = \lambda_{0,n}^2 \delta\phi_n$

and: $\delta\phi_n \equiv c_n f_n$, such that $\int_{x_E} f_n f_m = \delta_{mm}$

One can write

$$\left[\det\left(\frac{\delta^2 S_E}{\delta\phi^2}\right)_{\phi=0}\right]^{-1/2} = N \int_{-\infty}^{\infty} \prod \frac{dc_n}{\sqrt{2\pi}} \exp\left\{-\sum \frac{1}{2}\lambda_n^{(0)} c_n^2\right\} = \prod_n \sqrt{\frac{1}{\lambda_n^{(0)}}}.$$

2) Around the saddle point configuration we can do the same:

$$\frac{\delta^2 S_E}{\delta\phi^2}\Big|_{\bar\phi} \delta\bar\phi_n \equiv \bar\lambda_n \delta\bar\phi_n ; \quad \delta\bar\phi_n \equiv \sum c_n \bar f_n , \quad \text{with } \int_x \bar f_n \bar f_m = \delta_{nm}$$

However, the Gaussian integration does not work always, because:

- there is one negative mode $\frac{\delta^2 S_E}{\delta\phi^2}\Big|_{\bar\phi} \delta\bar\phi_1 = -\bar\lambda_1 \delta\phi_1$
- there are a number of zero modes: $\frac{\delta^2 S_E}{\delta\phi^2}\Big|_{\bar\phi} \delta\bar\phi_i = 0$

Zero modes

Zero-modes correspond to arbitrary placing of the SP-configuration $\bar\phi$ in the space, and so they correspond to translations: $\delta\varphi_n \propto \partial_\mu\varphi$. Indeed: $a^\mu\partial_\mu\bar\phi \approx \bar\phi(x+a)-\bar\phi(x)$.

Also

$$S_E[\phi] = \int_x \left[\frac{1}{2}(\partial_\mu\phi)^2 + V(\phi)\right].$$

$$\frac{\delta S_E}{\delta\phi} = 0 \Rightarrow -\partial_\mu^2\bar\phi + V'(\bar\phi) = 0 \Rightarrow \underbrace{\left(-\partial_\mu^2 + V''(\bar\phi)\right)}_{\delta^2 S/\delta\bar\phi^2} \partial_\alpha\bar\phi = 0.$$

zero modes!

- The zero-modes do not exist (make no sense) for the homogeneous $\phi = 0$- configuration.
- Gaussian integration over zero modes would be a disaster. To define them properly one must go to finite volume.

First, from $\int d^dx\, \overset{\sqrt{(\nabla\phi)^2}}{(\partial_\mu\bar\phi)^2} = d\,\bar S_E$, and the $O(d)$-symmetry of $\bar\phi$, we get $\int d^dx\,(\partial_{x_n}\bar\phi)^2 = \bar S_E$ for each individual translation.

Then, setting $\partial_{x_n}\bar\phi = \alpha f_{0n}$ and requiring $\int d^4x\, f_{0n} f_{0m} = \delta_{nm}$, we get

$$1 = \frac{1}{\alpha^2}\int d^4x\,(\partial_{x_n}\varphi)^2 = \frac{1}{\alpha^2}\bar S_E \Rightarrow f_{0n} = (\bar S_E)^{-1/2}\partial_{x_n}\bar\phi$$

Now $\lim_{k\to\infty}\left(1+\frac{C_{0n} f_{0n}}{k}\right)^k\bar\phi = e^{C_{0n}\bar S_E^{-1/2}\partial_x}\bar\phi = \bar\phi(x + C_{0n}\bar S_E^{-1/2})$, so we should restrict $C_{0n}\bar S_E^{-1/2}$ to range $[0, L]$. $\Rightarrow C_{0n} \in [0, L\bar S_E^{1/2}]$.

$$\Rightarrow \prod_n^d \int \frac{dC_{n0}}{\sqrt{2\pi}} = \left(\frac{\bar S_E}{2\pi}\right)^{d/2} V_d$$

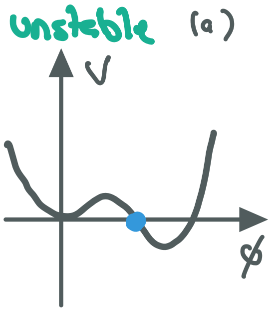
Thus zero-modes guarantee that $\text{Im}F \propto V$.

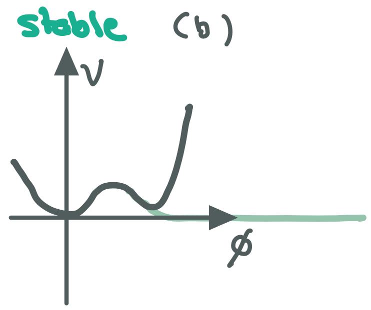
The negative mode

{hard to prove

The result is quite expected. As one mode becomes negative $\pi\sqrt{\frac{1}{\lambda_i}}$ becomes complex. However, there must be just one negative mode, and there is an additional factor $1/2$, which comes from the analytic continuation. These are sticky issues. Following Callan & Coleman 1977, consider a particular path in the configuration space, labelled by c_1, which passes through the saddle point along the unstable direction. This contributes a term J to partition function:

