1. Quantum statistical physics

- Ensembles, thermal potentials, partition function.
- SHO, SFO, particle in box, pressure, J_r-integrals
- Path integral for SHO. Matsubara sums. Exact PI.

Thermal ensembles

Microcanonical:

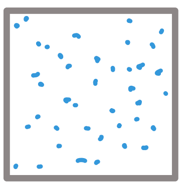
$$\hat{\rho} = \delta(\hat{H} - E) \quad ; \quad U(S,V,N) = TS - PV + \mu N$$

$$dU = TdS - PdV + \mu dN$$

density operator

$$\varrho = Ts - p + \mu n$$

$$dp = Tds - \mu dn$$



$\hat{=}$ Isolated system

Canonical:

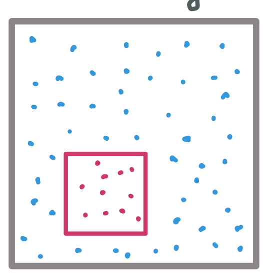
$$\hat{\rho}_c = e^{-\beta \hat{H}} \quad ; \quad F(T,V,N) = U - TS = -PV + \mu N$$

Hamiltonian

$$dF = -SdT - PdV + \mu dN$$

In thermal contact with the surroundings:

Heat exchange



$$U \rightarrow \langle E \rangle = \frac{1}{\mathrm{Tr}\,\hat{\rho}_c} \mathrm{Tr}\left(\hat{H} e^{-\beta \hat{H}}\right)$$

$$\mathrm{Tr}\,\hat{\rho}_c \equiv Z_c(T,V,N)$$

Statistical physics: $\quad F = -T \log Z_c$

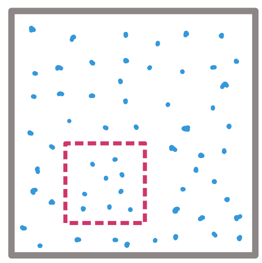
Indeed: $\quad dF = -S\,dT - P\,dV + \mu\,dN$

$$\Rightarrow \quad S = -\frac{\partial F}{\partial T} = \log Z + \frac{T}{Z}\frac{\partial Z}{\partial T} = \log Z + \frac{1}{TZ}\,\mathrm{Tr}\,\hat{H} e^{-\beta\hat{H}}$$

$$= -\frac{F}{T} + \frac{\langle E \rangle}{T} \quad \Rightarrow \quad F = U - TS$$

Grand canonical: $\quad \hat{\rho}_{gc} = e^{-\beta(\hat{H} - \mu\hat{N})}$ number operator

Heat & particle exchange



$$\Omega(T, V, \mu) = F - \mu N = -PV$$

$$d\Omega = -S\,dT - P\,dV - N\,d\mu$$

$$Z_{gc}(V, T, \mu) = \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu\hat{N})}$$ Grand canonical partition function

Again:

$$\Omega = -T \log Z_{gc} \quad \Rightarrow \quad P = -\left(\frac{\partial \Omega}{\partial V}\right)_{T,\mu} = T\frac{\partial}{\partial V}\log Z_{gc}$$

also

$$N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V} = T\frac{\partial}{\partial \mu}\log Z_{gc}$$

$$P = -\frac{\Omega}{V} = \frac{T}{V}\log Z_{gc}$$

$\uparrow$ extensive $\sim V$

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} = \log Z_{gc} + T\frac{\partial}{\partial T}\log Z_{gc}$$

- $N = T \frac{\partial}{\partial\mu} \log Z_{gc} = T \frac{1}{Z_{gc}} \mathrm{Tr}\left(\beta \hat{N} e^{-\beta(\hat{H}-\mu\hat{N})}\right) = \langle N \rangle$

- $\log Z_{gc} \propto V \Rightarrow P = T \frac{\partial}{\partial V} \log Z_{gc} = \frac{T}{V} \log Z_{gc} = -\frac{\Omega}{V}$.

will see

Again: $d\Omega = -S dT - P dV - N d\mu$

$$\Rightarrow S = -\frac{\partial\Omega}{\partial T} = \log Z + \frac{T}{Z}\frac{\partial Z}{\partial T} = \log Z + \frac{1}{TZ}\mathrm{Tr}(\hat{H}-\mu\hat{N})e^{-\beta(\hat{H}-\mu\hat{N})}$$

$$= -\frac{F}{T} + \frac{1}{T}(\langle E\rangle - \mu\langle N\rangle) \Rightarrow \Omega = \langle E\rangle - TS - \mu\langle N\rangle$$

Note: Gibbs entropy: def: $\tilde{\rho} \equiv \frac{\hat{\rho}}{\mathrm{Tr}\hat{\rho}} = \frac{\hat{\rho}}{Z}$

$$\Rightarrow S_{gibbs} \equiv -\mathrm{Tr}(\tilde{\rho}\log\tilde{\rho}) = -\frac{1}{Z}\mathrm{Tr}\left(\hat{\rho}\left(\underbrace{\log\hat{\rho}}_{-\beta H} - \log Z\right)\right)$$

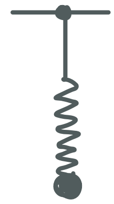
$$= +\log Z + \frac{\beta}{Z}\mathrm{Tr}\left((\hat{H}-\mu\hat{N})\hat{\rho}\right)$$

$$= +\log Z + \frac{1}{T}(\langle E\rangle - \mu\langle N\rangle)$$

$$= -\frac{\Omega}{T} + \frac{\langle E\rangle - \mu\langle N\rangle}{T} \Rightarrow \Omega = \langle E\rangle - TS_{gibbs} - \mu\langle N\rangle$$

ok

Simple harmonic oscillator



$F = -kq = -m\omega^2 q$

$$\hat{H}_{SHO} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2$$

where $[\hat{q},\hat{p}] = i$, $[\hat{q},\hat{q}] = [p,\hat{p}] = 0$

$$\hat{q} \equiv \frac{1}{\sqrt{2m\omega}}(a+a^\dagger) \quad \Rightarrow \quad \frac{1}{2}m\omega^2\hat{q}^2 = \frac{\omega}{4}(a+a^\dagger)^2$$

$$\hat{p} \equiv -i\sqrt{\frac{m\omega}{2}}(a-a^\dagger) \quad \Rightarrow \quad \frac{1}{2m}\hat{p}^2 = -\frac{\omega}{4}(a-a^\dagger)^2$$

$$\Rightarrow \quad H_{SHO} = \frac{\omega}{4}\left((a+a^\dagger)^2 - (a-a^\dagger)^2\right) = \frac{\omega}{2}(aa^\dagger + a^\dagger a) = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

given that $[a,a^\dagger] = 1$ (Ex.).

Number states: $a^\dagger|0\rangle = |1\rangle$; $a^\dagger a|1\rangle = a^\dagger a a^\dagger|0\rangle = a^\dagger|0\rangle = |1\rangle$

$$\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = |n\rangle, \qquad \langle n|m\rangle = \delta_{nm}$$

$$\langle n|n\rangle = \frac{1}{n!}\langle 0|a^n a^{\dagger n}|0\rangle$$

$$= \frac{1}{n!}\langle 0|\, n\, a^{n-1}(a^\dagger)^{n-1} + \cancel{a^{n-1}a^{\dagger n}a}\,|0\rangle$$

$$= \langle n-1|n-1\rangle \Rightarrow \boxtimes.$$

also $\hat{N}|n\rangle \equiv a^\dagger a|n\rangle = \frac{1}{\sqrt{n!}}a^\dagger a(a^\dagger)^n|0\rangle = n|n\rangle.$

$\Rightarrow$ Number operator $\hat{N} = a^\dagger a$

Partition function

$$Z_{SHO} = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \mathrm{Tr}\, e^{-\beta(\omega-\mu)\hat{N}-\frac{1}{2}\beta\omega}$$

$$= e^{-\frac{1}{2}\beta\omega} \sum_n \langle n | e^{-\beta(\omega-\mu)\hat{N}} | n\rangle$$

$$= e^{-\frac{1}{2}\beta\omega} \sum_n \left(e^{-\beta(\omega-\mu)}\right)^n = \frac{e^{-\frac{1}{2}\beta\omega}}{1-e^{-\beta(\omega-\mu)}}$$

$$\Rightarrow \ln Z = -\frac{\beta\omega}{2} - \log\left(1-e^{-\beta(\omega-\mu)}\right)$$

Particle number:

$$N = \langle \hat{N} \rangle = \frac{1}{\mathrm{Tr}\,\hat{\rho}} \mathrm{Tr}(\hat{N}\hat{\rho}) = T\frac{\partial}{\partial\mu}\log Z$$

$$= -T \frac{1}{1-e^{-\beta(\omega-\mu)}}\left(-e^{-\beta(\omega-\mu)}\cdot\beta\right) = \frac{\quad}{\quad}$$

$$= \frac{1}{e^{\beta(\omega-\mu)}-1} = f_{BE}(\omega) \quad ; \quad N \in [0,\infty[$$

Energy. Either directly computing from trace $E = \langle \hat{H} \rangle = \frac{1}{Z}\sum_n \langle n | \hat{H}\hat{\rho} | n\rangle$ .. or

$$\langle E \rangle = \Omega + Ts + \mu\langle N\rangle = \cancel{-T\log Z_{gc}} + T\left(\cancel{\log Z_{gc}} + T\frac{\partial}{\partial T}\log Z_{gc}\right) + \mu T\frac{\partial}{\partial\mu}\log Z_{gc}$$

$$= \frac{\omega}{2} - T^2\frac{\partial}{\partial T}\log\left(1-e^{-\beta(\omega-\mu)}\right) + \mu f_{BE} = \frac{\omega}{2} + \frac{\omega}{e^{\beta(\omega-\mu)}-1}$$

Simple fermionic oscillator

Pauli exclusion principle: $\Rightarrow$ anticommutation rules:

$\Rightarrow$ $a \rightarrow \alpha$, with $\{\alpha, \alpha^+\} = 1$; $\{\alpha, \alpha\} = \{\alpha^+, \alpha^+\} = 0$

Now only two states! $\alpha^+|0\rangle = |1\rangle$ $\alpha|1\rangle = 0$

$\alpha^+|1\rangle = \alpha^+\alpha^+|0\rangle = \frac{1}{2}\{\alpha^+, \alpha^+\}|0\rangle = 0$

Hamiltonian function: $H_{SFO} \equiv \frac{1}{2}\omega(\alpha^+\alpha - \alpha\alpha^+) = \omega(\alpha^+\alpha - \frac{1}{2}) = \omega(\hat{N} - \frac{1}{2})$

Partition function

$$Z_{SFO} = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = e^{\frac{1}{2}\beta\omega} \sum_{n=0,1} \langle n| e^{-\beta(\omega-\mu)\hat{N}} |n\rangle = e^{\frac{1}{2}\beta\omega}\left(1 + e^{-\beta(\omega-\mu)}\right)$$

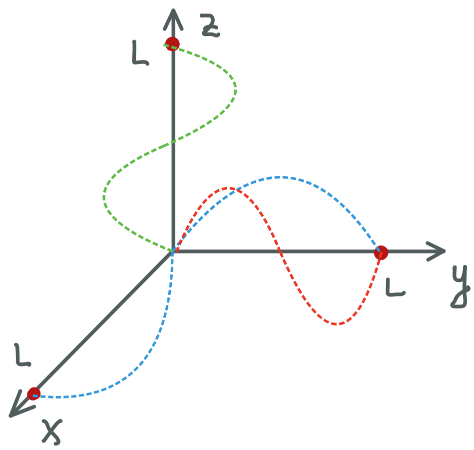
$$\Rightarrow \ln Z_{SFO} = \frac{1}{2}\beta\omega + \log\left(1 + e^{-\beta(\omega-\mu)}\right).$$

$$\langle N \rangle = T\frac{\partial}{\partial\mu}\log Z = \frac{1}{1+e^{-\beta(\omega-\mu)}}\, T\beta e^{-\beta(\omega-\mu)} = \frac{1}{e^{\beta(\omega-\mu)}+1} = n_{FD}(\omega)$$

$$\langle E \rangle = \frac{1}{Z}\mathrm{Tr}\left(\hat{H} e^{-\beta(\omega-\mu)\hat{N} + \frac{1}{2}\beta\omega}\right) = \frac{e^{-\frac{1}{2}\beta\omega}}{1+e^{-\beta(\omega-\mu)}}\left(-\frac{\omega}{2}e^{\frac{\beta\omega}{2}} + \frac{\omega}{2}e^{-\beta(\omega-\mu)+\frac{1}{2}\beta\omega}\right)$$

$$= \frac{1}{1+e^{-\beta(\omega-\mu)}}\left(-\frac{\omega}{2}\left(1+e^{-\beta(\omega-\mu)}\right) + \omega e^{-\beta(\omega-\mu)}\right) = -\frac{\omega}{2} + \omega f_{FD}(\omega)$$

Particles in a box



Boundary cond: $\psi(x,y,z=0,L) = 0$

$$\Rightarrow L = \frac{n_i \lambda_i}{2}$$

$$\Rightarrow |p_i| = \frac{2\pi}{\lambda_i} = \frac{\pi n_i}{L} .$$ Number of quantized states

Each mode is equivalent with a SHO with $\omega = \frac{|\vec{p}|^2}{2m}$; $\vec{p} = \frac{\pi}{L}(n_1, n_2, n_3)$.

$$\hat{H} = \sum_{\{i\}} \hat{H}_{\{i\}} \quad ; \quad \hat{N} = \sum_{\{i\}} \hat{N}_{\{i\}}$$

and

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu\hat{N})} = \prod_{\{i\}} \mathcal{Z}_{\{i\}} .$$

Now

$$(\Delta p_i = \tfrac{\pi}{L})$$

$$T \log \mathcal{Z} = T \sum_{\{i\}} \log \mathcal{Z}_{\{i\}} \xrightarrow{L \to \infty} \left(\frac{L}{\pi}\right)^3 T \int_0^\infty \prod_{i=1}^{3} d|p_i| \log \mathcal{Z}_{\vec{p}}$$

$$= VT \int \frac{d^3p}{(2\pi)^3} \log \mathcal{Z}_p$$

$\Omega = -T \log \mathcal{Z}$

$P = -\frac{\Omega}{V} = +\frac{T}{V} \log \mathcal{Z}$

$$= V \int \frac{d^3p}{(2\pi)^3} \left(\mp \frac{\omega_p}{2} \mp T \log\left(1 \mp e^{-\beta(\omega - \mu)}\right)\right) \quad \begin{matrix} -B \\ +F \end{matrix}$$

$$\Rightarrow P_\pm = -\frac{\Omega}{V} = \int \frac{d^3p}{(2\pi)^3} \left(\mp \frac{\omega_p}{2} \mp T \log\left(1 \mp e^{-\beta(\omega - \mu)}\right)\right) . \quad \begin{matrix} +B \\ -F \end{matrix}$$

$$= P_T^{\mp} = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3ET} \frac{1}{e^{\beta(\omega-\mu)} \pm 1}$$

$P_v = -E_v$; $w = -1$

Number densities & $\langle E \rangle / V$:

$$n = \frac{N}{V} = \frac{T}{V} \frac{\partial}{\partial \mu} \log Z_{gc} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(\omega-\mu)} \pm 1}$$

$$E_{\mp} \equiv \langle \hat{H} \rangle = V \int \frac{d^3p}{(2\pi)^3} \left(\mp \frac{\omega_p}{2} + \frac{\omega_p}{e^{\beta(\omega-\mu)} \pm 1} \right) = \mp E_0 + E_T^{\pm}$$

upper signs : fermions
lower —"— : bosons

Antiparticles.

In relativistic field theory those are included automatically. Here we must put them in by hand. We can do this using Dirac hole interpretation. A state with n antiparticles corresponds to a positive energy state with lack of n particles. Then

same as for particles

$$\langle \bar{n} | \hat{\rho} | \bar{n} \rangle = e^{-\beta(\omega - \mu(-1))n \mp \frac{1}{2}\beta\omega} = e^{\mp\frac{1}{2}\beta\omega} e^{-\beta(\omega+\mu)n}$$

$$\Rightarrow \quad E_{\pm} = V \int \frac{d^3p}{(2\pi)^3} \left(\mp \omega_p + \frac{\omega_p}{e^{\beta(\omega_p-\mu)} \pm 1} + \frac{\omega_p}{e^{\beta(\omega_p+\mu)} \pm 1} \right) = \pm 2E_0 + E_T^{\pm}$$

upper signs fermion

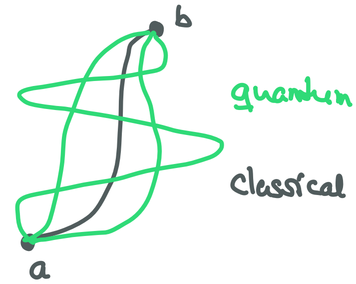
Similarly, if particles carry a conserved charge

$$Q_{\pm} = V \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{e^{\beta(\omega_p-\mu)} \pm 1} - \frac{1}{e^{\beta(\omega_p+\mu)} \pm 1} \right).$$

Chemical potential $\hat{=}$ free energy difference due to adding / removing a particle ($d\Omega = \cdots + \mu dN$). Adding antiparticle requires pair creation so $\mu \simeq m + \mu_{NR}$.
$\Rightarrow$ at low $T \ll m$ (anti)particles suppressed by $e^{-2\beta m}$.

Path integral methods

We may define QM transition amplitudes with PI.



Start from the standard QFT case (conceptually simple)

- $P(b,a) \equiv |K(b,a)|^2$ (1)

when

- $F(b,a) \equiv \sum_{\forall \text{ paths}} k\, e^{\frac{i}{\hbar}S}$ (2)

where S = Classical action and

- $K(c,a) = \sum_{\forall b} K(c,b)\, K(b,a)$ (3)

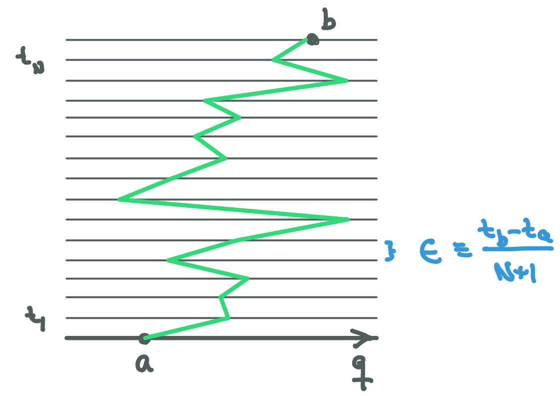
Correspondence: $\frac{\delta}{\delta q(t)} S[q] = 0 \Rightarrow$ constructive interference only around q_{cl}

$\hbar$ small $\Rightarrow$ classical physics at macro-scales.

Discretization:

why extra? See later.

$$\sum_{\forall \text{paths}} = \lim_{N\to\infty} k_N \prod_{i=1}^{N} \int dq_i\, k_i \equiv \int [\mathcal{D}x]$$



$$S = \int dt\, L = \int dt \left(\tfrac{1}{2} m \dot{q}^2 - V(q) \right)$$

$$\longrightarrow \sum_{k=0}^{N+1} \left(\frac{m}{2} \frac{(q_{k+1} - q_k)^2}{\epsilon} - \epsilon V\left(\frac{q_k + q_{k+1}}{2}\right) \right)$$

$q_0 = q_a$

$q_{N+1} = q_b$

$\epsilon = \frac{t_b - t_a}{N+1}$

↑ why average?