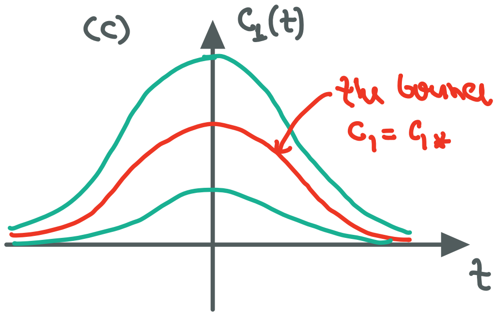
$$J = \int_{c_{10}}^{\infty} \frac{dc_1}{\sqrt{2\pi}} e^{-S[c_1]}$$

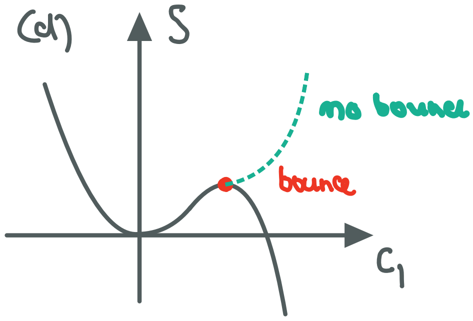




to Z. The path is chosen such that

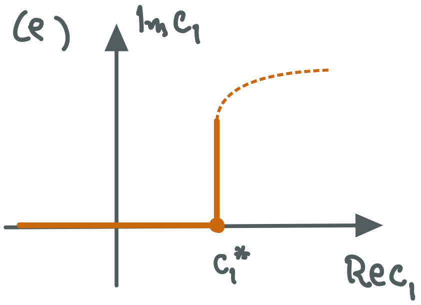
- $\phi \equiv 0$ occurs at, say $c_{10} = 0$,
- $\bar{\phi}$, the bounce, occurs at $c_{10} = c_{10}^*$
- ϕ_1, the negative mode, is tangent to the path at $c_{10} = c_{10}^*$.





⇒

- $\phi_0 = 0$ is the global minimum of S
- $\bar{\phi}$ is the local maximum of S, along the path



For $c_{10} > c_{10}^*$ $S \to -\infty$ because V is negative there.

"starting from" the stable situation, one deforms the path, as shown in fig. e, the integral along c_{10} remains finite for $\lambda_1 < 0$, but J picks a complex part.

$$\mathrm{Im}\, J = \int_{c_1^*}^{c_1^* + i\infty} \frac{dc_1}{\sqrt{2\pi}} e^{-S[c_{1*}] - \frac{1}{2} S_1''[c_1^*](c_1 - c_1^*)^2 + \ldots}$$

$$\simeq e^{-S[\bar{\phi}]} \int_{\substack{0 \\ \cancel{-\infty}}}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-\frac{1}{2}|S''[\bar{\phi}]| y^2} = \frac{1}{2} |S''[\bar{\phi}]|^{-1/2} e^{-S[\bar{\phi}]} = \frac{1}{2} \sqrt{\frac{1}{|\lambda_1|}} e^{-S[\bar{\phi}]}$$

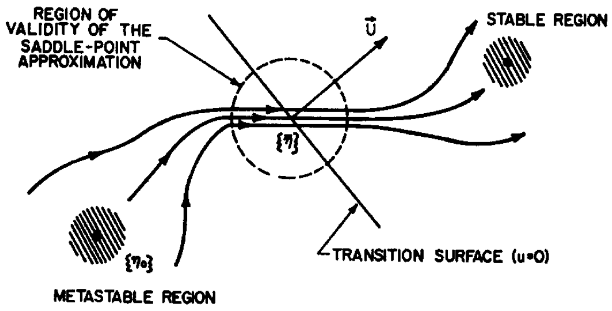
So the negative mode gives $\frac{1}{2}$ of the "full fluctuation correction", because the Gaussian integral passes over only "half" of the saddle point path.

We should still relate ImF to the decay rates. This part is even lengthier (albeit more accurately developed than finding ImF), and we only go through some simple examples. More generally:

$T=0$: (or $T \ll V''$) case. Callan & Coleman nicely show that summing over ∞ number of bounces, leads to a correction to the ground state energy $\delta E = F \Rightarrow -2\text{Im}F$ gives the decay rate as suggested initially. [Callan & Coleman Phys. Rev. D16 (1977) 1762]

$$\Gamma = -2\,\text{Im}\,F$$

$T \gg V''$ case. Langer [J.S. Langer, Annals of Physics 54 (1967) 258] gives a general proof that the decay rate of a metastable state via thermal activation is dominated by the flow across the saddle point configuration, where the rate can again be related to ImF, this time by:



$$\Gamma = \frac{|\lambda_1|}{2\pi T}\, 2\,\text{Im}\,F \qquad \frac{\lambda_i}{2i\pi}$$

↑ first Matsubara frequency

(Of course F in the $T \ll V''$ and $T \gg V''$ cases are very different.). Langers general proof beautiful but lengthy.

At any rate we have the result for the nucleation rate at finite T ($\lambda_1 < 2\pi T$):

' means exclude zero-modes (but not negative mode)

$$\boxed{\frac{\Gamma}{V} = \frac{\lambda_-}{2\pi}\left|\frac{\det'(\partial_\mu^2 - V''(\bar\phi))}{\det(\partial_\mu^2 - V''(0))}\right|^{-1/2}\left(\frac{S_{3d}}{2\pi T}\right)^{3/2} e^{-S_{3d}(\bar\phi)/T}}$$

where in the end we used $S_E = \beta S_{3d}$. Usually at the time of nucleation $S_{3d}/T \gg 1$, so one usually drops to $\mathcal{O}(1)$-terms $\frac{\lambda_-}{2\pi T}$ and $\left|\frac{\det'}{\det}\right|^{1/2}$, and solves Γ/V from

$$\boxed{\frac{\Gamma}{V} \simeq T^4\left(\frac{S_{3d}}{2\pi T}\right)^{3/2} e^{-S_{3d}(\bar\phi)/T}}$$

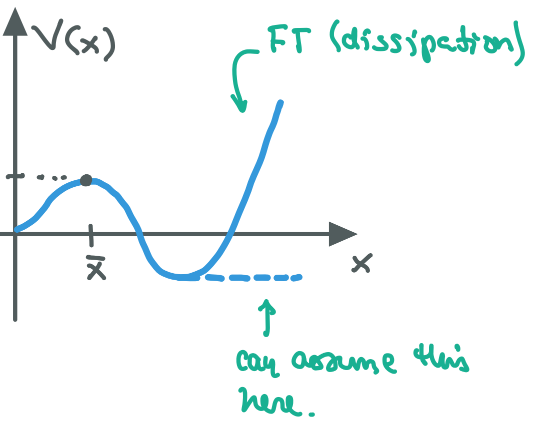
We set $\frac{\lambda_-}{2\pi} \sim T$ and also approximate dimensionally $\left|\frac{\det'(\,)}{\det(\,)}\right|^{-1/2} \sim T^3$

This may be off by a factor $\mathcal{O}(10^{\pm 1})$.

Simple 1D-example (Affleck)

Consider the 1d example of particle in the potential at right.

Classical flux over the barrier is



must pass the barrier

only right movers

flux

$$\Gamma_{class} = \frac{\int \frac{dx\,dp}{2\pi} \exp\left(-\beta\left(\frac{1}{2}p^2 + V(x)\right)\right)\delta(x-\bar x)\,\theta(p)\,p}{\int \frac{dx\,dp}{2\pi} \exp\left(-\beta\left(\frac{1}{2}p^2 + V\right)\right)}$$

- denominator is dominated by $x \simeq 0$-fluctuations, where $V \approx \frac{1}{2}V''(0)x^2 \equiv \frac{1}{2}\omega_0^2 x^2$

  It gives classically:

$$Z_0 \simeq \frac{1}{2\pi}\int dp\, e^{-\frac{1}{2}\beta p^2}\int dx\, e^{-\frac{1}{2}\beta\omega_0^2 x^2} = \frac{1}{2\pi}\sqrt{\frac{2\pi}{\beta}}\sqrt{\frac{2\pi}{\beta\omega_0^2}} = \frac{1}{\beta\omega_0}$$

- Nominator has been designed to evaluate the flux over the maximum at $x=\bar{x}$

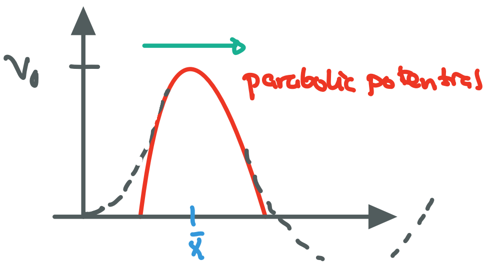
$$\simeq e^{-\beta V_0}\frac{1}{2\pi}\int_0^\infty dp\, p\, e^{-\frac{\beta}{2}p^2} = e^{-\beta V_0}\frac{1}{2\pi}\left(-\frac{1}{\beta}\right)\int_0^\infty dp\,\partial_p e^{-\frac{\beta}{2}p^2} = \frac{e^{-\beta V_0}}{2\pi\beta}$$

$$\Rightarrow \underline{\Gamma_{class} = \frac{\omega_0}{2\pi}e^{-\beta V_0}}$$

Quantum mechanically: The flux is given by

transmission coefficient

$$\Gamma = \frac{\mathrm{Tr}\,\hat{\rho}\,|t|^2}{\mathrm{Tr}\,\hat{\rho}} \simeq \frac{1}{Z_0}\int\frac{dE}{2\pi}|t(E)|^2 e^{-\beta E}$$



Consider the case $\frac{\beta\omega}{2\pi}<1$, where $\omega^2 \equiv -V''(\bar{x})$

In this case the flux is dominated by thermal activation by modes with $E \gtrsim V_0$. In this region (solving QM. Schrödinger equation. Not simple: eg. Landau & Lifshitz)

$$V(x) = V_0 + \frac{1}{2}V''(\bar{x})(x-\bar{x})^2$$
$$\equiv V_0 - \frac{1}{2}\omega^2(x-\bar{x})^2.$$

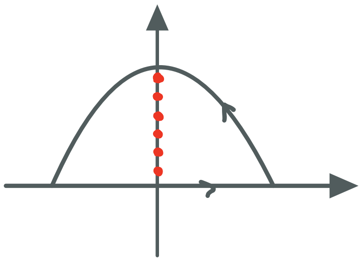
$$|t|^2 = \left(1+e^{-2\pi\frac{E-V_0}{\omega}}\right)^{-1}.$$

Then $$\int\frac{dE}{2\pi}|t(E)|^2 e^{-\beta E} = \frac{1}{2\pi}e^{-\beta V_0}\int_{-\infty}^{\infty}d\epsilon\,\frac{e^{-\beta\epsilon}}{1+e^{-\frac{2\pi}{\omega}\epsilon}}$$

$$= \frac{1}{2\pi}e^{-\beta V_0}\frac{\omega}{2\pi}\int_{-\infty}^{\infty}dx\,\frac{e^{-\left(\frac{\beta\omega}{2\pi}\right)x}}{1+e^{-x}}$$

$$= \frac{1}{2\pi}e^{-\beta V_0}\frac{\omega}{2\pi}2\pi i\sum_{n=0}^{\infty}e^{-i\left(\frac{\beta\omega}{2\pi}\right)(2n+1)\pi}$$