Determine k: by applying (3) to interval $[t_b-\epsilon, t_b]$

$$K(q_{t_b}, t_b; q_{t_a}, t_a) = \int_{-\infty}^{\infty} dq' \, k_N \, e^{i\left(\frac{m}{2}\left(\frac{q_{t_b}-q'}{\epsilon}\right)^2 - \epsilon V\left(\frac{q_{t_b}+q'}{2}\right)\right)} K(q', t_b-\epsilon; q_{t_a}, t_a)$$

$$\simeq \int_{-\infty}^{\infty} dq' \, k_N \, e^{\frac{im}{2\epsilon}(q_{t_b}-q')^2}\left(1-i\epsilon V(q_{t_b})+\dots\right)\left(1+\overbrace{(q-q')}^{0:\text{ odd}}\frac{\partial}{\partial q_{t_b}} + \frac{1}{2}\underbrace{(q-q')^2}_{\sim\epsilon}\frac{\partial^2}{\partial q_{t_b}^2}+\dots\right)K(q_{t_b}, t_b-\epsilon; q_{t_a}, t_a)$$

This is where we need to tilt the time path: convergence of the gaussian integral can be guaranteed by $t \to t(1-i\delta)$ $\Rightarrow$ $\epsilon \to \epsilon(1-i\delta')$; $\delta' = \frac{\delta}{N+1}$

Then using

$$\int_{-\infty}^{\infty} dy \, y^{2n} e^{\frac{ib_\epsilon}{1-i\delta'}y^2} = \lim_{\delta'\to 0}\int_{-\infty}^{\infty} dy \, y^{2n} e^{-(\delta'-ib_\epsilon)y^2} \qquad (4)$$

$$= \lim_{\delta'\to 0}(-1)^n \partial_a^n \sqrt{\frac{\pi}{a}}\,\Big|_{a=\delta'-ib_\epsilon} = (-1)^n \partial^n_{-ib_\epsilon}\sqrt{\frac{\pi}{-ib_\epsilon}} = \sqrt{i\pi}\,(-i)^n \partial^n_{b_\epsilon}\frac{1}{\sqrt{b_\epsilon}}$$

$$\Rightarrow K(q_{t_b}, t_b; q_{t_a}, t_a) \simeq \underbrace{k_N\sqrt{\frac{2i\pi\epsilon}{m}}}_{=1}\left(1-i\epsilon V(q_{t_b}) + \underbrace{\frac{1}{2}\left(\frac{i}{2}\right)\left(\frac{2\epsilon}{m}\right)}_{=\frac{i\epsilon}{2m}}\frac{\partial^2}{\partial q_{t_b}^2} + O(\epsilon^2)\right)K(q_{t_b}, t_b-\epsilon; q_{t_a}, t_a)$$

$\Rightarrow$ LHS = RHS when $\epsilon \to 0$ $\Rightarrow$ $k_N\sqrt{\frac{2i\pi\epsilon}{m}} = 1$ $\Leftrightarrow$ $k_N = \sqrt{\frac{m}{2\pi i\epsilon}}$

$\Rightarrow$ $i\frac{\partial}{\partial t_b}K(q_{t_b}, t_b; q_{t_a}, t_a) = \left(-\frac{1}{2m}\frac{\partial^2}{\partial q_{t_b}^2} + V(q_{t_b})\right)K(q_{t_b}, t_b; q_{t_a}, t_a)$

Also: $\lim_{t_b\to t_a} K(q_{t_b}, t_b; q_{t_a}, t_a) = \lim_{\epsilon\to 0} k_N \, e^{i\frac{m(q_{t_b}-q_{t_a})^2}{2\epsilon}} = \delta(q_{t_b}-q_{t_a})$

only one slice-interval, no integration

$\Rightarrow$ K obeys the same equation as $U(q_{t_a}, t_a; q_{t_b}, t_b) = \langle q_{t_b}, t_b| e^{-i\hat{H}(t_b-t_a)}|q_{t_a}, t_a\rangle$

& has the same initial condition.

Formal equivalence

Using the integral relation (4): $e^{-ib^2/4a} = \sqrt{\frac{ia}{\pi}} \int dp\, e^{iap^2+ibp}$

one can write

$$K(q_{t_b}, t_b ; q_{t_a}, t_a) = \int [\mathcal{D}q]\, e^{i\int_{t_a}^{t_b} dt\,(\frac{1}{2}m\dot{q}^2 - V(q))}$$

$$= \int [\mathcal{D}q\, \mathcal{D}p]\, e^{i\int_{t_a}^{t_b} dt\,(p\dot{q} - \frac{p^2}{2m} - V(q))} \qquad (5)$$

Going backwards, discretizing & treating p just as V(q) we see $a = \frac{\epsilon}{2m}$. Then it is easy to show that 2nd line in (5) reproduces discretized $K(q_{t_b}, t_b; q_{t_a}, t_a)$.

However, we see also that $K_N^p = \sqrt{\frac{ia}{\pi}} = \sqrt{\frac{i\epsilon}{2m\pi}} \Rightarrow K_N K_N^p = \sqrt{\frac{i\epsilon}{2m\pi}} \cdot \sqrt{\frac{m}{2\pi i}} = \frac{1}{2\pi}$?

one p-integral for each q-interval including enpoints. eats the extra K_N

$$\Rightarrow K(q_{t_b}, t_b ; q_{t_a}, t_a) = \lim_{N\to\infty} \prod_{i=1}^{N} \prod_{k=1}^{N+1} \int dq_i \frac{dp_k}{2\pi}\, e^{-i\left(H(q_i, p_k)\epsilon - p_k(q_{i+1} - q_i)\right)}$$

$$= \lim_{N\to\infty} \prod_{i=1}^{N} \prod_{k=1}^{N+1} \int dq_i \frac{dp_k}{2\pi}\, \underbrace{e^{-iH(q,p)_i\epsilon}}\, e^{ip_i \overbrace{(q_{i+1} - q_i)}^{\equiv r_i}}$$

$$\equiv e^{-i\hat{H}(\bar{x}_i, -i\partial_{r_i})}\, e^{ip_i r_i} \qquad \bar{x}_i = \frac{x_{i+1}+x_i}{2}$$

$e^{A+B} = e^A e^B$ iff $[A,B]=0$

c-numbers ok.

Operators ?

$[\bar{x}, \partial_{\bar{r}}] = 0$

Given this prescription for $\hat{H}$ in terms of commuting $[\bar{x}, \partial_{\bar{r}}] = 0$:

↑ Not conjugate variables

$$\int \frac{dp_i}{2\pi}\, e^{-i\hat{H}(\bar{x}, \partial_{\bar{r}})_i\epsilon}\, \underbrace{e^{ip_i(q_{i+1} - q_i)}}_{\langle q_{i+1}|p_i\rangle\langle p_i|q_i\rangle} = \langle q_{i+1}|\, e^{-i\hat{H}(\hat{q},\hat{p})\epsilon}\, |q_i\rangle$$

The last step is only valid if $\langle q_{i+1}|\, \hat{H}(\hat{q},\hat{p})\,|q_i\rangle \equiv \langle q_i|\, \hat{H}(\frac{q_{i+1}+q_i}{2}, \hat{p}\,|q_i\rangle$

$\Rightarrow$ operators must be Weyl-ordered. For example

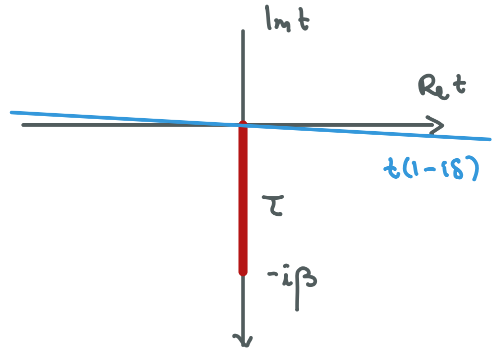
$$\mathrm{Weyl}(\hat{q}^2\hat{p}^n) = \hat{q}^2\hat{p}^n + \hat{p}^n\hat{q}^2 + 2\hat{q}\hat{p}^n\hat{q} \Rightarrow \langle q_{i+1}|\,\mathrm{Weyl}(\hat{q}^2\hat{p}^n)\,|q_i\rangle$$
$$= \left(q_{i+1}^2 + 2q_{i+1}q_i + q_i^2\right)\langle q_{i+1}|\hat{p}^n|q_i\rangle$$
$$= (q_{i+1}+q_i)^2\,\hat{p}^n$$

$$\Rightarrow K(q_b,t_b\,;q_a,t_a) = \lim_{N\to\infty}\prod_{i=1}^{N}\int dq_i\,\langle q_{i+1}|e^{-i\hat{H}\epsilon}|q_i\rangle$$
$$= \lim_{N\to\infty}\langle q_b|e^{-i\hat{H}\epsilon}\overbrace{\int dq_N|q_N\rangle\langle q_N|}^{=1}e^{-i\hat{H}\epsilon}|q_{N-1}\rangle\cdots\langle q_1|e^{-i\hat{H}\epsilon}|q_a\rangle$$
$$= \langle q_b,t_b|e^{-iH(t_b-t_a)}|q_a,t_a\rangle = U(q_b,t_b\,;q_a,t_a).$$

Finite-T: we are interested in computing traces

$$Z = \mathrm{Tr}\,\hat{\rho} = \sum_n\langle n|e^{-\beta\hat{H}}|n\rangle = \sum_n\langle n|e^{-\beta\hat{H}}\int dq\,|q\rangle\langle q|n\rangle$$
$$= \int dq\,\langle q|\sum_n|n\rangle\langle n|e^{-\beta\hat{H}}|q\rangle = \int dq\,\langle q|e^{-\beta H}|q\rangle.$$

We can write $\langle q'|e^{-\beta H}|q\rangle = \langle q'|e^{-i\hat{H}(-i\beta)}|q\rangle = K(t_a-i\beta,q'\,;t_a,q)$, and in particular

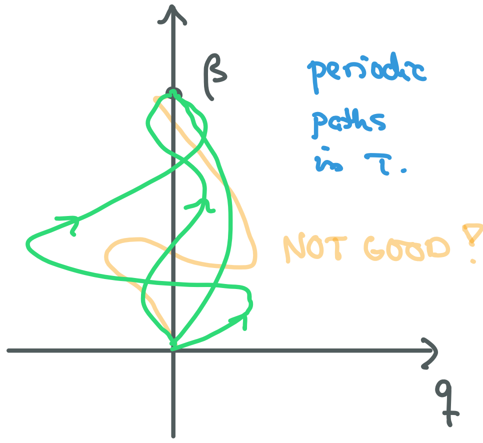


$$Z = \int dq\,K(-i\beta,q\,;0,q)$$
$$= \int[\mathcal{D}q]_{\substack{q_i=q\\ q_f=q}}\,e^{-i\int(-id\tau)\,\mathcal{L}_M(\partial_t\to i\partial_\tau)}$$
$$= \int[\mathcal{D}q]_\beta\,e^{-\int_0^\beta d\tau\,\mathcal{L}_E}\,.$$

In measure $[\mathcal{D}q]_\beta$ index β refers to periodicity in τ.

where

$$\mathcal{L}_E = \frac{1}{2}(\partial_\tau q)^2 + V(q)$$



Of course we could have derived the Euclidean path integral directly without the recourse to real time PI. That would be perfectly analogous to what we did, except that one does not need the time-path tilting argument.

(Also in this case $d\tau$ must be positive along the path, for PI to exist.)

Generating function & propagator

We now have a PI expression for Z:

$\dot{q} \equiv \partial_\tau q$

$$Z(\beta) = \int [\mathcal{D}q]_\beta \, e^{-S_E[q]} \quad ; \quad S_E[q] = \int_0^\beta d\tau \, L_E = \int_0^\beta d\tau \left(\frac{1}{2}\dot{q}^2 + V(q) \right)$$

We can generalize this to a generating function

$$Z[\beta, j] = \int [\mathcal{D}q]_\beta \, e^{-S_E[q] + \int_0^\beta d\tau \, j(\tau) q(\tau)}$$

Then $Z(\beta) = Z[\beta, 0]$. We get the 2-point correlation function.

$$\Delta(\tau_1, \tau_2) \equiv \frac{1}{Z(\beta)} \frac{\delta^2 Z[\beta, j]}{\delta j(\tau_2) \delta j(\tau_1)}\bigg|_{j=0} = \frac{1}{Z(\beta)} \int [\mathcal{D}q]_\beta \, q(\tau_1) \, q(\tau_2) \, e^{-S_E[q]}$$

Based on the derivation of the PI & its connection to operator formalism on p.12, it is obvious that $\Delta(\tau_1,\tau_2)$ is the τ-ordered propagator:

$$\Delta(\tau_1,\tau_2) = \frac{1}{\mathrm{Tr}\hat\rho}\,\mathrm{Tr}\left[\hat\rho\,\overset{\tau\text{-ordered}}{\mathcal{T}}(\hat q_1(\tau_1)\hat q(\tau_2))\right] = \langle \mathcal{T}(\hat q(\tau_1)\hat q(\tau_2))\rangle$$

where $\quad \mathcal{T}(\hat q(\tau_1)\hat q(\tau_2)) = \theta(\tau_1-\tau_2)\,\hat q(\tau_1)\hat q(\tau_2) + \theta(\tau_2-\tau_1)\,\hat q(\tau_2)\hat q(\tau_1)$

i.e PI - expectation values are automatically time-ordered (or τ-ordered).

Translation invariance & KMS relation $\quad (p=\dot q = \partial_t q = -i\partial_\tau q)$

Noting that $\quad \hat q(\tau) = e^{\hat H\tau}\hat q(0)\,e^{-H\tau}$

$$[\hat q,\hat p] = i \quad ; \quad \hat H = \frac{\hat p^2}{2} + V(\hat q)$$

$$\Rightarrow \hat H\hat q = \hat q\hat H + [\hat H,\hat q] = \hat q\hat H + i\hat p$$

$$\Rightarrow \hat q\hat H = (\hat H - i\partial_t)\hat q = (\hat H - \partial_\tau)\hat q$$

$$\Rightarrow \hat q\hat H^n = (H-\partial_\tau)^n\hat q$$

$$\Rightarrow e^{\hat H\tau}\hat q(0)\,e^{-\hat H\tau} = e^{\hat H\tau}e^{-\tau(\hat H-\partial_\tau)}\hat q(0) = e^{\tau\partial_\tau}\hat q(0) = \hat q(\tau)$$

$\Rightarrow \quad \Delta(\tau_1,\tau_2)$

$$\begin{aligned}
&= \langle \mathcal{T}(\hat q(\tau_1)\hat q(\tau_2))\rangle \qquad \tau_1>\tau_2\\
&= \mathrm{Tr}\left(e^{-\beta\hat H}\hat q(\tau_1)q(\tau_2)\right)/\mathrm{Tr}\hat\rho\\
&= \mathrm{Tr}\left(e^{-\beta\hat H}\hat q(\tau_1)\,e^{\hat H\tau_2}\hat q(0)e^{-H\tau_2}\right)/\mathrm{Tr}\hat\rho\\
&= \mathrm{Tr}\left(e^{-\beta\hat H}e^{-H\tau_2}\hat q(\tau_1)\,e^{\hat H\tau_2}\hat q(0)\right)/\mathrm{Tr}\hat\rho\\
&= \mathrm{Tr}\left(e^{-\beta\hat H}\hat q(\tau_1-\tau_2)\hat q(0)\right) = \Delta(\tau_1-\tau_2,0) = \Delta(\tau_1-\tau_2) \qquad \mathrm{Tr}\text{ invariant.}
\end{aligned}$$

KMS:

$$\begin{aligned}
\Delta(\tau) &= \frac{1}{\mathrm{Tr}\hat\rho}\mathrm{Tr}\left(e^{-\beta\hat H}\hat q(\tau)\,\hat q(0)\right), \quad \tau>0\\
&= \frac{1}{\mathrm{Tr}\hat\rho}\mathrm{Tr}\left(e^{-\beta\hat H}\hat q(\tau)\,e^{\beta\hat H}e^{-\beta\hat H}\hat q(0)\right) = \frac{1}{\mathrm{Tr}\hat\rho}\mathrm{Tr}\left(\hat q(\tau-\beta)\,e^{-\beta\hat H}\hat q(0)\right)\\
&= \frac{1}{\mathrm{Tr}\rho}\mathrm{Tr}\left(\mathcal{T}(\hat q(\tau-\beta)\hat q(0)\right) = \Delta(\beta-\tau).
\end{aligned}$$

KUBO – MARTIN – SCHWINGER (KMS) - Relation

Propagator as greens function

$$\left[\tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}\omega^2 q^2 - jq\right]$$

Now take

$$V_0 = \tfrac{1}{2} m\omega^2 q^2 \xrightarrow{q \to m^{-1/2} q} \tfrac{1}{2}\omega^2 q^2$$

$$\Rightarrow \quad Z[\beta, j] = \int [\mathcal{D}q]_\beta \, e^{-\int_0^\beta d\tau \left[\frac{1}{2} q(-\partial_\tau^2 + \omega^2) q - jq\right]}$$

$$\int d\tau' \, \Delta^{-1}(\tau,\tau') q(\tau') \equiv (-\partial_\tau^2 + \omega^2) q(\tau)$$

$$= \int [\mathcal{D}q]_\beta \, e^{\underbrace{-\int_0^\beta d\tau d\tau' q(\tau) \Delta^{-1}(\tau,\tau') q(\tau')}\, - \int d\tau \, j(\tau) q(\tau)}$$

$$= \tfrac{1}{2} q \Delta^{-1} q - jq = \tfrac{1}{2}(\overbrace{q - j\Delta}^{q'}) \Delta^{-1} (q - \Delta j) - \tfrac{1}{2} j \Delta j$$

$$\longrightarrow \tfrac{1}{2} q' \Delta^{-1} q' - \tfrac{1}{2} j \Delta j$$

$$= Z(\beta) \, e^{\frac{1}{2}\int_0^\beta\int_0^\beta d\tau d\tau' \, j(\tau') \Delta(\tau - \tau') j(\tau)}$$

Thus indeed (as notation already suggests) $\displaystyle \frac{1}{Z(\beta)} \frac{\delta^2 Z[\beta, j]}{\delta j(\tau_2) \delta j(\tau_1)}\bigg|_{j=0} = \Delta_0(\tau)$.

So $\Delta_0(\tau)$ is the Greens function, that obeys

$$(-\partial_\tau^2 + \omega^2) \Delta_0(\tau, \tau') = \delta(\tau - \tau').$$

Setting
$$\Delta_0(\tau, \omega) \equiv T \sum_{n=-\infty}^{\infty} \Delta_0(\omega_n) e^{-i\omega_n \tau}$$

$$\Rightarrow \quad (\omega_n^2 + \omega^2) \Delta_0(\omega_n, \omega) = 1 \quad \Rightarrow \quad \boxed{\Delta_0 = \frac{1}{\omega_n^2 + \omega^2}}$$

Also KMS:
$$\Delta(\tau - \beta) = T \sum_{n=-\infty}^{\infty} \Delta_0(\omega_n) e^{-i\omega_n \tau} \underbrace{e^{i\omega_n \beta}}_{=1} = \Delta(\tau)$$

$$\Rightarrow \quad e^{i\omega_n \tau} = 1 \quad \Rightarrow \quad \omega_n = 2\pi n T$$

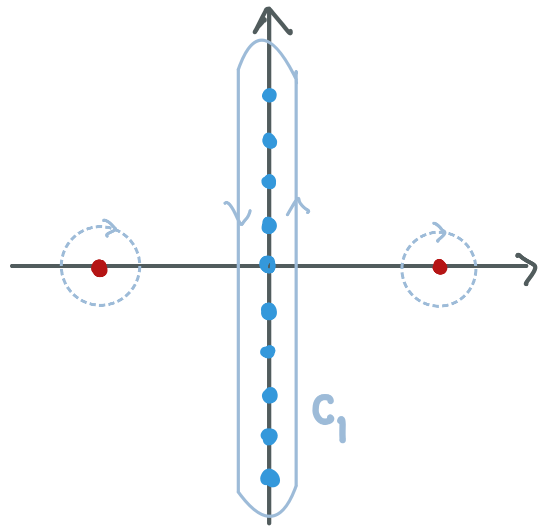
Bosonic Matsubara frequencies

Matsubara sums.

First compute propagator

$$\Delta_0(\tau,\omega) = T\sum_{n=-\infty}^{\infty} \Delta_0(\omega_n)\, e^{-i\omega_n\tau}$$

$$= T\sum_{n=-\infty}^{\infty} \frac{1}{\omega^2+\omega_n^2}\, e^{-i\omega_n\tau}$$



Note that $\Delta(-\tau,\omega) = \Delta(\tau,\omega)$, b

because $\tau \to -\tau$ & $n \to -n$ leaves $\Delta_0(\tau,\omega)$ invariant.

Now use the fact that $\frac{e^{\beta z}}{e^{\beta z}-1} \simeq \frac{1}{\beta\delta z}$ for $z \simeq 2n\pi T i + \delta z$ (escentral)

and $\frac{e^{(\beta-\tau)z}}{e^{\beta z}-1} \longrightarrow 0 \quad \forall\ |z| \to \infty$

$$\Rightarrow \Delta_0(\tau,\omega) = \frac{1}{2\pi i}\oint_{C_1} \frac{e^{-z|\tau|}}{\omega^2-z^2}\,\frac{e^{\beta z}}{e^{\beta z}-1}$$: sum is given by residues inside C_1

$$= \sum_{\pm} \lim_{z\to\pm\omega} \frac{\cancel{z\mp\omega}}{(\cancel{z\mp\omega})(z\pm\omega)}\, e^{\mp\omega|\tau|}\, \frac{e^{\pm\beta\omega}}{e^{\pm\beta\omega}-1}$$: sum is given by residues outside pole.