; $x \equiv \frac{2\pi}{\omega}\epsilon$



arc integral → 0 when $\frac{\beta\omega}{2\pi}<1$

$$= \frac{1}{4\pi} e^{-\beta V_0} \, 2i\omega \, e^{-i\beta\frac{\omega}{2}} \sum_{n=0}^{\infty} \left(e^{-i\beta\omega}\right)^n$$

$$= \frac{1}{4\pi} e^{-\beta V_0} \omega \, \frac{2i e^{-i\beta\frac{\omega}{2}}}{1-e^{-i\beta\omega}} = \frac{\omega}{2\pi} \frac{1}{2\sin(\frac{\beta\omega}{2})}$$

The denominator is familiar to us: (now computed as quantum)

$$Z_0 = \sum_n e^{-(n+\frac{1}{2})\beta\omega_0} = \frac{1}{2\sinh\frac{1}{2}\beta\omega_0} = \left[\det\left(\partial_t^2+\omega_0^2\right)\right]^{-1/2}$$

Thus

$$\Gamma = \frac{\omega}{2\pi} \frac{\sinh(\frac{\beta\omega_0}{2})}{\sin(\frac{\beta\omega}{2})} e^{-\beta V_0} \xrightarrow{\beta\omega\ll 1} \frac{\omega_0}{2\pi} e^{-\beta V_0}$$

We observe that

$$\frac{1}{2\sin(\frac{\beta\omega}{2})} = \frac{1}{2\sinh(i\frac{\beta\omega}{2})} = \left[\det\left(\partial_t^2+(i\omega)^2\right)\right]^{-1/2} = \left[\det\left(\partial_t^2-\omega^2\right)\right]^{-1/2}$$

analytically continued to $\omega \to i\omega$

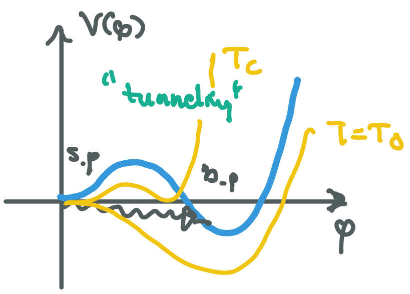
So we do have

$$\Gamma = \frac{\omega}{2\pi} \left| \frac{\det\left(\partial_t^2+V''(\bar{x})\right)}{\det\left(\partial_t^2+V''(0)\right)} \right|^{-1/2} e^{\overset{\overset{\approx -\beta S_{3d}}{\downarrow}}{-\beta V_0}} = \frac{\omega}{2\pi}\left(2\,\mathrm{Im}\,F\right)$$

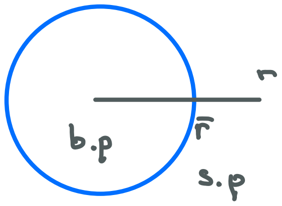
Here we had no zero modes, but the negative mode behaved as expected. Also, note that here the ratio of the fluctuation determinants really is small, assuming $\frac{\beta\omega}{2} \lesssim \mathcal{O}(1)$ (calculation assumed it was $\lesssim \pi$). This is really the case in almost all relevant applications.

Bubble nucleation in 3d

We argue that by symmetry the bounce solution in 3d Euclidian case ($T \gg \omega$) is $O(3)$-symmetric bubble. The classical EoM, corresponding to 3d-action

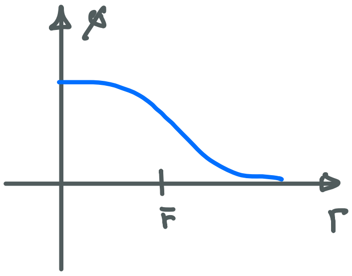


$$S_{3d} = \beta \int d^3r \left(\frac{1}{2}(\nabla\varphi)^2 + V(\varphi,T)\right)$$



For such configuration, $\bar{\varphi} = \bar{\varphi}(r)$ is

$$\frac{d^2\bar{\varphi}}{dr^2} + \frac{2}{r}\frac{d\bar{\varphi}}{dr} = V'(\bar{\varphi},T)$$



with $\bar{\varphi}(\infty) = 0$ and $\left.\frac{d\bar{\varphi}}{dr}\right|_{r=0} = 0$. Such equation can always be solved numerically, when the potential $V'(\varphi,T)$ is defined. Let us pause to do that:

- A good analytic model potential is

$$m^2(\varphi,T) = -\mu^2 + \gamma T^2 \quad \text{eg} \quad T_0^2 \equiv \frac{\mu^2}{\gamma}$$

$$V(\varphi,T) = \frac{1}{2}\gamma(T^2 - T_0^2)\varphi^2 - \frac{1}{3}\delta T\varphi^3 + \frac{\lambda}{4}\varphi^4$$

Clearly at $T = T_0$ $\partial_\varphi^2 V_T(\varphi) \equiv 0$. We can rewrite $V_T(\varphi)$ as

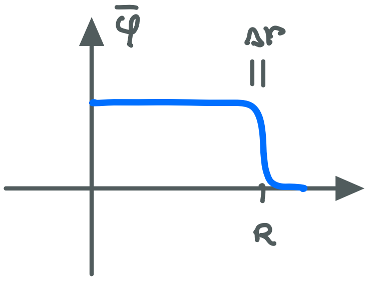
$$V(\varphi,T) = \varphi^2\left[\frac{\lambda}{4}\Big(\overbrace{\varphi - \frac{2\delta}{3\lambda}T}^{=0 \text{ at } T=T_c}\Big)^2 + \frac{1}{2}\gamma\Big(\underbrace{\Big(1 - \frac{2\delta^2}{9\lambda\gamma}\Big)T^2 - T_0^2}_{\equiv 0 \text{ at } T=T_c}\Big)\right]$$

$$\Rightarrow T_c^2 = \frac{T_0^2}{1 - \frac{2\delta^2}{9\lambda\gamma}} \quad \text{and} \quad \varphi_c = \frac{2\delta}{3\lambda}T_c \quad \Rightarrow \quad \boxed{\frac{\varphi_c}{T_c} = \frac{2\delta}{3\lambda}}$$

$$\hookrightarrow \; 1 - \left(\frac{T_0}{T_c}\right)^2 = \frac{2\delta^2}{9\lambda\gamma}$$

We continue study potential more a little later. For now we observe that near T_c we expect that the nucleating would be very large, such that $\partial_r^2\bar{\varphi} \gg \frac{2}{r}\partial_r\bar{\varphi}$ near the boundary. We may therefore use

$$\frac{d^2\bar{\varphi}}{dr^2} \simeq V'(\bar{\varphi}) \qquad (1)$$



thin wall bubble

If $V = \frac{\lambda}{4}\varphi^2(\varphi-\varphi_c)^2 = \frac{\lambda\varphi_c^4}{4} g^2(g-1)^2$ $(\varphi \equiv \varphi_c g)$ it is easy to write the equation in form

$$\varphi_c \frac{d^2 g}{dr^2} = \frac{\lambda\varphi_c^3}{4}\partial_g\left[g^2(g-1)^2\right]$$

$$\Leftrightarrow \quad \frac{d^2 g}{dy^2} = 4g(2g^2-3g+1)$$

$$\partial_g[\ \ ] = 2g\overbrace{(1-g)^2}^{g^2-2g+1} + 2\overbrace{(g-1)g^2}^{g^3-g^2}$$

$$= 4g^3-6g^2+2g$$

where $y = r/l_w$, with $l_w^{-1} = \sqrt{\frac{\lambda}{2}}\varphi_c = \sqrt{m^2(T_c)}$.

It is now easy to see that

Indeed: $m^2(\varphi_c, T_c) = (T_c^2 - T_0^2)\gamma$

$$= \gamma\frac{2\delta^2}{9\lambda\gamma}T_c^2 = \frac{2\delta^2}{9\lambda}\frac{9\lambda^2}{4\delta^2}\varphi_c^2$$

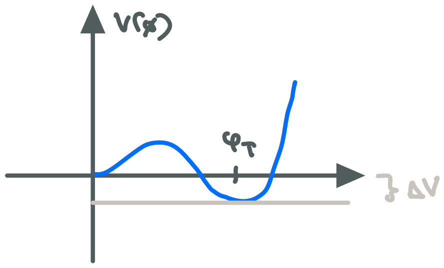
$$= \frac{\lambda}{2}\varphi_c^2 \Rightarrow m_c = \sqrt{\frac{\lambda}{2}}\varphi_c$$

$$\bar{\varphi} = \frac{\varphi_c}{2}\left(1 - \tanh\frac{r-R_c}{l_w}\right)$$

is a solution. This is exact at T_c and a good approximation near T_c. For our current approximation scheme it is essential that $R_c \gg l_w$ (to be seen shortly). We want to use the nucleation formula $\Gamma = T\left(\frac{S_{3d}}{2\pi T}\right)^3 \exp(-S_{3d}/T)$ to estimate the nucleation rate. For this we need an approximation for $S_{3d}(\bar{\varphi})$ for the bounce (critical bubble). Let us compute S_{3d} for arbitrary radius bubble in thin wall approximation (1). The action has two contributions: 1) volume contribution δS_{3d}^V and 2) surface contribution δS_{3d}^σ

1) **Volume contribution** inside the bubble $\partial\bar{\varphi}/\partial r \simeq 0$ and $V = -\Delta V$

$$\delta S^{v}_{3d} \simeq -\int d^3r\, \Delta V = -\frac{4\pi}{3}R^3\Delta V$$

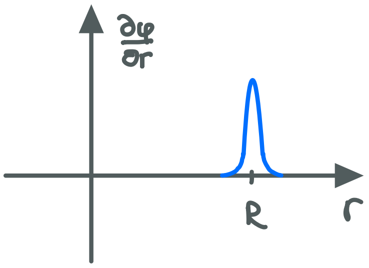


2) **Surface contribution.**

$$\delta S^{\sigma}_{3d} \simeq 4\pi R^2 \int dr\, \frac{1}{2}\left[(\partial_r\varphi)^2 + V(r)\right]$$

1d action

$$\equiv 4\pi R^2 \sigma$$

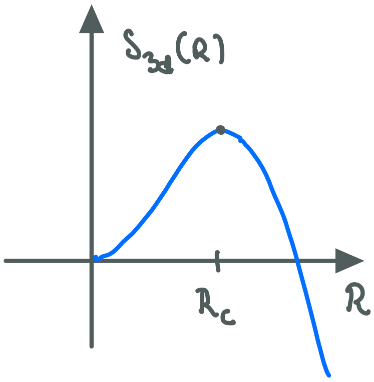


where $\sigma \equiv \int dr\, (\partial_r\varphi)^2$ = surface tension

Altogether then

$$S_{3d}(R) = -\frac{4\pi}{3}R^3\Delta V + 4\pi R^2\sigma.$$

$$\frac{dS_{3d}}{dR} = -4\pi R^2\Delta V + 8\pi R\sigma \equiv 0$$



Now R is the parameter that labels the steepest descent path across the saddle point.