– sign due to clockwise path is absorbed to $\omega^2-z^2 \to z^2-\omega^2$

$$= \frac{1}{2\omega}\left(e^{-\omega|\tau|}\frac{e^{\beta\omega}}{e^{\beta\omega}-1} - e^{\omega|\tau|}\frac{e^{-\beta\omega}}{e^{-\beta\omega}-1}\right)$$

$$= \frac{1}{2\omega}\Big((1+n_B(\omega))\, e^{-\omega|\tau|} + n_B(\omega)\, e^{\omega|\tau|}\Big),$$ with $\boxed{n_B(\omega) \equiv \frac{1}{e^{\beta\omega}-1}}$

In particular $T\sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2+\omega^2} = \frac{1}{2\omega}\left(1+2n_B(\omega)\right)$

Partition function This is a very important topic. We will do the calculation exactly in Ex. 1.6. Here we do a more heuristic QFT-evaluation, followed by a regularization trick. The usual QFT-evaluation uses F-space:

"QFT"-evaluation (Fourier space evaluation)

$$Z(\beta) = \int [\mathcal{D}q]_\beta \exp\Big(\int_0^\beta d\tau \, \tfrac{1}{2} q(\tau)(-\partial_\tau^2+\omega^2) q(\tau) + \underbrace{C_s}_{\text{surface-term}}\Big)$$

- Scale: $\tau \equiv \beta\eta$ & $q \equiv \beta^{1/2}\tilde{q}$; $[q] = [\tau]^{1/2} = [\omega]^{-1/2}$
  - $-\int_0^\beta d\tau \, \frac{1}{2} q(\tau)(-\partial_\tau^2+\omega^2) q(\tau) \longrightarrow \int_0^1 d\eta \, \frac{1}{2}\tilde{q}(\eta)(-\partial_\eta^2 + (\beta\omega)^2)\tilde{q}(\eta)$
  - Since $k_{N\tau} = \left(\frac{m}{2\pi i \epsilon_{N\tau}}\right)^{1/2}$, where $\epsilon_{N\tau} = \frac{\beta}{N+1}$ $\Rightarrow$ $k_{N\eta} = \beta^{-1/2} k_{N\tau}$

$$\Rightarrow \quad [\mathcal{D}q]_\beta = \prod_i^{N+1} k_{N\tau} dq_i = \prod_i^{N+1} k_{N\eta} d\tilde{q}_i = [\mathcal{D}\tilde{q}]_1 \quad \text{invariant}$$

(N+1' th integration is the Trace over q).

- $\tilde{q}(\eta) \equiv \sum_n \hat{\tilde{q}}_n e^{-i\tilde{\omega}_n \eta}$, $\tilde{\omega}_n \equiv 2\pi n$ complex $\hat{\tilde{q}}_n = \hat{\tilde{q}}^*_{-n}$ because $\tilde{q} \in \mathbb{R}$.
  - $$\int_0^1 d\eta \, \frac{1}{2}\tilde{q}(\eta)(-\partial_\eta^2+(\beta\omega)^2)\tilde{q}(\eta) = \sum_{n,m} \frac{1}{2}\hat{\tilde{q}}_n \hat{\tilde{q}}_m (\tilde{\omega}_n^2 + (\beta\omega)^2) \underbrace{\int_0^1 d\eta \, e^{i(\tilde{\omega}_n+\tilde{\omega}_m)\eta}}_{\delta_{n,-m}}$$
    $$= \sum_n \frac{1}{2}(\tilde{\omega}_n^2 + (\beta\omega)^2)|\hat{\tilde{q}}_n|^2 = \sum_n \frac{1}{2}(\tilde{\omega}_n^2+(\beta\omega)^2)\left[(\mathrm{Re}\,\hat{\tilde{q}}_n)^2 + (\mathrm{Im}\,\hat{\tilde{q}}_n)^2\right]$$
  - $q_i \equiv q(\eta_i) = \sum_n e^{-i\tilde{\omega}_n \eta_i} \hat{\tilde{q}}_n \equiv \sum_n U_{in} \hat{\tilde{q}}_n$ Unitary transformation

$$\Rightarrow \prod_i dq_i = \det\left(\frac{\partial q_i}{\partial \hat{q}_n}\right)\prod_n d\hat{q}_n = \det(U)\prod_m d\hat{q}_n$$

$$= \prod_{n\geq 0} d\hat{q}_n d\hat{q}_n^* = \prod_{n\geq 0} d\,\mathrm{Re}\,\hat{q}_n \, d\,\mathrm{Im}\,\hat{q}_n$$

we are a bit sloppy with the zero mode here ...

After these preliminaries we can evaluate $Z(\beta)$:

$$Z(\beta) = C \det\Big(\overbrace{\left(-\partial_\eta^2 + (\beta\omega)^2\right)}^{\Delta_\eta^{-1}}\Big)^{-1} = C\, e^{-\mathrm{Tr}\log \Delta_\eta^{-1}} \quad ; \quad C = \lim_{N\to\infty}\left(\sqrt{\pi}\, e^{C(\epsilon)}\right)^N$$

$$\Rightarrow -\log Z(\beta) = \frac{1}{2}\mathrm{Tr}\log\Delta_\eta^{-1} + C' = \frac{1}{2}\sum_n \log\left(\tilde{\omega}_n^2 + (\beta\omega)^2\right) + C'$$

$$= \int_1^{\beta\omega} d\theta \sum_n \frac{\theta}{\theta^2 + \tilde{\omega}_n^2} + \frac{1}{2}\sum_n \log(1+\tilde{\omega}_n) + C'$$

$$= \int_0^\beta d\omega'\, \omega' \beta\, \Delta_0(\tau=0,\omega') + C''$$

drop

$$= \frac{1}{2}\int_0^{\beta\omega} d\theta\, \left(1 + 2 n_{BE}(\theta)\right) = \frac{\beta\omega}{2} + \log\left(1 - e^{-\beta\omega}\right) \quad \square$$

We were careful not to drop any β-dependent constants on the way. This means that we know $Z(\beta)$ and $Z[\beta;j]$ up to an overall constant. Usually that is good enough as all correlation functions evaluated from $Z[\beta;j]$ are independent of such constant.

However $P = \frac{1}{\beta V}\log Z$ would seem to depend on C'' : $\delta P'' = \frac{C''}{\beta V}$. Also, we know from our quantum statistics calculation that $C'' = 0$. How does this result emerge from a more rigorous calculation?

- Most straightforward way is to perform PI consistently in the direct space. This is the topic of Exercise 1.6.

Laine & Vuorinen - trick.

The idea is to evaluate the infinite coefficient in the limit $\omega = 0$, where Z can be evaluated also without PI. However, zero mode becomes unbounded in this limit and needs special care. Hence one writes:

$$q(\tau) = \frac{1}{\beta}\hat{q}_0 + \frac{1}{\beta}\sum_{n\neq 0}\hat{q}_n e^{-i\omega_n\tau}$$

$$\Rightarrow (-\partial_\tau^2 + \omega^2)\, q(\tau) = \omega^2 \hat{q}_0 + \sum_{n\neq 0}(\omega_n^2+\omega^2)\hat{q}_n e^{-i\omega_n\tau}.$$

Then using $\int_0^\beta d\tau\, e^{i(\omega_n+\omega_m)\tau} = \beta\,\delta_{mn}$, one finds

$$S_E = \int d\tau\, q(\tau)\,(-\partial_\tau^2+\omega^2)\,q(\tau) = \frac{1}{2\beta}\omega^2\tilde{q}_0^2 + \frac{1}{2\beta}\underbrace{\sum_{n\neq 0}}_{=2\sum_{n>0}}(\omega_n^2+\omega^2)$$

Performing Gaussian integrals we get

$$Z(\beta) = C(\beta)\,\underbrace{\frac{\sqrt{2\pi\beta}}{\omega}}_{\text{zero-mode contribution}}\prod_{n=1}^{\infty}\frac{1}{\omega_n^2+\omega^2}$$

independent of ω

all ω-indep. terms for non-zero modes absorbed to $C(\beta)$. Contains also surface terms

The goal is to determine $C(\beta)$ in the $\omega \to 0$ limit. But zero mode blows out here, because it contributes a gaussian integral $\int dq_0\, e^{-\frac{1}{2\beta}\omega^2 q_0^2}$

To circumvent this we consider a regulated system, where the average variation over $\tau \in [0,\beta]$ is restricted to some range Δq. Indeed:

$$\int_0^\beta d\tau\, q(\tau) = \int d\tau \left(\frac{1}{\beta} \hat{q}_0 + \frac{1}{\beta} \sum_{n\neq 0} \hat{q}_n e^{-i\omega_n \tau} \right) = \hat{q}_0$$

So that $\frac{\hat{q}_0}{\beta} = \langle q(\tau) \rangle$. The constraint only restricts the zero mode, which now gives a contribution $\beta \Delta q$. So we have:

$$Z_{reg}(\beta, \omega=0) = C(\beta)\, \beta \Delta q \prod_{n=1}^{\infty} \frac{1}{\omega_n^2}$$

On the other hand:

$$Z_{reg}(\beta,\omega=0) = \int_{\Delta q} dq \langle q | e^{-\frac{1}{2}\beta \hat{q}^2} | q \rangle$$

$\ulcorner \hat{p}^2$

$$= \int_{\Delta q} dq \int \frac{dp}{2\pi} \langle q | e^{-\frac{1}{2}\beta \hat{p}^2} | p \rangle \langle p | q \rangle$$

$$= \int_{\Delta q} dq \int \frac{dp}{2\pi} e^{-\frac{1}{2}\beta p^2} \underbrace{|\langle p | q \rangle|^2}_{=1} = \frac{\Delta q}{\sqrt{2\pi\beta}}$$

Combining the two results we get

$$\boxed{C(\beta) = \frac{1}{\sqrt{2\pi\beta^3}} \prod \omega_n^2}$$

which then gives

$$Z(\beta) = \frac{1}{\beta\omega} \prod_{n>0} \frac{\omega_n^2}{\omega_n^2 + \omega^2} = \frac{1}{\beta\omega} \prod_{n>0} \frac{1}{\left(1 + \left(\frac{\beta\omega}{2\pi n}\right)^2\right)} = \frac{1}{2\sinh \frac{1}{2}\beta\omega}$$

where one at last used $\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \frac{\sinh \pi x}{\pi x}$.

(This of course hides some of the burden of proof.. Exercise 1.7)

2. Free bosonic field theory

We will study simple Klein-Gordon field and then move to complex scalar field with nonvanishing charge, and finish with a study of Bose condensation

2.1. Klein-Gordon field

Consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 \quad \Rightarrow \quad \pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}$$

Canonical quantization

$$\boxed{\begin{aligned} &[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \\ &[\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)] = [\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = 0 \end{aligned}} \qquad (2.1)$$

Field operator

density of the states

$$\hat{\phi}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} D_p \left(\hat{a}_{\vec{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}} e^{ip\cdot x} \right)$$

$$\hat{\pi}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} D_p \left(-i\omega_{\vec{p}}\, \hat{a}_{\vec{p}} e^{-ip\cdot x} + i\omega_{\vec{p}}\, \hat{a}^\dagger_{p} e^{ip\cdot x} \right)$$

Canonical rules (2.1) imply that $[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}] = [\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = 0$, while

norm

$$\boxed{[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = C_{\vec{p}}\, \delta^3(\vec{p}-\vec{p}')}$$

where $\boxed{D_{\vec{p}}^2 C_{\vec{p}} = \frac{(2\pi)^3}{2\omega_p}}$. Beyond this restriction one is free to choose $C_{\vec{p}}$ at will. It is interesting to keep D_p & $C_{\vec{p}}$ free for now, and write down the Hamiltonian

$$\frac{\hat{H}}{V} = \frac{1}{V}\int d^3x\,(\hat{\pi}\dot{\hat{\phi}} - \hat{\mathcal{L}}) \overset{HT}{=} \int \frac{d^3p}{(2\pi)^3}\frac{2\omega_p^2 D_p^2}{V}\Big(\hat{a}_p^\dagger \hat{a}_p + \frac{1}{2}\underbrace{[\hat{a}_p,\hat{a}_p^\dagger]}_{= C_p\delta^3(0) = \frac{C_p V}{(2\pi)^3}}\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{2\omega_p^2 D_p^2}{V}\hat{a}_p^\dagger \hat{a}_p + \frac{\omega_p^2 D_p^2 C_p}{(2\pi)^3}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{2\omega_p^2 D_p^2}{V}\hat{a}_p^\dagger \hat{a}_p + \frac{\omega_p}{2}\right)$$

$\hookrightarrow$ Vacuum part, independent of choice for C_p & D_p

If we want to set $\frac{\hat{H}}{V} = \int \frac{d^3p}{(2\pi)^3}\,\omega_p\left(\hat{a}_p^\dagger \hat{a}_p + \frac{1}{2}\right)$, we need to choose

$$\frac{2\omega_p^2 D_p^2}{V} = \omega_p \Rightarrow D = \sqrt{\frac{V}{2\omega_p}} \quad \& \quad C_p = \frac{(2\pi)^3}{V}$$

$$\Rightarrow \langle p|p\rangle = \langle 0|a_p a_p^\dagger|0\rangle = C_p\delta(0) = \frac{(2\pi)^3}{V}\frac{V}{(2\pi)^3} = 1$$

This is the normalization used above with SHO. In QFT one often uses a covariant normalization $C_p = (2\pi)^3 2\omega_p$, which then implies $D_p = \frac{1}{2\omega_p}$.
With this normalization

$$\Rightarrow \quad \bullet \quad \hat{\phi}(\vec{x},t) = \int \underbrace{\frac{d^3p}{(2\pi)^3 2\omega_p}}^{L\text{-invariant}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

$$\bullet \quad [a_p, a_{p'}^\dagger] = (2\pi)^3 2\omega_p\,\delta^3(p-p') \quad ; \quad \langle p|p\rangle = \overbrace{2\omega}^{\rho} V = N_p^{vac}$$

$$\bullet \quad \hat{H} = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2}a_p^\dagger a_p + V\frac{\omega_p}{2}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3 2\omega_p}\Big(\omega_p a_p^\dagger a_p + \overbrace{2\omega_p V}^{N_p^{vac}}\frac{\omega_p}{2}\Big)$$

usually $V \to 1$ (unit volume)

Partition function & generating functional

Now

$$Z(\beta) = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \int [\mathcal{D}\phi_a] \langle \phi_a(\vec{x}) | e^{-\beta\hat{H}} | \phi_a(\vec{x})\rangle$$

$$= \int [\mathcal{D}\phi]_{\phi(\vec{x},0) = \phi(\vec{x},\beta) = \phi_a(\vec{x})} \, e^{-\int_0^\beta d\tau \int d^3x \, \mathcal{L}_E(\phi, \partial_\mu\phi)}$$

where we formally made the same replacements as with SHO partition function:

$$\frac{i}{\hbar}\int dt\, d^3x \, \frac{1}{2}\left((\partial_t\phi)^2 - (\nabla\phi)^2 - m^2\phi^2\right)$$

$$\xrightarrow{t\to -i\tau} -\underbrace{\int_0^\beta d\tau \int d^3x}_{\equiv \int_{X_E^\beta}} \, \underbrace{\frac{1}{2}\left((\partial_\tau\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right)}_{\equiv \mathcal{L}_E(\phi, \partial_\mu\phi)} = -\int_{X_E^\beta} \mathcal{L}_E[\phi, \partial_\mu\phi] \equiv -S_E[\phi]$$

Alternatively, we can use the Hamiltonian form

$$Z(\beta) = \int [\mathcal{D}\phi]_\beta [\mathcal{D}\pi] \, e^{\int_{X_E^\beta}\left(i\pi(\partial_\tau\phi) - \mathcal{L}_E(\phi,\pi)\right)}$$

where we used $\pi = \partial_t\phi = i\partial_\tau\phi$. One can also write generating functional:

$$Z[\beta, j] = \int [\mathcal{D}\phi]_\beta \exp\left[-S_E(\phi) + \int_{X_E^\beta} j\phi\right]$$

Whence

$$= \int [\mathcal{D}\phi]_\beta \exp\left[-\int_{X_E X_E'} \phi(x_E)\, \Delta_0^{-1}(x_E, x_E')\, \phi(x_E') + \int_{X_E} j(x_E)\, \phi(x_E)\right]$$

⋮ discretize → matrix $x^T A x$

$$= Z(\beta) \exp\left(-\frac{1}{2}\int_{X_E^\beta X_E'} j(x_E)\, \Delta_0(x_E - x_E')\, j(x_E')\right).$$

Propagator

$$\Delta_0(\tau,\vec{x}) = \frac{1}{Z(\beta)} \frac{\delta^2 Z[\beta,j]}{\delta j(0)\,\delta j(\tau,\vec{x})}\Big|_{j=0} = \langle \mathcal{T}[\hat{\phi}(\tau,\vec{x}), \hat{\phi}(0)] \rangle_\beta$$

This propagator is also the Greens function for equation

$$(-\partial_\tau^2 - \nabla^2 + m^2)\,\Delta_0(\tau,\vec{x}) = \delta(\tau)\,\delta^3(\vec{x})$$

Fourier space $\Delta(\tau,\vec{p}) \equiv \int_{\vec{p}} \Delta(\tau,\vec{x})\, e^{i\vec{p}\cdot\vec{x}}$, gives

$$(-\partial_\tau^2 + \omega_{\vec{p}}^2)\,\Delta_0(\tau,\vec{p}) = \delta(\tau)$$

This is the same equation solved for SHO earlier with $\omega \to \omega_{\vec{p}}$. So we know the result

$$\boxed{\Delta_0(\omega_n,\vec{p}) = \frac{1}{\omega_n^2 + \omega_{\vec{p}}^2}} \quad ; \quad \omega_n = 2\pi n T.$$

and also $\Delta_0(\tau,\vec{p}) = \frac{1}{2\omega_{\vec{p}}}\left((1+n_{BE}(\omega_{\vec{p}}))\, e^{-\omega_{\vec{p}}|\tau|} + n_{BE}(\omega_{\vec{p}})\, e^{\omega_{\vec{p}}|\tau|} \right).$

Evaluating KG-free Z.

$[d\tau\, d^3x] = \beta L^3 \quad [S_E] = \beta^0 L^0$

$\Rightarrow [(\partial_\tau\phi)^2] = \beta^{-2}[\phi]^2 \equiv \beta^{-1}L^{-3} \Rightarrow [\phi] = \beta^{1/2}L^{-3/2}$

Introducing

$$\phi(\vec{x},t) = \beta^{1/2} \sum_n \int \frac{d^3p}{(2\pi)^3}\, \hat{\phi}_n(\vec{p})\, e^{-i(\omega_n\tau - \vec{p}\cdot\vec{x})} \qquad [\hat{\phi}] = L^{3/2}$$

$$= \beta^{3/2} \sum\!\!\!\!\!\!\int_B \hat{\phi}_n(\vec{p})\, e^{-i(\omega_n\tau - \vec{p}\cdot\vec{x})}.$$

$$\Rightarrow S_E[\hat{\phi}(\vec{x},\tau)] = \beta \sum_{n,m} \int_{\vec{p},\vec{p}'} \hat{\phi}_n(\vec{p}) \left[\omega_m^2 + \vec{p}^2 + m^2\right] \hat{\phi}_m(p) \underbrace{\int_0^\beta d\tau \, e^{-i(\omega_n+\omega_m)\tau}}_{\beta \delta_{n,-m}} \underbrace{\int d^3x \, e^{+i(\vec{p}+\vec{p}')\cdot x}}_{(2\pi)^3 \delta^3(\vec{p}+\vec{p}')}$$

$$= \sum_n \int_{\vec{p}} \left(\tilde{\omega}_n^2 + (\beta\omega_p)^2\right) |\hat{\phi}_n(\vec{p})|^2$$

Again, measure is invariant in F-transform, which is unitary, and we get directly

$$\log Z(\beta) = \log \Big\{ \int \underbrace{[\mathcal{D}\hat{\phi}]_\beta}_{[\mathcal{D}\mathrm{Re}\,\phi][\mathcal{D}\mathrm{Im}\,\phi]_{n\geq 0}} \exp\Big[-\tfrac{1}{2}\beta \sum_{n,\vec{p}} \beta^2(\omega_n^2+\omega_{\vec{p}}^2) \underbrace{|\phi_{\vec{p}}|^2}_{\mathrm{Re}\,\phi^2+\mathrm{Im}\,\phi^2}\Big] \Big\}$$

$$= \log \Big\{ \prod_{n,\vec{p}} \int d|\phi_{n,p}| \exp\Big[-\tfrac{1}{2}\beta^2(\omega_n^2+\omega_{\vec{p}}^2)|\phi_{\vec{p}}|^2\Big]$$

$$= \frac{1}{2} \log \prod_{n,\vec{p}} \frac{2\pi}{\beta^2(\omega_n^2+\omega_{\vec{p}}^2)} = -\frac{1}{2} \sum_{n,\vec{p}} \log\left(\beta^2(\omega_n^2+\omega_{\vec{p}}^2)\right)$$

$$= V \int \frac{d^3p}{(2\pi)^3} \left(-\frac{\beta\omega_p}{2} - \log\left(1-e^{-\beta\omega_p}\right)\right) = \underline{\frac{V}{T} J_T^-(m,T)}$$

$$\Rightarrow \boxed{P = \frac{T}{V} \log Z = J_T^-(m,T)}$$

While we again derived this result schematically in "QFT"-fashion, we know from our earlier work with SHO PI that this is an exact result.