$$\Rightarrow R = 0 \vee R = \frac{2\sigma}{\Delta V} \equiv R_c$$

$$\Rightarrow S_{3d}(R) = -\frac{4\pi}{3}\left(\frac{2\sigma}{\Delta V}\right)^3\Delta V + 4\pi\left(\frac{2\sigma}{\Delta V}\right)^2\sigma = \frac{16\pi}{3}\frac{\sigma^3}{(\Delta V)^2}$$

: dimensionless

We thus get for the nucleation rate including fluctuations around the bounce:

$$\frac{\Gamma}{V} \simeq T^4\left(\frac{8}{3T}\frac{\sigma^3}{\Delta V_T^2}\right)^{3/2} e^{-\frac{16\pi}{3}\frac{\sigma^3}{\Delta V_T^2 T}}$$

We now see that Γ/V is defined by ΔV_T and σ. These both depend on the temperature, so Γ/V is also (strongly) dependent on T. Let us study how we get these quantities from $V(\varphi, T)$.

a) Surface tension. We compute the 1d-action using

$$\partial_r \bar{\varphi} \, \partial_r^2 \bar{\varphi} = \partial_r \bar{\varphi} \, V'(\bar{\varphi}) \iff \partial_r \left[\tfrac{1}{2} (\partial_r \bar{\varphi})^2 - V \right] = 0$$

$$\Rightarrow \quad \sigma = \int dr \, (\partial_r \bar{\varphi})^2 = \int d\varphi \, |\partial_r \bar{\varphi}| = \int_0^{\varphi_*} \sqrt{2V} \, d\varphi$$

$$= \sqrt{\frac{\lambda}{2}} \, \varphi_c^3 \int_0^1 dg \, g(1-g) = \frac{\varphi_c^3}{6\sqrt{2}} \sqrt{\lambda} = \frac{2\sqrt{2}}{81} \frac{\delta^3}{\lambda^{5/2}} T_c^3$$

b) Because $V(\varphi, T) \equiv 0$, $\Delta V = -V(\varphi_T, T)$, where φ_T is the broken minimum at $T \neq 0$:

$$\partial_\varphi V(\varphi, T) = 0 \iff \varphi \left(\gamma (T^2 - T_0^2) - \delta T \varphi + \lambda \varphi^2 \right) = 0$$

$$\Rightarrow \quad \varphi = 0 \quad \text{or} \quad \varphi_T = \frac{\delta T}{2\lambda} \left(1 + \sqrt{1 - \frac{8}{9} \bar{\lambda}(T)} \right),$$

$$\text{where} \quad \bar{\lambda}(T) = \frac{9}{8} \frac{4 \lambda \gamma}{\delta^2} \left(1 - \frac{T_0^2}{T^2} \right) = \frac{9 \lambda \gamma}{2 \delta^2} \left(1 - \frac{T_0^2}{T_c^2} \right) = \frac{T^2 - T_0^2}{T_c^2 - T_0^2} .$$

One could just compute $\Delta V(\varphi_T, T)$ numerically. However, we can get a tractable approximation defining the latent heat

$\downarrow$ $s = \frac{\partial p}{\partial T}$; $p = -V$; $\rho + p = sT$

$$L(T) \equiv \Delta \rho \equiv \Delta(-p + sT) = \left[V(\varphi, T) - T \frac{dV}{dT} \right]_{\varphi_T}^{0}$$

Latent heat is the internal energy released in the transition. We may in particular define $L_c = L(T_c)$

$$V(\varphi,T) = \frac{1}{2}\gamma(T^2-T_0^2)\varphi^2 - \frac{1}{3}\delta T\varphi^3 + \frac{\lambda}{4}\varphi^4$$

$$L_c = T_c \frac{d}{dT}V(\varphi,T)\Big|_{\varphi=\varphi_T} = T_c\left(\gamma T_c\varphi_c^2 - \frac{1}{3}\delta\varphi_c^3\right) =$$

$$1-\left(\frac{T_0}{T_c}\right)^2 = \frac{2\delta^2}{9\lambda\gamma}$$

$$= \left[\gamma\left(\frac{2\delta}{3\lambda}\right)^2 - \frac{1}{3}\delta\left(\frac{2\delta}{3\lambda}\right)^3\right]T_c^4 = \frac{4\delta^2\gamma}{9\lambda^2}T_0^2T_c^2$$

Anticipating now that $T_c - T_n \ll T_c$ (T_n = nucleation temperature), we can write

$$\Delta V_T = -V(\varphi_T,T) \simeq \frac{1}{T_c}L_c(T_c-T) \simeq \frac{4\delta^2\gamma}{9\lambda^2}T_0^2T_c^2\left(\frac{T_c-T}{T_c}\right)$$