Noninteracting complex scalar field, with action

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

Decomposing $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \Rightarrow \mathcal{L} = \sum_{i=1}^{2} \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{m^2}{2}\phi_i^2$. So we already know that $Z(\beta)_\phi = [Z(\beta)_{KG}]^2$, if no charge. However, we have continuous symmetry

$$\phi \to e^{i\alpha}\phi \quad ; \quad \mathcal{L} \to \mathcal{L}$$

$\overset{\text{Noether}}{\Longrightarrow}$ $\exists$ conserved current & charge.

$\delta S = 0 \Rightarrow \frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}$

& complex conj

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi}\delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta(\partial_\mu \phi) + h.c$$

$$= \alpha \cdot \left(\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} i\phi\right) - h.c\right) = 0$$

$\Rightarrow \partial_\mu j^\mu = 0$, where $j^\mu \equiv i[\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi]$

irrelevant, depends on def. of charge

$\longrightarrow$ if $\phi \simeq e^{-ip\cdot x} \Rightarrow j^\mu \propto -p^\mu$

Charge:

$$Q = \int d^3x\, j^0(x) = \int d^3x \left(i(\phi\pi - \phi^*\pi^*)\right)$$

where $\pi = \delta\mathcal{L}_M/\delta(\partial_t\phi) = \partial^t \phi^* = \partial_t \phi^* = i\partial_t\phi^*$.

Note that in component notation $\pi_i = \partial_t \phi_i \Rightarrow \pi = \frac{1}{\sqrt{2}}(\partial_t\phi_1 - i\partial_t\phi_2) = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)$

$\pi^* = \frac{1}{\sqrt{2}}(\partial_t\phi_1 + i\partial_t\phi_2) = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2)$

$$\Rightarrow Q = \int d^3x\, \frac{i}{2}\left((\phi_1 + i\phi_2)(\pi_1 - i\pi_2) - h.c\right) = \int d^3x (\phi_1\pi_2 - \phi_2\pi_1)$$

Partition function with $\mu \neq 0$

$$\mathcal{Z}(\beta,\mu) = \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]$$

$$= \int [\mathcal{D}\pi\mathcal{D}\pi^*][\mathcal{D}\phi\mathcal{D}\phi^*]_\beta \exp\left\{\int_{X_E^\beta}\left(\pi\dot{\phi}+\pi^*\dot{\phi}^* - \mathcal{H} + \mu Q\right)\right\}$$

we still kept the notation $\dot{\phi} \equiv \partial_t \phi$ despite moving to Euclidean space

Here $\mathcal{H} = \pi\pi^* + \nabla\phi\nabla\phi^* + m^2\phi\phi^*$

$Q = i\phi\pi - i\phi^*\pi^*$

Combining all terms containing π we get

$$-\pi\pi^* + \pi\dot{\phi} + \pi^*\dot{\phi}^* + i\mu(\phi\pi - \phi^*\pi^*)$$

$$= -\pi\pi^* + \pi(\dot{\phi}+i\mu\phi) + \pi^*(\dot{\phi}^*-i\mu\phi^*)$$

$$= -\underbrace{(\pi-\dot{\phi}^*+i\mu\phi^*)}_{\equiv\pi'}\underbrace{(\pi^*-\dot{\phi}-i\mu\phi)}_{\pi'^*} + (\dot{\phi}+i\mu\phi)(\dot{\phi}^*-i\mu\phi^*)$$

After shift $\pi \equiv \pi' + \dot{\phi}^* - i\mu\phi^*$, $\pi^* \equiv \pi' + \dot{\phi} + i\mu\phi$ the π'-integrals can be performed and give just a constant. We are left with

Also noting that $\dot{\phi} = i\partial_\tau\phi$, we are then left with

$$\boxed{\mathcal{Z}(\beta,\mu) = \int [\mathcal{D}\phi\,\mathcal{D}\phi^*] \exp\left\{\int_{X_E^\beta}[(\partial_\tau+\mu)\phi][(\partial_\tau-\mu)\phi^*] + |\nabla\phi|^2 + m^2|\phi|^2\right\}}$$

Now move to F-space

$$\phi(\tau,\vec{x}) \equiv \beta^{3/2} \SumInt \hat{\phi}_n(p)\, e^{-i\omega_n\tau + i\vec{p}\cdot\vec{x}},$$

one finds first from periodicity requirement $\phi(\tau+\beta,\vec{x}) = \phi(\tau,\vec{x}) \Rightarrow \omega_n = 2\pi n T$.
Also : $\partial_\tau\phi \longrightarrow -i\omega_n\phi_n \qquad \partial_\tau\phi^* \longrightarrow i\omega_n\phi_n$, whence:

$$Z(\beta,\mu) = \int [D\phi_n D\phi_m^*] \exp\left\{\beta \SumInt \phi_n^*(p)\left(\beta^2(\omega_n+i\mu)^2 + m^2 + \vec{p}^2\right)\phi_n(p)\right\}$$

We can read off the propagator

$$\boxed{\Delta_0(\omega_n,\vec{p},\mu) = \frac{1}{(\omega_n+i\mu)^2 + \omega_{\vec{p}}^2}}$$

So chemical potential appears as a shift of Matsubara-frequency $i\omega_n \longrightarrow i\omega_n - \mu$. Periodicity of $\phi(\tau,\vec{x})$ then implies $\omega_n = 2\pi n T$.

KMS-condition Derivation is identical to SHO. Define $\hat{K} = \hat{H} - \mu\hat{N}$, so that grand canonical $\hat{\rho} = e^{-\beta\hat{K}}$. Now assume $0 < \tau < \beta$:

$$\Delta_\phi(\tau,\vec{x}) = \frac{1}{\mathrm{Tr}\hat{\rho}} \mathrm{Tr}\left[\hat{\rho}\, T(\hat{\phi}(\tau,\vec{x})\,\hat{\phi}(0))\right]$$

$$\overset{\tau>\beta}{=} \frac{1}{\mathrm{Tr}\hat{\rho}} \mathrm{Tr}\left[e^{-\beta\hat{K}}\,\hat{\phi}(\tau,\vec{x})\, e^{\beta\hat{K}} e^{-\beta\hat{K}}\,\hat{\phi}(0)\right]; \qquad e^{-\beta\hat{K}}\hat{\phi}(\tau,\vec{x})\, e^{\beta\hat{K}}$$

$$= e^{\mu\beta\hat{N}} e^{-\beta\hat{H}}\hat{\phi}(\tau,\vec{x})\, e^{\beta\hat{H}} e^{-\mu\beta\hat{N}}$$

$$= \frac{1}{\mathrm{Tr}\hat{\rho}} e^{\beta\mu}\, \mathrm{Tr}\left(e^{-\beta\hat{K}}\hat{\phi}(0)\,\hat{\phi}(\tau-\beta,\vec{x})\right) \qquad = e^{\mu\beta\hat{N}}\hat{\phi}(\tau-\beta,\vec{x})\, e^{-\mu\beta\hat{N}}$$

$$= e^{\beta\mu}\hat{\phi}(\tau-\beta,\vec{x})$$

$$= \frac{1}{\mathrm{Tr}\hat{\rho}} e^{\beta\mu}\, \mathrm{Tr}\left[\hat{\rho}\, T(\hat{\phi}(\tau-\beta)\,\hat{\phi}(0))\right] = e^{\beta\mu}\,\Delta_\phi(\tau-\beta,\vec{x})$$

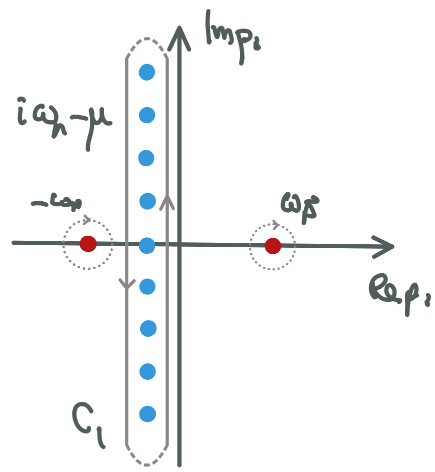
If one only had this information to go by, one could now introduce Fourier transformation

$$\Delta_\beta(\tau,\vec{x}) = \sum\!\!\!\!\!\!\int \Delta_n(\omega,\vec{p})\, e^{-ip_0\tau + i\vec{p}\vec{x}}$$

Then imposing the KMS-condition gives: (without solving Δ explicitly)

$$1 = e^{\beta\mu} e^{ip_0\tau} \quad \Rightarrow \quad p_0 = 2\pi n T + i\mu$$

Evaluating $Z(\beta,\mu)$: Chemical potential poses but a minor complication. Now already with some experience we may write



$$P = \frac{1}{\beta V}\log Z(\beta,\mu) = \frac{1}{\beta V}\operatorname{Tr}\log\Delta_\beta$$

$$= -\frac{1}{\beta V}\sum_{n,\vec{p}} \log\left(\beta^2\left((\omega+i\mu)^2+\omega_{\vec{p}}^2\right)\right)$$

$$= -\frac{1}{\cancel{\beta}}\int_{\vec{p}}\int_0^{\omega_{\vec{p}}} d\omega' \underbrace{\sum_{n=-\infty}^{\infty}\frac{\cancel{2\omega'}}{(\omega_n+i\mu)^2+\omega'^2}}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{-1}{z^2-\omega^2}\,\frac{\beta}{e^{\beta(z+\mu)}-1}$$

$$= \frac{\beta}{2\omega'}\left(\frac{1}{e^{\beta(\omega'+\mu)}-1} + \frac{1}{e^{-\beta(\omega'-\mu)}-1}\right) = \frac{\cancel{\beta}}{\cancel{2\omega'}}\left(1+\sum_{\pm}\frac{1}{e^{\beta(\omega\pm\mu)}-1}\right)$$

$$= -\int_{\vec{p}}\int_0^{\omega_{\vec{p}}} d\omega'\left(1+\sum_{\pm}\frac{1}{e^{\beta(\omega\pm\mu)}-1}\right)$$

$$= \sum_{\pm}\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\omega_p}{2} - \frac{1}{\beta}\log\left(1-e^{-\beta(\omega_p\pm\mu)}\right)\right).$$

Bose condensation

If system has a charge, then it has conserved particle number. At high T all particles fit into available phase space. At very low T however, there may not be enough phase space & charge starts to accumulate to ground state, which has zero free energy.

We missed condensate above, when we moved from discretized $\vec{p}$ to continuous one. The correct way is to set

$$\phi(\tau,\vec{x}) = \underbrace{\xi e^{i\theta}}_{\text{complex condensate}} + \underbrace{\sum\!\!\!\!\!\!\int \phi_n(\vec{p})\, e^{-i\omega_n\tau + i\vec{p}\cdot x}}_{\text{fluctuating states}}$$

Using this, the evaluation of the previous section is corrected to

$$Z(\beta,\mu,\xi) = \int [\mathcal{D}\phi_{in\vec{p}}\, \mathcal{D}\phi^*_{in\vec{p}}] \exp\Big[\beta V(\mu^2 - m^2)\underbrace{\xi^2}_{\text{not integrated!}} - \sum\!\!\!\!\!\!\int \phi^*_n(p)\left((\omega_n + i\mu)^2 + \omega_{\vec{p}}^2\right)\phi_n(p)\Big]$$

$$\Rightarrow \frac{1}{V}\log Z(\beta,\mu,\xi) = \beta(\mu^2 - m^2)\underbrace{\xi^2}_{\text{what is this.}} - \sum_{\pm}\int_{\vec{p}}\left[\frac{\beta\omega_p}{2} + \log\left(1 - e^{-\beta(\omega\pm\mu)}\right)\right]$$

Treat ξ as a variational parameter, requiring

$$\frac{1}{V}\left(\frac{\partial \log Z}{\partial \xi}\right)_{\beta,\mu} = 2\beta(\mu^2 - m^2)\xi = 0 \;\Rightarrow\; \xi = 0 \text{ if } |\mu| \neq m.$$

So the condensate can only form if the free energy of the ground state

$$f_{gs} = \omega_{gs} - \mu = m - \mu = 0.$$

We now determine ξ from charge conservation.

$\Omega = -T \log Z$

$d\Omega = -SdT - PdV - Nd\mu$

$\Rightarrow N = -\frac{\partial \Omega}{\partial \mu} = T \frac{\partial \log Z}{\partial \mu}$

Denote $\quad q \equiv \frac{Q}{V} = -\frac{eN}{V} = -\frac{eT}{V}\frac{\partial \log Z}{\partial \mu}$

$$= -2e\mu\xi^2 - e\int_{\vec p}\left[\frac{1}{e^{\beta(\omega_{\vec p}-\mu)}-1} - \frac{1}{e^{\beta(\omega+\mu)}-1}\right]$$

$$\xrightarrow[\xi=0]{\text{high } T} -e\int_{\vec p}\left[\frac{1}{e^{\beta(\omega_{\vec p}-\mu)}-1} - \frac{1}{e^{\beta(\omega+\mu)}-1}\right] \equiv q(\mu,T)_{\text{particles}}$$

both get smaller for smaller T

and eventually $\rightarrow 0$ as $\beta \to \infty$ $(T\to 0)$

for $Q>0$ change sign of μ. $\longleftrightarrow$ (assume $Q>0 \Rightarrow$ need $\mu>0$.)

for a fixed β & ω_p. particle distribution increases if μ is increased.

$\Rightarrow$ to keep q fixed as T decreases must increase μ

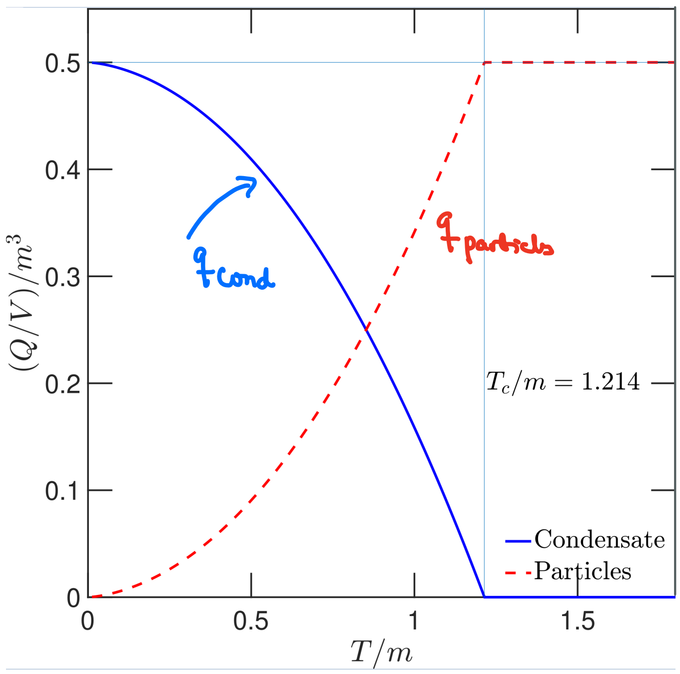
but this can be done only until $\mu = m$.

$\Rightarrow \exists$ lowest $T = T_c$ at which all particles are on fluctuating states.

We solve this by setting $\quad \boxed{q = -e\int_{\vec p}\left[\frac{1}{e^{\beta_c(\omega_{\vec p}-m)}-1} - \frac{1}{e^{\beta_c(\omega+m)}-1}\right]}$

$\Rightarrow T_c = T_c(q)$. must be solved numerically

At low temperatures $T<T_c$ we balance q by the condensate:



Here $\frac{Q}{V} \equiv 0{,}5\, m^3 \Rightarrow T_c \simeq 0{,}1214\, m$.

$$q = -2em\xi^2 + q(m,T)_{particles}$$

$$\Rightarrow \xi^2(T) = -\frac{1}{2me}\left(q - q(m,T)_{particles}\right)$$

$$\Rightarrow \xi(T) = \frac{1}{2m|e|}\left(|q| - |q(m,T)_{particles}|\right)$$

and finally: $\Rightarrow q_{cond} = -2em\xi^2 = q - q(m,T)_{particles}$.

One could attempt to treat condensate formally as a δ-function contribution to $f(p)$, setting (awqwardly)

$$\frac{1}{e^{\beta(\omega\pm\mu)}-1} \longrightarrow \pm\,\delta(\omega_p - m)\left(q - q(m,T)_{particles}\right)\theta\left(q - q(m,T)_{particles}\right) + \frac{1}{e^{\beta(\omega\pm\mu)}-1}$$

But can any IR enhanced distribution be thought of as a condensate?
No, if it is not associated with conserved charge.

3. Higher spin fields

Now move to Fermions & gauge fields. Still noninteracting..

Keywords: Anticommutation rules, grassmann numbers / fields, KMS-relation. gauge fixing, Abelian gauge field, non-abelian gf.

Fermions

Free Lagrangian

$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi$$

canonical momentum: $\pi = \frac{\delta \mathcal{L}}{\delta \dot{\Psi}} = i\Psi^\dagger$.

Canonical anticommutation rules

$$\{\hat{\Psi}_\alpha(t,\vec{x}), i\hat{\Psi}^\dagger_\beta(t,\vec{y})\} \equiv i\delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$$
$$\{\hat{\Psi}_\alpha(t,\vec{x}), \hat{\Psi}_\beta(t,\vec{y})\} \equiv 0$$
$$\{\hat{\Psi}^\dagger_\alpha(t,\vec{x}), \hat{\Psi}^\dagger_\beta(t,\vec{y})\} \equiv 0$$

Field operator:

$$\hat{\Psi}(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_s \left(a^s_{\vec{p}}\, u_s(p)\, e^{-ip\cdot x} + b^{s\,\dagger}_{\vec{p}}\, v_s(p)\, e^{ip\cdot x} \right)$$

Choosing normalization $u^\dagger(s,\vec{p})u(s',\vec{p}) = v^\dagger(s,\vec{p})v(s',p) = 2\omega_p\delta_{ss'}$ the canonical commutation relations imply

$$\{a^s_{\vec{p}}, a^{s'\,\dagger}_{\vec{p}'}\} = \{b^s_{\vec{p}}, b^{s'\,\dagger}_{\vec{p}'}\} = (2\pi)^3 2\omega_p\,\delta^3(\vec{p}-\vec{p}')$$

while other anticommutators vanish.

Hamiltonian function

$$\Rightarrow \quad H = \int d^3x\, \mathcal{H} = \int d^3x \left(\pi\dot\psi - \mathcal{L}\right)$$

$$= \int d^3x \left(\cancel{i\psi^\dagger\dot\psi} - \cancel{i\psi^\dagger\partial_t\psi} - i\bar\psi(+i\vec\gamma\cdot\nabla - m)\psi \right)$$

$$= \int d^3x\, \bar\psi(-i\vec\gamma\cdot\nabla + m)\psi = \int d^3x\, i\psi^\dagger\partial_t\psi$$

Inserting field operators into this expression one finds

$$\hat H = \int \frac{d^3p}{(2\pi)^3 2\omega_p}\left(\omega_{\vec p}\left(a_{\vec p}^{s\dagger}a_{\vec p}^{s} + b_{\vec p}^{s\dagger}b_{\vec p}^{s}\right) - \omega_{\vec p} \right)$$

Conserved charge

$\mathcal{L}_\psi$ is symmetric under $\psi \to e^{i\alpha}\psi$

$$\Rightarrow \quad \partial_\mu j^\mu = 0 \qquad \text{where} \quad j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)$$

$$\& \quad \frac{\partial Q}{\partial t} = 0 \qquad \text{for} \quad Q = \int d^3x\, j^0 = \int d^3x\, \psi^\dagger\psi \equiv \int d^3x\, \mathcal{Q}(t,\vec x)$$

We can now write $\quad (\mathcal{H} = \pi\dot\psi - \mathcal{L} \rightarrow i\pi\partial_\tau\psi - \mathcal{L})$

$$\mathcal{Z}(\beta,\mu) = \mathrm{Tr}\left(e^{-\beta(\hat H - \mu\hat Q)}\right)$$

$$= \int [\mathcal{D}\psi^\dagger\mathcal{D}\psi]_\beta \exp\Big(\int_{x_E^\beta} \overbrace{i\pi\partial_\tau\psi}^{-\psi^\dagger\partial_\tau\psi} - \mathcal{H}(\pi,\psi) + \mu\, \overbrace{\mathcal{Q}(\pi,\psi)}^{\psi^\dagger\psi = \bar\psi\gamma^0\psi} \Big)$$

$$= \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]_{\bar{\beta}} \exp\Big(-\int_{X_E^\beta} \bar{\psi}\,\underbrace{\big(\gamma^0(\partial_\tau-\mu) - i\vec{\gamma}\cdot\nabla + m\big)}_{\equiv \Delta_F^{-1}(\tau,\vec{x})}\psi \quad (A)$$

$[\;]_{\bar{\beta}}$ refers to fact that integration is over antiperiodic field configurations: $\psi(\beta,\vec{x}) = -\psi(0,\vec{x})$ and $\psi^\dagger(\beta,\vec{x}) = -\psi^\dagger(\beta,\vec{x})$. Here $\psi(x)$ is mathematically a Grassmann valued field.