We now have eventually an estimate for S_{3d}/T_c in thin wall limit:

$$\frac{\bar{S}_{3d}}{T_c} = \frac{16\pi}{3}\frac{\sigma^3}{\Delta V T_c} \simeq \frac{16\pi}{3}\left(\frac{2\sqrt{2}}{81}\frac{\delta^3}{\lambda^{5/2}}\right)^3\left(\frac{9\lambda^2}{4\delta^2\gamma}\right)^2\overbrace{\left(\frac{T_c}{T_0}\right)^2}^{\simeq 1}\left(\frac{T_c}{T_c-T}\right)^2$$

$$= \frac{16\sqrt{2}\pi}{3^8}\frac{\delta^5}{\sqrt{\lambda}\gamma^2}\left(\frac{T_c}{T_c-T}\right)^2 \simeq 0{,}011\frac{\delta^5}{\sqrt{\lambda}\gamma^2}\left(\frac{T_c}{T_c-T}\right)^2$$

Note the strong sensitivity on δ: the larger bump, the more larger is $S_{3d}(\bar{\varphi})/T_c$ ⇒ more supercooling is needed to make $\bar{S}_{3d}/T_c$ small enough ⇒ $T_n - T_c$ grows

Indeed, we can get a rough estimate for T_n just by setting $S_{3d}/T_c \equiv 1$ which gives

$$T_n - T_c \simeq 0{,}1\frac{\delta^{5/2}}{\lambda^{1/4}\gamma}T_c$$

Nucleation temperature.

A more appropriate condition for T_n could be defined by setting T_n to be the temperature at which one nucleates one bubble/hubble horizon.

$$1 \equiv \int dt\, V_H \frac{\Gamma}{V}(t) = \int dt \left(\frac{4\pi}{3} H^{-3}\right) T^4 \left(\frac{\bar{S}_{3d}}{2\pi T}\right)^{3/2} \exp\left(-S_{3d}/T\right)$$

To evaluate this carefully note first that in radiation dominance ($H \propto a^{-2}$)

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} = \# T^2 \Rightarrow \text{below } T_c: \frac{t - t_c}{t_c} = 2\frac{T_c - T}{T_c} \quad ;$$

Then we see that $S_{3d}/T_c \propto \frac{1}{\delta T_c^2} \propto \frac{1}{\delta t^2}$ $\Rightarrow \partial_t(\beta S) \simeq -2(\beta S)\,\delta t^{-1}$. These quantities diverge at $t = t_c$ but we can expand βS_{3d} around the yet unknown t_n

$$\beta \bar{S}_{3d} \simeq \beta \bar{S}_{3d}(t_n) + (\beta \bar{S}_{3d})'(t_n)\,(t - t_n) + \dots$$

Obviously $(\beta S')(t_n) = -\frac{1}{\delta t_n}(\beta S)(t_n)$. By far dominant t-dependence of Γ is in the exponent and the integral is overwhelmingly dominated by $t \approx t_n$. We then have:

$$1 \simeq \left(\frac{4\pi}{3}\right)\left(\frac{T_n}{H_n}\right)^4 \left(\frac{\bar{S}_{3d}}{2\pi T_n}\right)^{3/2} e^{-(\beta\bar{S})_n} \underbrace{H_n}_{=1/2t_n} \overbrace{\int_{t_c}^{t_n} dt\, e^{-(\beta\bar{S})'_n (t - t_n)}}^{\simeq -\frac{1}{(\beta S')_n} \simeq \frac{\delta t_n}{2(\beta S)_n}}$$

Which gives simply:

$$e^{(\beta\bar{S})_n} = \frac{1}{6\sqrt{2\pi}} \left(\frac{T_n}{\underbrace{H_n}_{\simeq \left(\frac{T_c}{H_c}\right)^2}}\right)^4 \overbrace{\frac{t_n - t_c}{t_n}}^{\simeq 2\frac{T_c - T_n}{T_c}} (\beta\bar{S})_n^{1/2}$$

($\frac{\delta T}{T} \ll 1$; so to first we may set $T_n = T_c$ in the denominator)

$$\Leftrightarrow \quad (\beta\bar{S}_{3d})_n = -\log\left(3\sqrt{2\pi}\right) + 4\log\left(\frac{T_c}{H_c}\right) + \log\frac{T_c-T_n}{T_c} + \frac{1}{2}\log(\beta\bar{S}_{3d})$$

This equation can be solved iteratively for any $\bar{S}_{3d}$. Plugging in

$$H \simeq \left(\frac{4\pi^3}{45} g_* \frac{T^4}{M_{pl}^2}\right)^{1/2} \simeq 17\,\frac{T^2}{M_{pl}} \Rightarrow \frac{T_c}{H_c} \simeq \frac{1}{17}\frac{\overset{\swarrow\, 1.22\cdot 10^{19}\,\text{GeV}}{M_{pl}}}{T_c} \simeq 7.18\cdot 10^{15}\left(\frac{100}{T_c}\right)$$

Then we already see that $\beta\bar{S}_{3d}$ is rather large $\gtrsim 100$:

$$\boxed{(\beta\bar{S}_{3d})_n \simeq 144.0 + 4\log\left(\frac{100}{T_c}\right) + \log\frac{T_c-T_n}{T_c} + \frac{1}{2}\log(\beta\bar{S}_{3d})}$$

If we now set: $(\beta\bar{S}_{3d})_n \simeq 0.011\,\alpha\left(\frac{T_c}{T_c-T_n}\right)^2$, where in our model $\alpha = \delta^5/\sqrt{\lambda}\gamma$, we can further set

$$\frac{\Delta T_n}{T_c} = 0.1\sqrt{\alpha}\left(139.5 + 4\log\left(\frac{100}{T_c}\right) + \frac{1}{2}\log\alpha\right)^{-1/2}$$

$$\simeq 8.92\times 10^{-3}\sqrt{\alpha}\left\{1 - \underbrace{0.014\left\{\log\right.}_{\text{small correction}}\left(\frac{100}{T_c}\right) + \frac{1}{8}\log\sqrt{\alpha}\,\Big)\right\}$$