Digression **Grassmann numbers**

Assume θ_i and θ_j are G-numbers $\Rightarrow$ $\boxed{\theta_i\theta_j \equiv -\theta_j\theta_i}$

$$\Rightarrow \theta_i^2 = 0 \qquad \Rightarrow \phi(\theta) = \underbrace{a + b}_{\text{c-numbers}}\theta \qquad \text{most general function}$$

Thus for example $e^{a\theta} = 1 + a\theta = \dfrac{1}{1-a\theta} = \frac{1}{2}(1+a\theta)^2 = \ldots$

Integration:

$$\int d\theta\,\phi(\theta) \equiv \int d\theta\,\phi(\theta+\xi)$$

$\leftarrow$ another G-number

$$\phi(\theta) = a + b\theta \Rightarrow \cancel{a\int d\theta} + \cancel{b\int d\theta\,\theta} = (\cancel{a} - b\xi)\int d\theta + \cancel{b\int d\theta\,\theta}$$

where we used $d\theta\,\xi = -\xi\,d\theta$. This must hold for all ξ $\Rightarrow$ $\int d\theta = 0$.

Furthermore we set $\int d\theta\,\theta \equiv 1$ $\Rightarrow$ $\int d\theta\,\phi(\theta) = \int d\theta\,(a+b\theta) = b$.

Grassmann integration is then formally equivalent to G-derivative

$$\frac{\partial}{\partial\theta}\phi(\theta) = \frac{\partial}{\partial\theta}(a+b\theta) = b.$$

Note also that $\int d\eta\, d\theta\, \theta\eta = 1$ due to anticommutation rule. Any odd permutation of $d\eta\, d\theta\, \theta\eta$ changes the sign of integration.

Now consider complex G-numbers

$$\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \quad \text{and} \quad \theta^* = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)$$

Then $\int d\theta\, d\theta^*\, \theta^*\theta = 1$ and in particular $\int d\theta^* d\theta\, \underbrace{e^{-\theta^* b\theta}}_{1-\theta^* b\theta = 1+\theta\theta^* b} = b$

Generalization:

$$\int \prod_{i=1}^{N} d\theta^* d\theta \; e^{-\sum_{j,k}\theta_j^* A_{jk}\theta_k} = \int \prod_{i=1}^{N} d\theta_i^* d\theta_i \frac{(-1)^N}{N!}\left(\sum_{j,k}\theta_j^* A_{jk}\theta_k\right)^N$$

$$= \int \prod_{i=1}^{N} d\theta_i d\theta_i^* \prod_{j=1}^{N} \theta_j^* (A_{jk_j}\theta_{k_j})$$

$$= \int \prod_{i=1}^{N} d\theta_i d\theta_i^* \sum_{\text{perm}} A_{1k_1}\cdots A_{Nk_N}\, \theta_1^*\theta_{k_1}\cdots\theta_N^*\theta_{k_N}$$

$$= \epsilon_{k_1\cdots k_N} A_{1k_1}\cdots A_{Nk_N} \equiv \det(A).$$

Fermionic path integral

Ket states $(\{\theta, \hat{a}\} \equiv 0$ etc. $)$
└ behave like G-numbers

- $|\theta\rangle \equiv e^{-\theta\hat{a}^\dagger}|0\rangle = (1-\theta\hat{a}^\dagger)|0\rangle \;\Rightarrow\; \hat{a}|\theta\rangle = \theta|\theta\rangle = \theta|0\rangle$
- $\langle\theta| \equiv \langle 0|e^{-a\theta^*} = \langle 0|(1-a^\dagger\theta^*) \;\Rightarrow\; \langle\theta|\hat{a}^\dagger = \langle\theta|\theta^* = \langle 0|\theta^*$

One then finds

$$\langle\theta'|\theta\rangle = \langle 0|(1-\hat{a}\theta'^*)(1-\theta\hat{a}^\dagger)|0\rangle = \langle 0|0\rangle + \langle 0|\hat{a}\theta'^*\theta\hat{a}^\dagger|0\rangle$$

$$= \qquad\qquad = 1+\theta'^*\theta = e^{\theta'^*\theta}$$

Unit operator, trace. With the θ-states then

- $$\int d\theta^* d\theta\, e^{-\theta^*\theta} |\theta\rangle\langle\theta| = \int d\theta^* d\theta\, (1-\theta^*\theta)(1-\theta a^\dagger)|0\rangle\langle 0|(1-a\theta^*)$$
$$= \int d\theta^* d\theta\, \left(-\theta^*\theta\,|0\rangle\langle 0| + \theta a^\dagger|0\rangle\langle 0| a\theta^*\right)$$
$$= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}.$$

- $$\int d\theta^* d\theta\, e^{-\theta^*\theta} \langle -\theta|\hat{A}|\theta\rangle = \int d\theta^* d\theta\, (1-\theta^*\theta)\langle 0|(1+\hat{a}\theta^*)\hat{A}(1-\theta\hat{a}^\dagger)|0\rangle$$
$$= \int d\theta^* d\theta\, \left(-\theta^*\theta\,\langle 0|\hat{A}|0\rangle - \langle 0|\hat{a}\,\theta^*\theta\, a^\dagger|0\rangle\right)$$
$$= \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle = \mathrm{Tr}\,\hat{A}.$$

Where we assumed that $[\theta,\hat{A}] = 0$ etc. Eg. $\hat{H} \propto \hat{a}^\dagger\hat{a}$, *

Tr corresponds to antiperiodic PI over θ weighted by $e^{-\theta^*\theta}$.

Path integral. for SFO

$$Z = \int d\theta^* d\theta\, e^{-\theta^*\theta}\langle -\theta| e^{-\beta\hat{H}}|\theta\rangle$$

$\int d\theta_i^* d\theta_i\, e^{-\theta_i^*\theta_i} |\theta_i\rangle\langle\theta_i|$

$$= \int d\theta^* d\theta\, e^{-\theta^*\theta}\langle -\theta| e^{-\epsilon\hat{H}}\, \mathbb{1}_N\, e^{-\epsilon\hat{H}} \cdots \mathbb{1}_1\, e^{-\epsilon\hat{H}}|\theta\rangle$$

Now
$$e^{-\theta_{i+1}^*\theta_{i+1}}\langle\theta_{i+1}| e^{-\epsilon\hat{H}(\hat{a}^\dagger,\hat{a})}|\theta_i\rangle = e^{-\theta_{i+1}^*\theta_{i+1}}\langle\theta_{i+1}|\theta_i\rangle e^{-\epsilon H(\theta_{i+1}^*,\theta_i)}$$
$$= \exp\left(-\theta_{i+1}^*\theta_{i+1} + \theta_{i+1}^*\theta_i - \epsilon H(\theta_{i+1}^*,\theta_i)\right)$$
$$= \exp\left(-\epsilon\left[\theta_{i+1}^*\left(\frac{\theta_{i+1}-\theta_i}{\epsilon}\right) + H(\theta_{i+1}^*,\theta_i)\right]\right)$$

The rightmost point is OK. Nothing special. The leftmost point requires some care because it has $-\theta$:

$$\int d\theta^* d\theta\, e^{-\theta^*\theta} \langle -\theta | e^{-\epsilon \hat{H}(a^\dagger, a)} \int d\theta_N^* d\theta_N\, |\theta_N\rangle$$

$$= \int d\theta^* d\theta \int d\theta_N^* d\theta_N\, e^{-\theta^*\theta} \langle -\theta | \theta_N \rangle\, e^{-\epsilon H(-\theta^*, \theta_N)}$$

$$= \int d\theta^* d\theta \int d\theta_N^* d\theta_N\, \exp\left(-\theta^*\theta - \theta^*\theta_N - \epsilon H(-\theta^*, \theta_N)\right)$$

$$= \int d\theta^* d\theta \int d\theta_N^* d\theta_N\, \exp\left[-\epsilon\left((-\theta^*)\frac{(-\theta) - \theta_N}{\epsilon} + H(-\theta^*, \theta_N)\right)\right]$$

OK

So, in total

$$\mathcal{Z} = \int d\theta^* d\theta\, e^{-\theta^*\theta} \langle -\theta | e^{-\beta \hat{H}} | \theta \rangle$$

$$\overset{\theta_0 \equiv \theta}{=} \int \prod_{i=0}^{N} d\theta_i^* d\theta_i\, \exp\left[-\epsilon \sum_{i=1}^{n} \theta_{i+1}^* \left(\frac{\theta_{i+1} - \theta_i}{\epsilon}\right) + H(\theta_{i+1}^*, \theta_i)\right]_{\substack{\theta_N = -\theta_0 \\ \theta_N^* = -\theta_0^*}}$$

$$\overset{N\to\infty}{=} \int [\mathcal{D}\theta^* \mathcal{D}\theta]_{\bar{\beta}}\, \exp\left[-\int_0^\beta d\tau \left(\theta^* \partial_\tau \theta + H(\theta^*, \theta)\right)\right]$$

Periodic path integral over Grassmann field $\theta = \theta(\tau)$.

Fermionic generating functional

$$\mathcal{Z}[\beta, \mu; \eta, \bar{\eta}] = \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]_{\bar{\beta}}\, \exp\left[-\int_{x_E^\beta} \left(\underbrace{\bar{\psi}\Delta_F^{-1}\psi + \bar{\psi}\eta + \bar{\eta}\psi}_{= (\bar{\psi} + \bar{\eta}\Delta_F)\Delta_F^{-1}(\psi + \Delta_F \eta) - \bar{\eta}\Delta_F \eta}\right)\right.$$

$$= \mathcal{Z}(\beta, \mu)\, \exp\left[\int_{x_E^\beta} \int_{x_E^{\beta\prime}} \bar{\eta}(x_E)\, \Delta_F(x_E - x_E')\, \eta(x_E')\right.$$

$$\Delta_F(\tau,\vec{x}) = \frac{1}{Z(\beta,\mu)} \frac{\delta^2 Z[\beta,\mu;\eta,\bar{\eta}]}{\delta\bar{\eta}(\tau,\vec{x})\,\delta\eta(0)}\bigg|_{\eta=\bar{\eta}=0}$$

$$= \frac{1}{Z(\beta,\mu)} \int [\mathcal{D}\bar{\Psi}\mathcal{D}\Psi]_\beta\, \Psi(\tau,\vec{x})\bar{\Psi}(0)\, e^{-\int_{xy}\bar{\Psi}\Delta_F^{-1}\Psi}$$

$$= \frac{1}{\mathrm{Tr}\hat{\rho}} \mathrm{Tr}\left(\hat{\rho}\,\mathcal{T}[\hat{\Psi}(\tau,\vec{x})\hat{\bar{\Psi}}(0)\right) = \underline{\langle \mathcal{T}(\hat{\Psi}(\tau,\vec{x})\hat{\bar{\Psi}}(0))\rangle}_\beta$$

Because Ψ and $\bar{\Psi}$ are anticommuting fields is $\mathcal{T}$ anti-time ordered product

$$\mathcal{T}[\hat{\Psi}(\tau,\vec{x})\hat{\bar{\Psi}}(0)] = \theta(\tau)\hat{\Psi}(\tau,\vec{x})\hat{\bar{\Psi}}(0) - \theta(-\tau)\hat{\bar{\Psi}}(0)\Psi(\tau,\vec{x})$$

Fermionic KMS-relation Again denote $\hat{K} = \hat{H} - \mu\hat{Q}$

$$\underline{\Delta_F(\tau,\vec{x})} = \frac{1}{\mathrm{Tr}\hat{\rho}} \mathrm{Tr}\left[\hat{\rho}\,\mathcal{T}[\hat{\Psi}(\tau,\vec{x}),\hat{\bar{\Psi}}(0)]\right] \qquad ; \; \tau > 0$$

$$= \frac{1}{\mathrm{Tr}\hat{\rho}} \mathrm{Tr}\Big(\underbrace{e^{-\beta\hat{K}}\hat{\Psi}(\tau,\vec{x})\, e^{\beta\hat{K}}}\, e^{-\beta\hat{K}}\hat{\bar{\Psi}}(0)\Big)$$

$$= \frac{e^{-\beta\mu}}{\mathrm{Tr}\hat{\rho}} \mathrm{Tr}\left[e^{-\beta\hat{K}}\hat{\bar{\Psi}}(0)\,\hat{\Psi}(\tau-\beta,\vec{x})\right] \qquad \tau-\beta<0$$

$$= -\frac{e^{-\beta\mu}}{\mathrm{Tr}\hat{\rho}} \mathrm{Tr}\left[e^{-\beta\hat{u}}\,\mathcal{T}(\hat{\Psi}(\tau-\beta,\vec{x})\hat{\bar{\Psi}}(0))\right] = \underline{-e^{-\beta\mu}\,\Delta_F(\tau-\beta,\vec{x})}$$

Here one word: $\hat{\mathcal{H}} = i\hat{\psi}^+\partial_t\hat{\psi} = -\hat{\psi}^+\partial_\tau\hat{\psi}$, $\hat{Q} = \hat{\psi}^+\hat{\psi}$ & $\{\hat{\psi},\hat{\psi}^+\} = \delta$

$$\Rightarrow \hat{\psi}_x\hat{H} = \int_y -\hat{\psi}_x\hat{\psi}_y^+\partial_\tau\hat{\psi}_y = \int_y \hat{\psi}_y^+\hat{\psi}_x\partial_\tau\hat{\psi}_y - \partial_\tau\hat{\psi}_x$$

$$= \int -\hat{\psi}_y^+(\partial_\tau\hat{\psi}_y)\hat{\psi}_x - \partial_\tau\hat{\psi}_x = (\hat{H}-\partial_\tau)\hat{\psi}_x$$

$$\Rightarrow \underline{e^{-\beta\hat{H}}\hat{\psi}(\tau,\vec{x})e^{\beta\hat{H}}} = e^{-\beta\hat{H}}e^{+\beta(\hat{H}-\partial_\tau)}\hat{\psi}(\tau,\vec{x}) = \underline{\hat{\psi}(\tau-\beta,\vec{x})}$$

- $$\hat{Q}\hat{\psi}_x = \int_y \hat{\psi}_y^\dagger\hat{\psi}_y\hat{\psi}_x = \int_x -\hat{\psi}_y^\dagger\hat{\psi}_x\hat{\psi}_y = \hat{Q}\hat{\psi}_x - \hat{\psi}_x = (\hat{Q}-1)\hat{\psi}_x$$

$$\Rightarrow \underline{e^{\beta\mu\hat{Q}}\hat{\psi}(\tau,\vec{x})e^{-\beta\mu\hat{Q}} = e^{-\beta\mu}\hat{\psi}(\tau,\vec{x})}.$$

$+\hat{\psi}_x$

One can now F-transform

$$\Delta_F(\tau,x) = \sum\!\!\!\!\!\!\int_p \Delta_F(p_0,\vec{p})\, e^{-ip_0\tau+i\vec{p}\cdot\vec{x}}$$

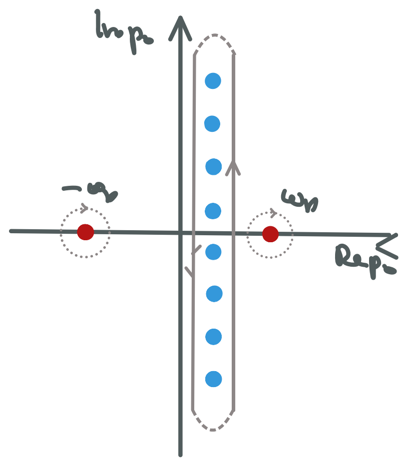
Then the KMS-condition implies

Fermionic Matsubara frequency $\equiv \omega_{Fn}$

$$-e^{-\beta\mu}e^{i\beta p_0} = 1 \quad \Rightarrow \quad \underline{p_0 = (2n+1)\pi T - i\mu}$$

$\Rightarrow \sum\!\!\!\!\!\!\int \to \sum\!\!\!\!\!\!\int_F$ where F-refers to fermionic frequencies. The Fermionic frequency requirement could have been seen also from antiperiodicity of ψ. Anyway one can read off from (A) on p. 35:

$$\Delta(p_0,p) = \frac{1}{\gamma^0(i(\omega_{Fn}+\mu)) + \vec{\gamma}\cdot\vec{p} + m}$$



In crucial difference to bosons, there are no Fermionic zero modes. That is, even the lightest thermalized fermionic excitation has a thermal mass πT.

$\Rightarrow$ at high T one can integrate fermions out from effective field theories that attempt to describe long-wave length modes (later).

Fermion gas pressure

$$P(\mu,\beta) = \frac{1}{\beta V}\log Z(\beta,\mu) = \frac{1}{\beta V}\log\int[\mathcal{D}\bar\psi_{n,\vec p}\,\mathcal{D}\psi_{n,\vec p}]\exp\left(\sum\!\!\!\!\!\!\!\int_F \bar\psi_n(p)\,\Delta_F^{-1}\,\psi_n(p)\right)$$

$\gamma^0 \times i\gamma^0$

$$= \frac{1}{\beta V}\log\prod_{n,\vec p}\det\left(\Delta_F^{-1}(n,\vec p)\right)$$

$\det AB = \det A \det B$

$\det i\gamma^0 = 1$

$$= \sum\!\!\!\!\!\!\!\int_F \log\det\left(-\omega_{Fn}+i\mu+i\vec\alpha\cdot\vec p+i\gamma^0 m\right)$$

$$= \sum\!\!\!\!\!\!\!\int_F \log\det\begin{pmatrix}-\omega_{Fn}+i\mu+i\vec\sigma\cdot\vec p & im\\ im & -\omega_{Fn}+i\mu-i\vec\sigma\cdot\vec p\end{pmatrix}$$

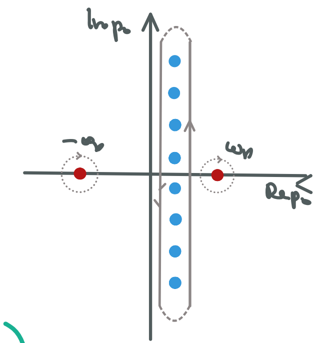
$$= 2\sum\!\!\!\!\!\!\!\int_F \log\left[(\omega_{Fn}-i\mu)^2+\vec p^{\,2}+m^2\right]$$

$$= 2\sum\!\!\!\!\!\!\!\int_F \int_0^{\omega_p}d\omega'\,\frac{2\omega'}{(\omega_{Fn}-i\mu)^2+\omega'^2}$$

$$= 2\int_{\vec p}\int_0^{\omega_p}d\omega'\,\cancel{2\omega'}\,\cancel{T}\underbrace{\sum_{n=-\infty}^{\infty}\frac{1}{(\omega_{Fn}-i\mu)^2+\omega'^2}}$$

$$= \frac{1}{2\pi i}\oint\frac{1}{\omega^2-z^2}\,\frac{-\beta}{e^{\beta(z-\mu)}+1}$$

$$= \frac{\cancel{\beta}}{\cancel{2\omega'}}\left(-\frac{1}{e^{\beta(\omega'-\mu)}+1}+\frac{1}{e^{-\beta(\omega'+\mu)}+1}\right)$$



$$= 2\int_{\vec p}\int_0^{\omega_p}d\omega'\left(1-\sum_{\pm}\frac{1}{e^{\beta(\omega'\pm\mu)}+1}\right)$$

spin $\rightarrow$

$$= 2\sum_{\pm}\int\frac{d^3p}{(2\pi)^3}\left(\frac{\omega_p}{2}+\frac{1}{\beta}\log\left(1+e^{\beta(\omega\pm\mu)}\right)\right)$$

$\uparrow$ particle-antiparticle

Abelian gauge theory

We quantize Gauge fields using the Faddeev-Popov method in path integral.
First note that

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$$E_i = F_{0i} = \partial_0 A_i - \partial_i A_0$$
$$B_i = \frac{1}{2}\epsilon_{ijk} F_{jk} = (\nabla \times \vec{A})_i$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant in gauge transformation

$$A_\mu \to A^\alpha_\mu = A_\mu + \partial_\mu \alpha \qquad \alpha \text{ arbitrary scalar field.}$$

All these configurations describe the same physics ($\vec{B}$ & $\vec{E}$) $\Rightarrow$ Huge degeneracy.

Canonical quantization: need to constrain to physical subspace
Path integral quantization: need to define PI for Z.

Indeed as a result of G-degeneracy, the partition function

$$Z_{\text{naive}} = \int [\mathcal{D}A]_\beta \, e^{\int_{x\beta} \mathcal{L}_A}$$

{ Moving to Euclidean space $A_\mu^E = (iA_0; \vec{A})$ of $A_\mu^M = (A_0; \vec{A})$ }

is not defined & one can not use this to create a generating function.
Problem is that writing

$$\Box \equiv -\delta_{\mu\nu}\partial^\mu\partial^\nu$$

$$\int_{X_E^P} \mathcal{L}_A = \int_{X_E^P} -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int_{X_E^P} A^\mu \left(-\Box_E \delta_{\mu\nu} - \partial_\mu\partial_\nu\right) A^\mu$$

$$-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$= -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right)$$
$$= \frac{1}{2}\left(A_\nu (\partial_\mu \partial^\mu) A^\nu - A_\nu \partial_\mu \partial^\nu A^\mu\right)$$
$$= \frac{1}{2} A_\nu \left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right) A_\mu$$
$$\longrightarrow \frac{1}{2} A_\nu^E \left(-\delta^{\mu\nu}[\Box_E] - \partial_E^\mu \partial_E^\nu\right) A_\mu^E$$

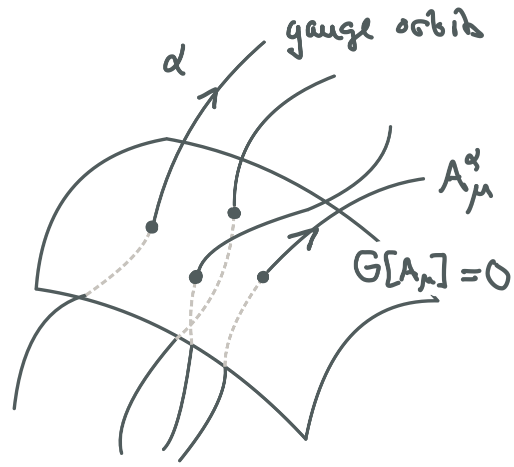
One can show that the operator

$$\Box_E \delta_{\mu\nu} - \partial_\mu\partial_\nu$$

does not have an inverse.

To fix this one has to impose a gauge fixing condition

$$\boxed{G[A_\mu^\alpha] \equiv 0}$$



which picks just one member of each gauge orbit.

respect the PI measure

But this must be done without biasing the PI- democracy. Each path is equally good. FP-method delicately extracts the G-dependence by introducing a unit operator:

$$1 = \Delta_{FP}[A_\mu] \int [\mathcal{D}\alpha]\, \delta(G[A_\mu^\alpha])$$

$$\int_{-\infty}^{\infty} dx\, \delta(x - x_0) = 1$$

$$\int dx\, \delta(f(x)) = \frac{1}{|f'(x)|_{f(x)=0}}$$

where

$$\Delta_{FP}[A_\mu] = \det\left(\frac{\delta G[A_\mu^\alpha]}{\delta\alpha}\right)$$

Faddeev-Popov functional determinant

First note that Δ_{FP} is gauge invariant $\qquad A^{\alpha\alpha'} = A^{\alpha'} - \partial\alpha$

$$\Delta^{-1}_{FP}[A^{\alpha'}_\mu] = \int [\mathcal{D}\alpha]\, \delta(G[A^{\alpha'\alpha}_\mu]) \qquad \text{integral over all gauges}$$

$$= \int [\mathcal{D}\alpha'\alpha]\, \delta(G[A^{\alpha'\alpha}_\mu]) \qquad \text{still integral over all gauges}$$

$$= \Delta_{FP}(A_\mu)$$

Then it is easy to see that

$$Z_{naive}(\beta) = \int [\mathcal{D}A]_\beta \Big(\underbrace{\Delta_{FP}[A_\mu]}_{\text{invariant}} \int [\mathcal{D}\alpha]\, \delta(G[A^\alpha_\mu]) \Big)\, e^{\overbrace{\int_{x_E^\beta} \mathcal{L}_A}^{\text{invariant}}}$$

invariant (for each A-integration range is $-\infty$ to ∞ & $\partial_\mu\alpha$ provides just a finite shift)

So, for each α we can transform $A^\alpha_\mu \to A_\mu$ everywhere

$$= \left(\int [\mathcal{D}\alpha]\right) \int [\mathcal{D}A]_\beta\, \Delta_{FP}[A_\mu]\, \delta(G[A_\mu])\, e^{\int_{x_E^\beta} \mathcal{L}_A}$$

FP det $\qquad$ constraint on shell

$$= \left(\int [\mathcal{D}\alpha]\right) Z_{phys}(\beta)$$

infinite gauge volume extracted

Eg.

$$\boxed{Z_{phys}(\beta) = \int [\mathcal{D}A]_\beta\, \Delta_{FP}[A_\mu]\, \delta(G[A_\mu])\, e^{\int_{x_E^\beta} \mathcal{L}_A}}$$

Precise form depends on gauge choice. (What is $G[A]$)

Black body radiation This is just a fancy name. We are again evaluating the free partition function. Here it is a little more interesting due to β-dependence.

Axial gauge choice $\boxed{A_3 \equiv 0}$ (special case of $\eta \cdot A = 0$ with $\eta \equiv (0,0,0,1)$)

In this case FP determinant is simple:

$$\Delta_{FP}[A^\mu] = \det\left(\frac{\delta(A_3 + \partial_3 \alpha)}{\delta \alpha}\right) = \det(\partial_3) \quad \not\propto A_\mu$$

Using this & the functional δ-constraint we get

$$Z(\beta) = \det(\partial_3) \int \mathcal{D}A_0 \mathcal{D}A_1 \mathcal{D}A_2 \, e^{\int_{X_E^\beta} \mathcal{L}_A \big|_{A_3=0}}$$

In zero T-theory we would drop $\det(\partial_3)$ as an irrelevant constant. Not here! Writing

$$\int_{X_E^\beta} \mathcal{L}_A = \frac{1}{2} \int_{X_E^\beta} A_\mu \left(-\delta_{\mu\nu} \Box_E - \partial_\mu \partial_\nu\right) A_\nu \Big|_{A_3=0} \qquad \sqrt{\partial_0 = \partial_\tau}$$

$$= \frac{1}{2} \int_{X_E^\beta} (A_0, A_1, A_2) \begin{pmatrix} -\Box - \partial_0\partial_0 & -\partial_0\partial_1 & -\partial_0\partial_2 \\ -\partial_1\partial_0 & -\Box - \partial_1\partial_1 & -\partial_1\partial_2 \\ -\partial_2\partial_0 & -\partial_2\partial_1 & -\Box - \partial_2\partial_2 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix}$$

Moving to F-space $\partial_0 \partial_i \to \omega_n p_i$, $\partial_i \partial_j \to -p_i p_j$; $\Box \to \omega_n^2 + \vec{p}^2$.

$e^{-i\omega_n \tau + i\vec{p}\cdot\vec{x}}$

$$\Rightarrow \log Z = \log\det(p_3) - \frac{1}{2} \log\det \begin{pmatrix} \vec{p}^2 & \omega_n p_1 & \omega_n p_2 \\ \omega_n p_1 & \omega_n^2 + \vec{p}^2 - p_1^2 & -p_1 p_2 \\ \omega_n p_2 & -p_1 p_2 & \omega_n^2 + \vec{p}^2 - p_2^2 \end{pmatrix}$$

Now

$$\det\begin{pmatrix} \vec p^{\,2} & \omega_n p_1 & \omega_n p_2 \\ \omega_n p_1 & \omega_n^2+\vec p^{\,2}-p_1^2 & -p_1p_2 \\ \omega_n p_2 & -p_1p_2 & \omega_n^2+\vec p^{\,2}-p_2^2 \end{pmatrix}$$

$$= \vec p^{\,2}\begin{vmatrix} \omega_n^2+\vec p^{\,2}-p_1^2 & -p_1p_2 \\ -p_1p_2 & \omega_n^2+\vec p^{\,2}-p_2^2 \end{vmatrix} - \omega_n p_1 \begin{vmatrix} \omega_n p_1 & -p_1p_2 \\ \omega_n p_2 & \omega_n^2+p^2-p_2^2 \end{vmatrix} + \omega_n p_2 \begin{vmatrix} \omega_n p_1 & \omega_n^2+\vec p^{\,2}-p_1^2 \\ \omega_n p_2 & -p_1p_2 \end{vmatrix}$$

$$= \vec p^{\,2}\left((\omega_n^2+\vec p^{\,2})(\omega_n^2+\overbrace{\vec p^{\,2}-p_1^2-p_2^2}^{p_3^2}) \right) - \omega_n p_1\left(\omega_n p_1(\omega_n^2+\vec p^{\,2}-\cancel{p_2^2}) + \cancel{\omega_n p_1 p_2^2}\right)$$

$$- \omega_n p_2\left(\omega_n p_2(\omega_n^2+\vec p^{\,2}-\cancel{p_1^2}) - \cancel{\omega_n p_1^2 p_2}\right)$$

$$= (\omega_n^2+p^2)\left(\vec p^{\,2}(\omega_n^2+p_3^2) - \omega_n^2p_1^2 - \omega_n^2p_2^2\right) = (\omega_n^2+\vec p^{\,2})(\omega_n^2p_3^2+p^2p_3^2) = p_3^2(\omega_n^2+\vec p^{\,2})^2$$

Thus we get

FP- determinant cancels this part

$$\Rightarrow \log Z = \frac{1}{2}\mathrm{Tr}\log \cancel{p_3^2} - \frac{1}{2}\mathrm{Tr}\log\left(\cancel{p_3^2}(\omega_n^2+p_i^2)^2\right) = -\mathrm{Tr}\log(\omega_n^2+\vec p^{\,2})$$

We recognize a familiar structure and write immediately:

2 phys. pol. states

$$P_\gamma = \frac{1}{\beta V}\log Z_\gamma = -2\int\frac{d^3p}{(2\pi)^3}\left(\cancel{\frac{|p|}{2}} + T\log\left(1-e^{-\beta|p|}\right)\right)$$

vacuum, drop

This can be in fact evaluated all the way (because $m_\gamma=0$)

$$\delta P_{\gamma T} = -2J_T^-(0) = \frac{T^4}{3\pi^2}\int_0^\infty dy\,\frac{y^3}{e^y-1} = \frac{T^4}{3\pi^2}\,3!\,\underbrace{\zeta(4)}_{\frac{\pi^4}{90}} = \frac{\pi^2}{45}T^4$$

It is customary to denote det-contribution by a vacuum loop:

$\log Z =$  $+$ 

gf

ghost kills extra longitudinal d.o.f

Photon propagator. Axial gauge can be cumbersome in PT. Covariant gauges can be induced by

$$G_\omega[A_\mu] = \partial^\mu A_\mu - \omega = 0$$

$$\Rightarrow \quad \Delta_{FP}[A_\mu] = \det\left(\frac{\delta\left(\partial^\mu(A_\mu + \partial_\mu\alpha) - \omega\right)}{\delta\alpha}\right) = \det(\partial^2)$$

Final trick is to do an integral over different choices of ω:

$$Z(\beta) = N_\xi \int [\mathcal{D}\omega]\, e^{-\frac{1}{2\xi}\int_{x_E^\beta}\omega^2} \int [\mathcal{D}A]_\beta\, \Delta_{FP}[A_\mu]\, \delta(G_\omega[A_\mu])\, e^{\int_{x_E^\beta}\mathcal{L}_E}$$

$$= N_\xi \det(\partial^2) \int [\mathcal{D}A]_\beta \exp\left[\int_{x_E^\beta} \underbrace{\left(\mathcal{L}_E - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)}_{\mathcal{L}_{\text{eff}}}\right]$$

Now

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2} A^\mu \left(\delta_{\mu\nu} \underbrace{\Box^2}_{= -\delta_{\alpha\beta} p^\alpha p^\beta} + \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu \right) A^\nu$$

$$\longrightarrow -\frac{1}{2} A^\mu \left(\delta_{\mu\nu} p^2 - \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu \right) A^\nu = -\frac{1}{2} A^\mu \Delta^{-1}_{\mu\nu} A^\nu$$

is invertible:

$$\boxed{\Delta_{\mu\nu} = \frac{1}{p^2}\left(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right),}$$

$p_\mu = (\omega_n; \vec{p})$ with $\omega_n = 2\pi n T$.

Photon pressure, again can be computed in covariant gauge. It is particularly simple in Feynman gauge where $\xi = 1$. Now directly (here $N_\xi = 1$)

$$P = \frac{1}{\beta V}\log Z = \frac{1}{\beta V}\log \det(\partial^2)\cdot\Big(\overbrace{\det(\delta_{\mu\nu}\partial^2)}^{[\det(\partial^2)]^4}\Big)^{-1/2}$$

$$= -2\,\frac{1}{\beta V}\,\frac{1}{2}\log\det(\partial^2)$$

$$= -2\,\frac{1}{\beta V}\,\frac{1}{2}\,\mathrm{Tr}\log(\omega_n^2+\vec{p}^{\,2}) = -2J_T^{-}(0).$$

For a general ξ one can write

$$\Delta_{\mu\nu} = \frac{1}{p^2}\Big(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\Big)+\xi\,\frac{p_\mu p_\nu}{p^4} = \frac{1}{p^2}\big(P^T_{\mu\nu}+\xi P^L_{\mu\nu}\big)$$

Noting that $P^T_{\mu\nu}P^{L\nu}{}_{\alpha} = 0$ & $P^T_{\mu\nu}P^{T\nu}{}_{\alpha} = P^T_{\mu\alpha}$ & $P^L_{\mu\nu}P^{L\nu}{}_{\alpha} = P^L_{\mu\alpha}$

$$\frac{1}{2}\log\big(\det(\Delta_{\mu\nu})\big) = \frac{1}{2}\log\Big(\det\frac{1}{p^2}\big(P^T_{\mu\nu}+\xi P^L_{\mu\nu}\big)\Big) = \frac{1}{2}\mathrm{Tr}\log\frac{1}{p^2}\big(P^T_{\mu\nu}+\xi P^L_{\mu\nu}\big)$$

$$= \frac{1}{2}\mathrm{Tr}\sum_{n=1}^{\infty}\Big(\frac{1}{p^2}\Big)^n\big(P^T_{\mu\nu}+\xi P^L_{\mu\nu}\big)^n = \frac{1}{2}\mathrm{Tr}\sum_{n=1}^{\infty}\Big(\frac{1}{p^2}\Big)^n\big(P^T_{\mu\nu}+\xi^n P^L_{\mu\nu}\big)$$

$$= -\frac{3}{2}\mathrm{Tr}\log p^2+\frac{1}{2}\mathrm{Tr}\log(\xi/p^2)$$

$$= -2\,\mathrm{Tr}\log p^2+\frac{1}{2}\mathrm{Tr}\log\xi$$

$=0$ in dim. reg. or infinite constant we divided out in definition of partition function

from ghost

$$\Rightarrow\quad P = -\underbrace{\mathrm{Tr}}_{1-2}\log p^2+\frac{1}{2}\cancel{\mathrm{Tr}\log\xi}+\frac{1}{\beta V}\log N(\xi)$$

$$\log N(\xi) \longmapsto = -\log\Big\{\int[\mathcal{D}\omega]\,e^{-\frac{1}{2\xi}\int_{x_E}\omega^2}\Big\} = -\frac{1}{2}\cancel{\mathrm{Tr}\log\xi}.$$

$$= \prod_x\sqrt{2\pi\xi} \doteq e^{\sum\log\xi^{1/2}}$$

$$= -\mathrm{Tr}\log p^2 \quad ✓.$$

Interacting bosonic field theory

The formal PT machinery is similar to T=0 QFT. Only the Feynman-rules are slightly different : continuous $p_0 \to \omega_n$ & $-i\lambda \to \lambda$.
This affects mainly the integrals coming from loop diagrams.
(and how depends on formulation (imaginary vs real time)).

Keywords: Perturbative expansion. Finite-T Feynman rules, renormalization.

Self-interacting scalar field

Consider first:

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + V_I(\phi) = \mathcal{L}_{E0} + V_I(\phi)$$

Now the generating function becomes

$$\mathcal{Z}[\beta, j] = \int [\mathcal{D}\phi]_\beta \exp\left[-\int_{x_E^\beta}\left(\mathcal{L}_{E0} + V_I(\phi) - j\phi\right)\right]$$

$$= \exp\left[-\int_{x_E^\beta} V_I\left(\frac{\delta}{\delta j}\right)\right] \int [\mathcal{D}\phi]_\beta \exp\left[-\int_{x_E^\beta}\left(\mathcal{L}_{E0} - j\phi\right)\right]$$

$$= \mathcal{Z}_0(\beta) \cdot \underbrace{\exp\left[-\int_{x_E^\beta} V_I\left(\frac{\delta}{\delta j}\right)\right] \exp\left[\iint j(x_E)\, \Delta_0(x_E - x_E')\, j(x_E')\right]}$$

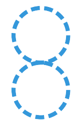
$$= \mathcal{Z}_0(\beta)\, \mathcal{Z}_1[\beta, j]$$

loop corrections

• This step is highly non-trivial: does PI converge to full result?

Now the Grand potential becomes even cleaner

$$\Omega = -\frac{1}{\beta}\log Z = -\frac{1}{\beta}\log Z_0 - \frac{1}{\beta}\log Z_1 = \Omega_0 + \delta\Omega$$

det:  connected loops  etc.

Quartic self-interaction

$$V_I(\phi) = \frac{\lambda}{4!}\phi^4$$

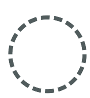
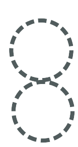
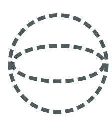
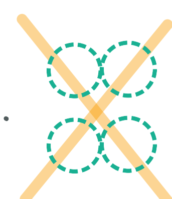
To compute an approximation for Z we need to expand

$$\log Z_1(\beta) = \log\left[e^{-\int V_I\left(\frac{\delta}{\delta j}\right)} e^{\frac{1}{2}\int\int j\Delta j} \right]_{j=0} = \log(1+\delta)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!}\left[-\int V_I\left(\frac{\delta}{\delta j}\right)\right]^k e^{\frac{1}{2}\int\int j\Delta j}\Big|_{j=0}^{\text{connected}}$$

$$= -\int V_I\left(\frac{\delta}{\delta j}\right) e^{\frac{1}{2}\int\int j\Delta j}\Big|_{j=0} + \dots$$

Where we used even more compact notation $\int_{x,t} = \int$. This creates the well known PT. Even more $\log Z(\beta)$ only picks the connected diagrams:

$\log Z(\beta) \sim$  +  +  + ... 

included in Z_1

Lowest order calculation

$$= 1 + \frac{1}{2!}\left(\frac{1}{2}\iint j\Delta j\right)^2 + \dots$$

$$\delta\Omega_1 = -\frac{1}{\beta}\int_{x_E^\beta} -\frac{\lambda}{4!}\left(\frac{\delta}{\delta j}\right)^4 e^{\frac{1}{2}\iint j\Delta j}\Big|_{j=0}$$

$$= \frac{1}{\beta}\,\frac{\lambda}{4!}\,\frac{1}{2!}\,\frac{1}{2^2}\int_{x_E^\beta}\left(\frac{\delta}{\delta j_x}\right)^4\left[\iint j\Delta j\right]^2$$

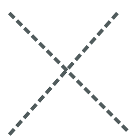
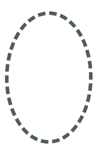
4! ways to connect

$$= \frac{\lambda}{8\beta}\int_{x_E^\beta}[\Delta(0)]^2 = \frac{\lambda}{8\cancel{\beta}}\,\cancel{\beta}V\,[\Delta(0)]^2$$

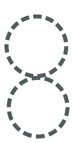
$$= \frac{\lambda V}{8}\left[\sum\!\!\!\!\!\!\int_B \frac{1}{\omega_n^2+\vec{p}^{\,2}}\right]^2$$

$$\Rightarrow \quad \delta P_1 = -\frac{\delta\Omega_1}{V} = -\frac{\lambda}{8}\left[\sum\!\!\!\!\!\!\int_B \frac{1}{\omega_n^2+\vec{p}^{\,2}}\right]^2$$

Of course, we can compute this also diagrammatically from FTFT-Feynman rules (by inspection or by Wick rotation from T=0-rules + propagator rule)

	$\hat{=}$	$\dfrac{1}{\omega_n^2+\omega_{\vec p}^2}$
	$\hat{=}$	$-\dfrac{\lambda}{4!}$
	$\hat{=}$	$\sum\!\!\!\!\!\!\int_B = T\sum_{2n}\int\dfrac{d^3p}{(2\pi)^3}$

add something

$$\Rightarrow \quad \delta P_1 = \frac{T}{V}$$  $$= -\frac{\lambda}{4!}\,3\left[\sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2}\right]^2 = -\frac{\lambda}{8}\left[\sum\!\!\!\!\!\!\int\frac{1}{\omega_n^2+\omega_p^2}\right]^2 \quad ✓.$$

This result has the problem of being infinite, of course.