It is now clear that even in very strong transitions, where $\alpha \sim \mathcal{O}(1)$ ΔT_n is at $\mathcal{O}(\%)$-level of T_c

Let us now make some numerical estimates: In the MSM: (Ex)

$$\lambda \simeq \frac{G_F m_h^2}{\sqrt{2}} \simeq 0{,}129$$

$$\gamma \simeq \frac{1}{48}\left(24\lambda + 9g^2 + 3g'^2 + 12y_t^2\right) \simeq 0{,}40$$

$$\delta \simeq \frac{3}{12\pi}\left((2\lambda)^{3/2} + 2\left(2\left(\frac{g^2}{4}\right)^{3/2} + \left(\frac{g^2+g'^2}{4}\right)^{3/2}\right)\right) \simeq 0{,}03$$

$m_W^2 = \frac{g^2\varphi^2}{4}$

$m_Z^2 = \frac{g^2+g'^2}{4}\varphi^2$

$m_t^2 = \frac{y^2}{2}\varphi^2$

$m_h^2 = 2\lambda v^2$

$\sqrt{2}G_F = \frac{g^2}{4M_W^2} = \frac{1}{v^2}$

$\Rightarrow \lambda = m_h^2/2v^2 = \frac{G_F m_h^2}{\sqrt{2}}$

$g \simeq \frac{2M_W}{v} \simeq 0{,}65$

$\sqrt{g^2+g'^2} \simeq \frac{2M_Z}{v} \simeq 0{,}74$

$g' = 0{,}35$

Because δ is so small, the MSM-transition is very weak. Formally $T_0^2 \equiv \frac{2\mu^2}{\gamma} = \frac{m_h^2}{\gamma}$;

$T_0 \simeq 197{,}7$ GeV (~ ok, lattice: $T_c \simeq 159{,}5$ GeV [d'Onofrio & al PRL 113, 141602 (2014)])

$T_c \simeq 198{,}4$ GeV

$T_c - T_n \simeq 1.2$ MeV Ridiculous!

↳ reflects the fact that transition very weak.

These numbers reflect the fact that SM-transition is very weak. It actually is not first order at all, but a cross-over. Also, we get

$$\frac{v_c}{T_c} \simeq \frac{2\delta}{3\lambda} \simeq 0{,}1531 .$$ EWBG needs $\frac{v_c}{T_c} \gtrsim 1$

or rather $\frac{v_n}{T_n} \gtrsim 1$,

but $\frac{v_n}{T_n} \simeq 0{,}1533$

↑ almost the same.

How to improve?

$m \lesssim T_c \approx 100$ GeV

- New light bosonic d.o.f's with large coupling to higgs
- 2-(or multi) step transitions.

For example, if one adds 6 new bosonic species (R-handed, light stops in the MSSM), then we must add

$$\Delta\gamma = 6/48 \quad \text{and} \quad \Delta\delta = 6/4\pi$$

$$\Rightarrow \quad T_0 \simeq 172{,}5 \text{ GeV} \qquad \text{and} \qquad \frac{v_c}{T_c} \simeq 0{,}769$$

$$T_c \simeq 186.1 \text{ GeV} \qquad \frac{v_n}{T_n} \simeq 0.772$$

$$T_c - T_n \simeq 0{,}04 \text{ GeV}$$

moderate change

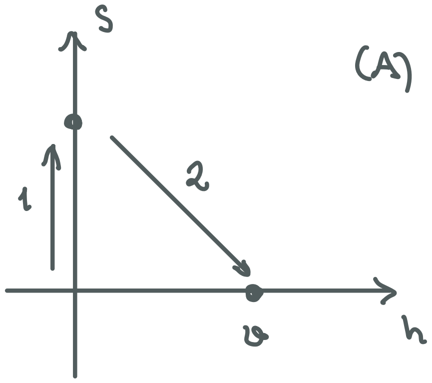
dramatic change!

Our approximations are a little crude. In a more careful evaluation the light stop scenario would work, but it is ruled out by experiment

2-step transitions

Add new scalar s coupled to h:

$$V(h,s,T) = -\mu^2|H|^2 + \lambda|H|^4 + \lambda_{hs}|H|^2 s^2 - \mu_s s^2 + \frac{\lambda_s}{4} s^4$$



correct just masses: $-\mu^2 \to -\mu^2 + c_h T^2$

$-\mu_s^2 \to -\mu_s^2 + c_s T^2$

Arrange $T_s \equiv \frac{\mu_s^2}{c_s} > T_h \equiv -\frac{\mu_h^2}{c_h}$, and yet $V(v,0,0) > V(0,w,0)$

Then transition progresses as in Fig A. In the second transition step the two minima at $(h,s) = (0,w(T))$ and $(h,s) = (v(T),0)$ are separated by a tree-level barrier, if $\lambda_{hs} > 0$. ⇒ Can have strong transition without large radiative corrections. This kind of mechanism is currently most promising & drives model building efforts.

Other topics that we have no time to address are

- Dynamical expansion of bubbles
  - Hydrodynamics:
  - Microscopic wall - particle interactions "friction"
    - ● deflagrations ?
    - ● detonations ?
      - ● Jouget ?
    - ● Runaway ?
- CP-violating dynamics at wall
  - Out-of-eq. FT.