Indeed, using our complex integration techniques we can compute:

$$\SumInt_Q \frac{1}{\omega_n^2+\omega_p^2} = \int_{\vec p} T\sum_n \frac{1}{\omega_n^2+\omega_p^2} = \qquad \underbrace{\qquad}_{=\frac{e^{\beta\omega}}{e^{\beta\omega}-1}}$$

$$= \int_{\vec p} \cancel{T}\frac{1}{2\pi i}\oint \frac{1}{\omega_p^2-z^2}\frac{\cancel{\beta}}{e^{\beta z}-1} = \int_{\vec p}\frac{1}{2\omega}\left(\frac{1}{e^{\beta\omega}-1} - \frac{1}{e^{-\beta\omega}-1}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2\omega_p} + \frac{1}{\omega_p}\frac{1}{e^{\beta\omega_p}-1}\right) \equiv I_0 + I_T$$

vacuum part: ↑ ↳ thermal part. finite.

$$\to \frac{\mu^\epsilon}{2}\int \frac{d^{3-\epsilon}p}{(2\pi)^3}\frac{1}{(p^2+m^2)^{1/2}} = \frac{\mu^\epsilon}{2}\Phi\left(m, 3-\epsilon, \tfrac{1}{2}\right) \quad : \quad \Phi(m,d,\alpha) = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\frac{1}{(m^2)^{\alpha-\frac{d}{2}}}$$

$$= \frac{1}{2}\frac{m^2}{(4\pi)^{3/2}}\left(4\pi\frac{\mu^2}{m^2}\right)^{\epsilon/2}\frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(\frac{1}{2})} = \frac{m^2}{(4\pi^2)^2}\left(4\pi\frac{\mu^2}{m^2}\right)^{\epsilon/2}\frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(1)} = iA_0(m^2)$$

$$= -\frac{\lambda_R m^2}{32\pi^2}\Big(\underset{\uparrow \text{diverges}}{\frac{2}{\epsilon}} - \gamma_E + \log 4\pi + 1 + \log\frac{\mu^2}{m^2}\Big) + O(\epsilon)$$

$\Rightarrow$ Need to renormalize.

Renormalization

Lagrangian defined in terms of local operators.

All observations have finite resolution $\Rightarrow$ Lagrangian parameters are not observable. In our model theory,

$$\mathcal{L}(\phi, m^2, \lambda) = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad (\text{Minkowski})$$

Redefine (formally so far)

$$\phi \equiv Z_\phi^{1/2}\phi_R\,, \quad \lambda = \lambda_R + \delta\lambda\,, \quad m^2 \equiv m_R^2 + \delta m^2$$

Then $$\mathcal{L}(\phi, m^2, \lambda) = \mathcal{L}(\phi_R, m_R^2, \lambda_R) + \frac{\delta_\phi}{2}(\partial_\mu \phi_R)^2 - \frac{1}{2}\delta_m \phi_R^2 - \frac{1}{4!}\delta_\lambda \phi_R^4$$
$$\equiv \underline{\mathcal{L}_0(\phi_R, m_R^2) + V_I(\lambda_R, m_R^2; \delta_\phi, \delta_m, \delta_\lambda)}$$

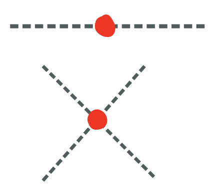
where

$\delta_\phi = Z_\phi - 1$ wave function renormalization

$\delta_m = Z_\phi(m_R^2 + \delta m^2) - m_R^2$ mass renormalization

$\delta_\lambda = Z_\phi^2(\lambda_R + \delta\lambda) - \lambda_R$ coupling constant renormalization

In BPHZ-scheme used here δ_i are treated as interactions, giving (Euclidean)



$-\frac{1}{2}(p^2\delta_\phi + \delta_m)$ $\quad$ $\left(= -ip^2\delta_\phi - i\delta_m \right.$

$-\frac{\delta_\lambda}{4!}$ $\quad$ $\left. -\frac{i\delta_\lambda}{4!} \right)_{\text{Minkowski}}$

Renormalization scheme: = choice of parameters. (coupling constant: later)

$$\Delta_R^{-1}(p) = p^2 - m_R^2 - \underline{\Pi_R(p^2)} \xrightarrow{p^2 = m_R^2} p^2 - m_R^2$$

self energy

—— + ——⊗—— + ...

$\frac{1}{p^2 - m_R^2} \quad \frac{1}{p^2-m^2}(-\Pi)\frac{1}{p^2-m^2}$

$= \frac{1}{p^2 - m^2 - \Pi}$

$\Delta^{-1} = p^2 - m^2 - \Pi$

eg: $$\begin{cases} \Pi_R(m_R^2) \equiv 0 \\ \frac{d}{dp^2}\Pi_R(p^2)\Big|_{p^2 = m_R^2} \equiv 0 \end{cases}$$ on-shell scheme

renormalization point

Alternatively $$\Delta_R^{-1}(p) \xrightarrow{p^2=0} -m_{R0}^2$$

$$\begin{cases} \Pi_{R0}(0) \equiv 0 \\ \frac{d}{dp^2}\Pi_{R0}(p^2)\Big|_{p^2 = 0} \equiv 0 \end{cases}$$ $p^2=0$ scheme

One-loop calculation

Minkowski rules

$$ \text{(dashed line)} \; = \; \frac{i}{p^2 - m_R^2} $$

$$ \times \; = \; -\frac{i\lambda_R}{4!} $$

$\mathbb{R}^4$ - Euclid rules

$$ \text{(dashed line)} \quad \frac{1}{p^2+m^2} \quad \left(\frac{1}{\omega_n^2+\omega_p^2}\right) \; 8 \cdot 0 \; ? $$

$$ \times \quad -\frac{\lambda_R}{4!} $$

Self-energy correction: what is the 'blob' in given set of rules, wrt Π:

$$ \frac{i}{p^2-m_R^2} + \frac{i}{p^2-m_R^2}(-i\Pi)\frac{i}{p^2-m_R^2} + \cdots $$
$$ = \frac{i}{p^2-m_R^2}\left(1 + \frac{\Pi}{p^2-m_R^2} + \cdots\right) = \frac{i}{p^2-m_R^2-\Pi} $$

$$ \frac{1}{p^2+m_R^2} + \frac{1}{p^2+m_R^2}(-\Pi)\frac{1}{p^2+m^2} + \cdots $$
$$ = \frac{1}{p^2+m_R^2}\left(1 - \frac{\Pi}{p^2+m_R^2} + \cdots\right) = \frac{1}{p^2+m_R^2+\Pi} $$

Eg: $\Pi_M = i$ -⊘- in Minkowski rules & $-\Pi_E =$ -⊘- in Euclidean rules.

$$ \Pi^{\text{vacuum}}_{\text{1-loop}} = i \,\bigcirc\, = i\left(\frac{-i\lambda_R}{2}\right)\int \frac{\overbrace{d^4p}^{\to\, i d^4 p_E}}{(2\pi)^4}\frac{i}{p^2-m_R^2} \overset{\text{Wick}}{=} -\left(-\frac{\lambda_R}{2}\right)\int\frac{d^4p_E}{(2\pi)^4}\frac{1}{p_E^2+m^2} = \frac{\lambda_R}{2} I_0 $$

same directly from 4d E-rules

$$ \xrightarrow[\text{reg.}]{\text{dim.}} \frac{\lambda_R}{2}\mu^{\epsilon}\int\frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}}\frac{1}{p_E^2+m^2} \equiv \frac{\lambda_R}{2}\, i A_0(m_R^2) $$

$$ = \frac{\lambda_R}{2}\mu^{\epsilon}\,\Phi(m_R, 4-\epsilon, 1) \simeq \frac{\lambda_R m_R^2}{32\pi^2}\left(4\pi\frac{\mu^2}{m_R^2}\right)^{\epsilon/2}\frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(1)} $$

$$ \simeq -\frac{\lambda_R m_R^2}{32\pi^2}\Bigg(\overbrace{\frac{2}{\epsilon} - \gamma_E + \log 4\pi}^{= 2/\epsilon_{\overline{MS}}} + 1 + \log\frac{\mu^2}{m_R^2}\Bigg) + \mathcal{O}(\epsilon) $$

$$ \equiv -\frac{\lambda_R m_R^2}{32\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}} + 1 + \log\frac{\mu^2}{m_R^2}\right) \simeq p^2\text{-independent constant} $$

Full self energy in vacuum:

$$\Pi = i\Big($$  $$+$$  $$\Big) = \Pi_{1\text{-loop}}(m_R^2) + \delta_\phi^{(1)} p^2 + \delta_m^{(1)}$$

On-shell scheme

$$\frac{d}{dp^2}\pi(p^2) = -\delta_\phi^{(1)} = 0$$

$$\pi(m_R^2) = \Pi_{1\text{-loop}}(m_R^2) - \delta_m^{(1)} = 0 \Rightarrow \delta_m = -\Pi_{1\text{-loop}}(m_R^2)$$

$\Rightarrow$ at one loop $\Pi_{\text{vacuum}}(p^2) = 0.$

Full thermal self-energy (now with finite-T-rules)

$$\Pi = -\Big($$  $$+$$  $$\Big) = \frac{\lambda_R}{2} \SumInt \frac{1}{\omega_n^2 + \omega_p^2} + \delta_m$$

$$= \frac{\lambda_R}{2}\left(\cancel{I_0} + I_T\right) - \frac{\lambda_R}{2} i \cancel{\Delta_0}(m_R^2) = \frac{\lambda_R}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}\frac{1}{e^{\beta\omega_p}-1} = \Pi_T$$

We thus get a finite T-dependent self-energy correction

$$\Rightarrow \Delta^{-1}(p^2;T)_{\text{ren}} \overset{\text{on-shell}}{=} \underbrace{p^2}_{\omega_n^2+\vec{p}^{\,2} \text{ in imag. T-rules}} + m_R^2 + \frac{\lambda_R}{2} I_T(m,T) \overset{\text{at pole}}{\equiv} 0$$

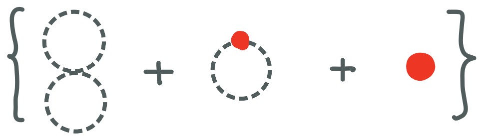
thermal correction to dispersion relation

For $T \gg m_R$

$$I_T \simeq \frac{1}{2\pi^2}\int dp \frac{p}{e^{\beta p}-1} = \frac{T^2}{2\pi^2}\underbrace{\zeta(2)}_{\frac{\pi^2}{6}} = \frac{T^2}{12} \Rightarrow \boxed{\Pi_T \simeq \frac{\lambda_R T^2}{24}}$$

$$\Rightarrow \Delta^{-1}(p^2)_{\text{ren}} = p^2 + m_R^2(T) \; ; \qquad m_R^2(T) \simeq m_R^2 + \frac{\lambda_R T^2}{24}$$ thermal mass.

Now go back to our evaluation of the pressure (Euclidean rules now)

$\Rightarrow \delta P_1 = \frac{T}{V}$ 

$$= -\frac{\lambda_R}{8}\left[\sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2}\right]^2 - \frac{1}{2}\,\delta_m^{(1)} \sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2} + \Lambda_{vac}$$

$$= -\frac{1}{2\lambda_R}\left[\Pi_{vac}^{1\text{-loop}}+\Pi_T\right]^2 + \Pi_{vac}^{1\text{-loop}}\,\frac{1}{\lambda_R}\left(\Pi_{vac}+\Pi_T\right) + \Lambda_{vac}$$

$$= -\frac{1}{2\lambda_R}\Pi_T^2 + \underbrace{\left(\Lambda_{vac} - \frac{1}{2\lambda_R}\Pi_{vac}^2\right)}_{\equiv 0} = -\frac{1}{2\lambda_R}\Pi_T^2$$

Vacuum counterterm is something one can add to Lagrangian any time

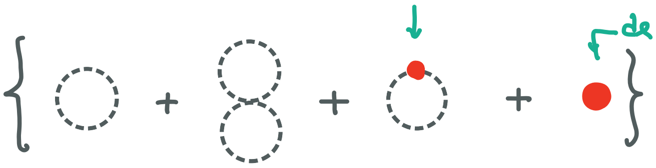
$$\int d^4x\sqrt{-g}\,\mathcal{L} \to \int d^4x\sqrt{-g}\,(\mathcal{L} + \Lambda) \Rightarrow$$

$\uparrow$ invariant

So, at 1-loop level the pressure is

definition of m

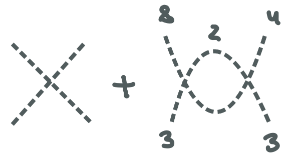
definition of free

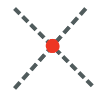
$P = \frac{T}{V}$ 

$$\overset{m=0}{\simeq} \frac{\pi^2}{90}T^4 + \frac{1}{2\lambda_R}\left(\frac{\lambda_R T^2}{24}\right)^2 \simeq \frac{\pi^2}{90}T^4\left(1 - \frac{5\lambda_R}{64}\right)$$

finite correction. Follows from defining the on-shell mass (here zero) and the vacuum energy.

Coupling constant renormalization Choose the scheme:

$$\lambda_R \equiv \Gamma^{(4)}(0,0,0) = i\Big($$


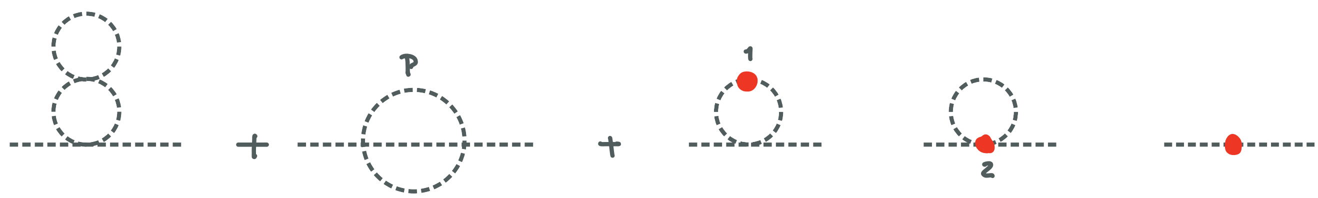
$$+ \; t\,\&\,u\text{-channels} \; +$$

$$\Big)$$

$$= \lambda_R + 3\cdot\frac{1}{2!}\frac{\lambda_R^2}{(4!)^2}\cdot 8\cdot4\cdot3\cdot3\cdot2\int\frac{1}{(p^2+m_R^2)} + \delta_\lambda^{(1)} \equiv \lambda_R$$

$$\Rightarrow \delta_\lambda^{(1)} = -\frac{3}{2}\lambda_R^2\int_p\frac{1}{(p^2+m_R^2)^2} \equiv -\frac{3}{2}\lambda_R^2 G_0 \qquad \big(G_0 = iB_0(m_R^2,m_R^2,0)\big)$$

Proof of no T-dep. divergences to 2 loops

I. Self energy:



$S = 8\cdot3\cdot4\cdot3$ $\qquad$ $8\cdot4\cdot3\cdot2$ $\qquad$ $4\cdot3\cdot2$ $\qquad$ $4\cdot3$ $\qquad$ 2

$$-\frac{\lambda_R^2}{4}\sum\!\!\!\!\!\!\int\frac{1}{p^2+m^2}\sum\!\!\!\!\!\!\int\frac{1}{(p^2+m^2)^2} - \frac{\lambda_R^2}{6}\underbrace{\sum\!\!\!\!\!\!\int_q\sum\!\!\!\!\!\!\int_p\frac{1}{(p^2+m^2)(q^2+m^2)((q-p)^2+m^2)}}_{\equiv H(m,m,m;p^2)}$$

$$-\frac{\delta_m^{(1)}\lambda_R}{2}\sum\!\!\!\!\!\!\int\frac{1}{(p^2+m^2)^2} - \frac{\delta_\lambda^{(1)}}{2}\sum\!\!\!\!\!\!\int\frac{1}{p^2+m^2} + \delta_\phi^{(2)}p^2 + \delta_m^{(2)}$$

$$\delta_\lambda^{(1)} = -\frac{3}{2}\lambda_R^2\int_p\frac{1}{(p^2+m_R^2)^2} \qquad \delta_m^{(1)} = -\frac{\lambda_R}{2}\int_p\frac{1}{p^2+m_R^2}$$

$$= -\frac{\lambda_R^2}{4}\Bigg(\Bigg(\underbrace{\Big(\sum\!\!\!\!\!\!\int - \int_p\Big)\frac{1}{p^2+m^2}}_{\frac{2}{\lambda_R}\Pi_T}\underbrace{\sum\!\!\!\!\!\!\int\frac{1}{(p^2+m^2)^2}}_{\int_p\frac{1}{(p^2+m^2)^2}+G_T(m,m)\;\equiv\; G_0+G_T} - 3\int_p\frac{1}{(p^2+m^2)^2}\underbrace{\sum\!\!\!\!\!\!\int\frac{1}{p^2+m^2}}_{=\int_p\frac{1}{p^2+m^2}+\frac{2}{\lambda_R}\Pi_T = I_0+I_T}\Bigg) - \frac{\lambda_R^2}{6}H(m,m,m,p^2) + \delta_\phi^{(2)}p^2 + \delta_m^{(2)}$$

$$= -\frac{\lambda_R}{2}\pi_T(G_0+G_T) + \frac{3\lambda_R^2}{4} G_0\left(\frac{2}{\lambda_R}\pi_T + I_0\right) - \frac{\lambda_R^2}{6} H(m,m,m,p^2) + \delta_\phi^{(2)} p^2 + \delta_m^{(2)}$$

$$= -\frac{\lambda_R}{2}\pi_T G_T + \lambda_R \pi_T G_0 + \frac{3\lambda_R^2}{4} G_0 I_0 - \frac{\lambda_R^2}{6}\left(H_0 + H_{0T} + H_T\right) + \delta_\phi^{(2)} p^2 + \delta_m^{(2)}$$

$$\xrightarrow{T\to 0} \frac{3\lambda_R^2}{4} G_0 I_0 - \frac{\lambda_R^2}{6} H_0 + \delta_\phi^{(2)} p^2 + \delta_m^{(2)}$$

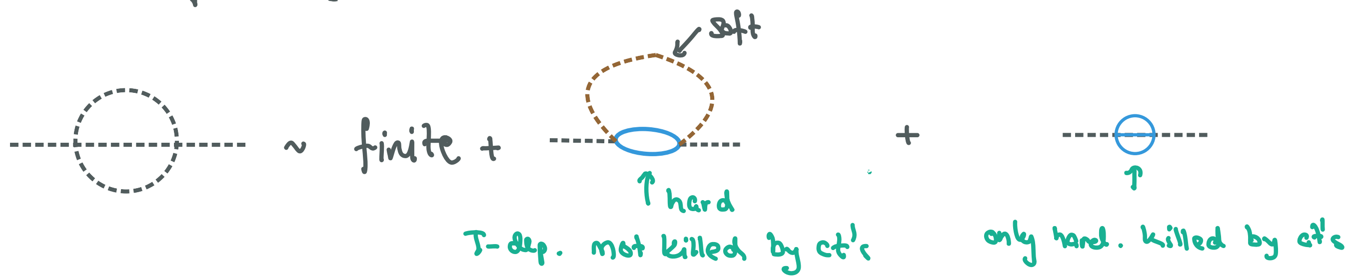
$$\Rightarrow \quad \delta_\phi^{(2)} = \frac{\lambda_R^2}{6}\frac{\partial}{\partial p^2} H_0\Big|_{p^2=m_R^2}$$

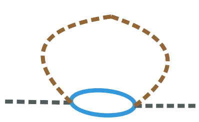
$$\delta_m^{(2)} = \frac{\lambda_R^2}{6}\left(1 + m_R^2\frac{\partial}{\partial p^2}\right) H_0\Big|_{p^2=m^2} - \frac{3\lambda_R^2}{4} G_0 I_0$$

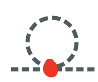
$$\Rightarrow -\frac{\lambda_R^2}{6} H_0 + \frac{3\lambda_R^2}{4} G_0 I_0 + \delta_\phi^{(2)} p^2 + \delta_m^{(2)} = -\frac{\lambda_R^2}{6}\left(H_0(p^2) - H_0(m_R^2) - (p^2+m_R^2)\frac{d}{dp^2} H_0\Big|_{p^2=m_R^2}\right)$$

This combination is finite. Remain the terms $\lambda_R \pi_T G_0$ and $-\frac{\lambda_R^2}{6} H_{0T}$.

Div. Structure of 



 $\simeq -3\,\frac{\lambda_R^2}{6}\,\frac{2}{\lambda_R}\pi_T \int \frac{1}{(p^2+m^2)^2} + \text{finite} = -\lambda_R \pi_T G_0 + \text{finite}$

The remaining ct-contribution ($\frac{2}{3}$'s of T-dep. part of ) exactly cancels this.

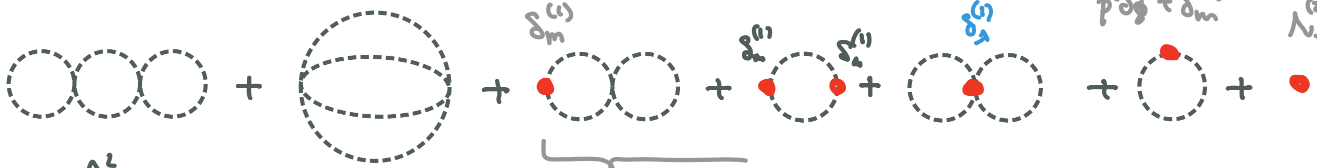
$\Rightarrow$ $\pi^{(2)}$ is finite. In particular all potentially T-dep. infinities cancel.

The 2-loop contribution to pressure comes from following terms:

$\int$ coupling constant renormalization

(1) (2) (1a) (1b)

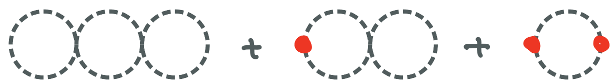
$\delta_m^{(1)}$ $\delta_m^{(1)}$ $\delta_m^{(1)}$ $\delta_\lambda^{(1)}$ $p^2\delta_\phi^{(2)}+\delta_m^{(2)}$ $\Lambda_{vac}^{(2)}$



λ^2 λ^2

UV- counter-terms & corrections. OK

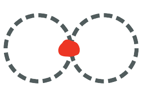
Adding (1), (1a) and (1b) gives:

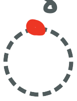
$\delta P_{(2a)} = \frac{T}{V}\Big\{$  $\Big\}$

$$= \frac{1}{2!}\left(\frac{\lambda_R}{4!}\right)^2 6\cdot 6\cdot 2\;\frac{4}{\lambda_R^2}\left(\Pi_{vac}^{(1)}+\Pi_T\right)^2 \sum\!\!\!\!\!\!\int \frac{1}{(p^2+m^2)^2} + \cancel{\frac{1}{2}}\underbrace{\delta_m^{(1)}}_{-\Pi_{vac}}\frac{\lambda_R}{4!}\,6\cdot 2\,\cancel{\frac{2}{\lambda_R}}\overbrace{\left(\Pi_{vac}^{(1)}+\Pi_T\right)}^{\sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2}}\sum\!\!\!\!\!\!\int_p \frac{1}{(p^2+m^2)^2}$$

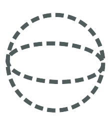
$$+\left(\frac{1}{2}\delta_m^{(1)}\right)^2 \sum\!\!\!\!\!\!\int \frac{1}{(p^2+m^2)^2}$$

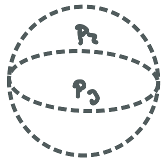
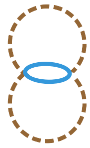
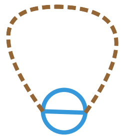
$$= \frac{1}{4}\Big(\overbrace{(\Pi_{vac}^{(1)}+\Pi_T^{(1)})^2 - 2\Pi_{vac}^{(1)}(\Pi_{vac}^{(1)}+\Pi_T^{(1)}) + \Pi_{vac}^{(1)\,2}}^{\Pi_T^2}\Big)\sum\!\!\!\!\!\!\int_p \frac{1}{(p^2+m^2)^2} = \frac{1}{4}\,\Pi_T^{(1)\,2}\left(G_0+G_T\right)$$

 $= -\frac{\delta_\lambda^{(1)}}{8}\left[\sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2}\right]^2 = -\frac{\delta_\lambda^{(1)}}{2\lambda_R^2}\left(\Pi_{vac}^{(1)}+\Pi_T^{(1)}\right)^2 = -\frac{3}{4}G_0\left(\Pi_{vac}^{(1)}+\Pi_T^{(1)}\right)^2$

$\delta^{(2)}$  $\hat{=}\ \frac{1}{2}\sum\!\!\!\!\!\!\int \frac{\delta_m^{(2)}+p^2\delta_\phi^{(2)}}{\omega_n^2+\omega_p^2} = \underbrace{\frac{3\lambda_R}{4}G_0 I_0}_{\frac{3}{2}G_0(\Pi_{vac}^{(1)}+\Pi_T)\Pi_{vac}}\left(\Pi_{vac}^{(1)}+\Pi_T\right)$

$$+\frac{\lambda_R^2}{12}\sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2}\left(H_\phi(m_R^2)+(p^2+m_R^2)\frac{\partial H}{\partial p^2}\Big|_{p^2=m^2}\right)$$

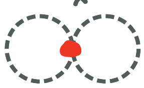
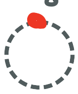
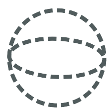
Remains the divergence structure of 

$$\mathrm{div}\Big[$$  $$\Big] \sim \binom{4}{2}$$  $$+ \binom{4}{3}$$  $$+ \; \lambda^{(2)}$$ 

$$= \frac{\lambda_R^2}{2\cdot 4!}\, 6\left(\frac{2}{\lambda_R}\right)^2 \pi_T^2\, G_0 + \frac{\lambda_R^3}{2\cdot 4!}\, 4 \sum\!\!\!\!\!\!\int \frac{H_0(p^2)}{\omega_n^2+\omega_p^2} +$$  $$+ \text{finite}$$

$$= \frac{1}{2} G_0 \pi_T^2 + \frac{\lambda_R^3}{12} \sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2} H_0(p^2) +$$  $$+ \text{finite}$$

We then get

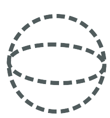
$$\delta P_{(2b)} = \frac{T}{V}\Big\{$$  ($\delta_\lambda^{(1)}$) $+$  ($\delta^{(2)}$) $+$  $+$  ($\lambda^{(2)}$) $\Big\}$

$$= -\frac{3}{4} G_0 \underbrace{(\pi_{vac}+\pi_T)^2}_{=\pi_{vac}^2 + \cancel{2\pi_T\pi_{vac}} + \pi_T^2} + \frac{3}{2} G_0 \pi_{vac}(\pi_{vac} + \cancel{\pi_T}) + \frac{1}{2} G_0 \pi_T^2$$

$$+ \frac{\lambda_R^2}{12} \sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2}\left(H_0(p^2) - H(m_R^2) - (p^2+m_R^2)\frac{\partial H}{\partial p^2}\Big|_{p^2=m_R^2}\right) +$$  $+$  ($\lambda^{(2)}$) $+ \text{finite}$

$$= -\frac{1}{4} G_0 \pi_T^2 + \text{finite}$$

Summing the two, we get $\delta P_{(2a)} + \delta P_{(2b)}$ is finite. The T-dependent divergences went away in particular. In all

$$\delta P_{(2)} = \frac{1}{4}\pi_T^2 G_T + \frac{T}{V}$$  $\Big|_{\text{finite},\, T\neq 0 \text{ part}}$

(T=0 part absorbed to $\lambda^{(2)}$)

Infrared divergence & Daisy resummation

We still have a problem, this time at IR. Indeed

$$\delta P^{\circ\circ\circ}_{(2)} = \frac{1}{4}\Pi_T^2 \Big(\underbrace{\sum\!\!\!\!\!\!\int_P}_{G_0+G_T} - \underbrace{\int_p}_{-G_0}\Big)\frac{1}{(p^2+m^2)^2} = \frac{1}{4}\Pi_T^2\, G_T$$

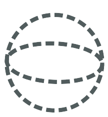
contains an IR divergence for $m=0$.

- $$I = \sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2+\omega_p^2} = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2\omega} + \frac{1}{\omega}\,\frac{1}{e^{\beta\omega}-1}\right)$$
- $$G = \sum\!\!\!\!\!\!\int \frac{1}{(\omega_n^2+\omega_p^2)^2} = -\frac{\partial}{\partial m^2} I = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{4\omega^3} + \underbrace{\frac{1}{2\omega^3}\big(n+\beta\omega(n^2+n)\big)}_{\sim \frac{1}{\beta\omega^4} \text{ for small } \omega}\right)$$
$$= G_0 + G_T$$

$$\Rightarrow \quad G_T \sim \frac{T}{2\pi^2}\int_{\Lambda_{IR}} \frac{dp}{p^2} \sim \frac{1}{\Lambda_{IR}} \quad \text{for } m=0$$

PT breaks down at IR at finite T.

Basketball is IR-safe however:



$$\Big|_{m_i=0;\ m=0} \sim \frac{\lambda_R^2}{6}T^3 \int_{\vec p}\int_{\vec q}\int_{\vec r} \frac{1}{\vec p^{\,2}\vec q^{\,2}\vec r^{\,2}(\vec p+\vec q+\vec r)^2}$$

$$\propto \frac{\lambda_R^2 T^3}{32\cdot 6\pi^5}\int dp\, dq\, dr\, dc\alpha\, dc\beta\, \overset{ds\phi}{} \frac{1}{(\vec p+\vec q+\vec r)^2} \propto \int dp \sim \text{finite}$$

The problem is that $T\neq 0$-corrections create an effective mass $\sim \Pi_T$, but we are still expanding PT using massless propagator. We already saw that thermal loop correction induced $m_R^2 \to m_R^2 + \frac{\lambda T^2}{24}$, so resumming the leading loop corrections should help. Question is how one does this consistently?

Daisy resummation for pressure

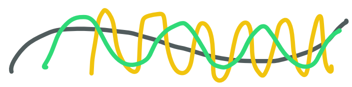
It is important to realize that IR-divergence comes entirely from the $n=0$-mode.

$$G_T^{n=0} = T\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p^2} \sim \frac{T}{2\pi^2}\int\frac{dp}{p^2}\,.$$

All other integrals $G_T^{n\neq 0}$ are IR-finite because of the thermal mass $2\pi n T$.

$\Rightarrow$ only need to resum the $n=0$-mode.

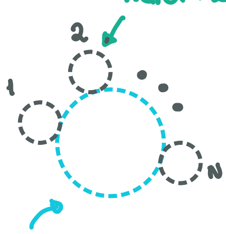
There is also intuitive reasoning for this: zero modes have long wave-length and it is natural that their interactions are screened by $n\neq 0$-modes.



$\lambda T^2/24$ = screening mass

Also, the IR-divergence in  -diagram was just the first in a series.

- To order λ_R the most IR-divergent diagram is



symm factor of subdiagram Ω

$$; \; S_N = \frac{1}{N!}\,\frac{(N-1)!}{2}\,(S_{\Pi_1})^N = \frac{(S_{\Pi_1})^N}{2N}$$

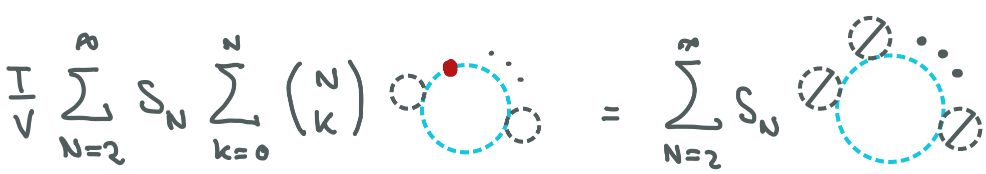
order of PT — ways to order N Ω's to circle

S_{Π_1} :  $\frac{\lambda}{4!}\,6\cdot 2\cdot I \equiv S_{\Pi_1}\lambda_R I$ oriented

- Add all possible c.t.'s

$\Leftrightarrow$ each sub 

$$(a+b)^N = \sum \binom{N}{k} a^k b^{N-k}$$

$\Rightarrow$ 

note that all graphs with $N\geq 3$ are finite

$$\bar\Delta_0 = \text{-------} = \frac{1}{\omega_0^2+\omega_{\vec p}^2}$$

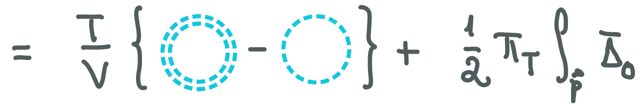
$$= \frac{1}{2}\int_{\vec p}\sum_{N=2}^{\infty}\frac{(-1)^N}{N}\left(\Pi_T\,\bar\Delta_0(\omega,\vec p)\right)^N$$

one - sign from each vertex $-\lambda$

$$= -\frac{1}{2}\int_{\vec p}\Big(\log\left(1+\Pi_T\bar\Delta_0\right) - \underbrace{\Pi_T\bar\Delta_0}_{N=1\text{ -term}}\Big)$$

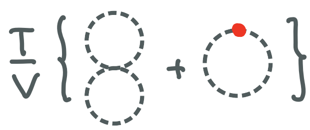
$$= -\frac{1}{2}\int_{\vec p}\Big(\log(\omega_n^2+\omega_{\vec p}^2+\Pi_T) - \log(\omega_n^2+\omega_{\vec p}^2) - \Pi_T\bar\Delta_0\Big)$$

denote this by $\Delta_0 \equiv \Delta_{n=0}$



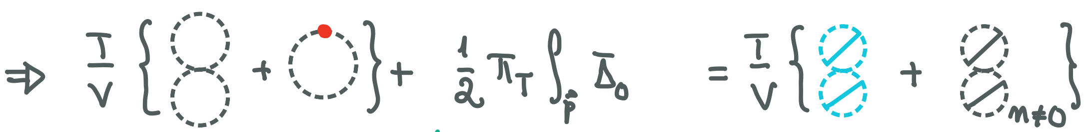
from zero-mode only

Here ========== refers to resummed zero-mode propagator $\Delta^{-1} = \omega_p^2 + \Pi_T$.
This is still valid also for $m\neq 0$. And that is fine. We may have $T \gg m$ $\neq 0$.
Now remember our one-loop result



$$= -\frac{1}{2\lambda_R}\Pi_T^2 = -\frac{1}{2\lambda_R}\left(\Pi_{T,0}+\Pi_{T,n\neq0}\right)^2$$

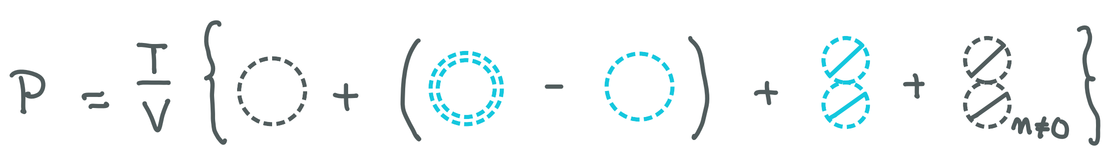
$$= -\frac{1}{2\lambda_R}\Pi_{T,0}^2 - \frac{1}{\lambda_R}\cancel{\Pi_{T,0}}\Pi_{T,n\neq0} - \frac{1}{2\lambda_R}\Pi_{T,n\neq0}^2$$



$$\underbrace{\frac{1}{2}\Pi_T\int_{\vec p}\bar\Delta_0}_{=\frac{1}{\lambda_R}\Pi_T\cancel{\Pi_{T,0}}}$$

both UV- & IR finite contributions.

So the full 1-loop ring-resummed result is diagrammatically:



$$= P_0 - \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\Big(\log\left(\vec{p}^{\,2}+m^2+\Pi_T\right) - \log\left(\vec{p}^{\,2}+m^2\right)\Big) - \underbrace{\frac{1}{2\lambda_R}\left(\Pi_{T,0}^2 + \Pi_{n\neq 0}^2\right)}_{\text{modified 1-loop correction}}$$

Now use $\displaystyle\int_{\vec{p}} \frac{1}{p^2+m^2} = -\frac{mT}{4\pi}$ (Excercise)

$$\Rightarrow \quad \Pi_{T,0} = -\frac{\lambda m T}{8\pi} \quad \Rightarrow \quad \Pi_{T,0} \ll \Pi_T \quad \text{if } T \gg m.$$

So for $T \gg m$ the 1-loop part is un-modified. Then use

$$-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\log\left(\vec{p}^{\,2}+m^2\right) = -\frac{1}{2}\int^{m^2} dm'^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2+m'^2} = +\frac{T}{4\pi}\int^{m} dm'\, m'^2 = +\frac{Tm^3}{12\pi} + C$$

$\Rightarrow \quad \frac{V}{T}\Big\{$  $-$  $\Big\}$ $= \dfrac{(m^2+\Pi_T)^{3/2}\,T}{12\pi} - \dfrac{m^3 T}{12\pi}$

$$\xrightarrow{m\to 0} \frac{T\,\Pi_T^{3/2}}{12\pi} = \frac{1}{12\pi}\,\frac{1}{(24)^{3/2}}\,\lambda^{3/2}\,T^4$$

Thus, the ring contribution is $\sim\lambda^{3/2}$, which is larger than the 2-loop corrections.

$$\Rightarrow \quad P \simeq \frac{\pi^2}{90}T^4\left(1 - \frac{5\lambda}{64\pi^2} + \frac{5\lambda^{3/2}}{384\pi} + \dots\right)$$

restricted to $T \gg m$

Full daisy resummation?

How about resumming all Matsubara modes? Sometimes one might like to do this, (Laine) so let's try:

$$\frac{P}{V}\Big\{\sum_{N=2}^{\infty} S_N$$  $$\Big\} = \frac{1}{2}\,\sum\!\!\!\!\!\!\int_{P} \sum_{N=2}^{\infty} \frac{(-1)^N}{N}\left(\pi_T\,\Delta_0(\omega_n,\vec p)\right)^N$$

m-dep. prop.

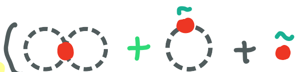
N=1 -term

$$= -\frac{1}{2}\,\sum\!\!\!\!\!\!\int_{P}\Big(\log\left(1+\pi_T\Delta_0\right)-\pi_T\Delta_0\Big)$$

$$= -\frac{1}{2}\,\sum\!\!\!\!\!\!\int_{P}\Big(\log\left(\omega_n^2+\omega_{\vec p}^2+\pi_T\right)-\log\left(\omega_n^2+\omega_{\vec p}^2\right)-\pi_T\Delta_0\Big)$$

$$= \frac{P}{V}\Big\{$$  $$\Big\} + \frac{1}{\lambda_R}\pi_T\left(\pi_T+\pi_{vac}\right)$$

- Now this is of course still divergent. In addition to $\frac{1}{\lambda_R}\pi_T(\pi_T+\pi_{vac})$, that got extracted there is the divergence $\frac{1}{4}\pi_T^2 G_0$ in the N=2-term.

- This is cancelled by $\frac{1}{3}\Big($  $\Big)$

  suitably chosen

  only G_0-div. parts

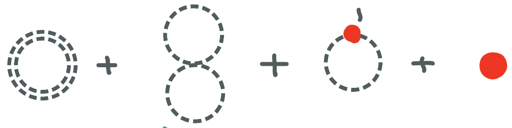
  – we did not need these above, because for the zero-mode

$$\int_{\vec p}\frac{1}{(\vec p^2+m^2)^2} = -\frac{\partial}{\partial m^2}\int_{\vec p}\frac{1}{\vec p^2+m^2} = \frac{\partial}{\partial m^2}\frac{mT}{4\pi} = \frac{T}{4\pi m} \qquad \text{(UV-finite)}$$

So, combining with δP_1 & subtracting the appropriate ct's one can write a finite ring corrected P

The mixed term from 2-loop contribution δP_2

$$P = \frac{V}{T}\sum_{N=0}^{\infty} S_N$$  $$-\frac{1}{4}\pi_T^2 G_0$$

$$= \frac{V}{T}\Big\{$$  $$\Big\} + \frac{1}{\lambda_R}\Pi_T\left(\Pi_T+\Pi_{vac}\right) + \frac{1}{4}\Pi_T^2 G_0$$

1-loop

$$= -\frac{1}{2}\sum\!\!\!\!\!\!\int \log\left(\omega_n^2+\omega_p^2+\Pi_T\right) - \frac{1}{2\lambda_R}\Pi_T^2 + \frac{1}{\lambda_R}\Pi_T\Pi_{full} + \frac{1}{4}\Pi_T^2 G_0 + \bullet$$

$$= -J_0\left(\sqrt{m^2+\Pi_T}\right) + \underbrace{J_T^-\left(\sqrt{m^2+\Pi_T},T\right)}_{\text{resummed thermal integral}} \overbrace{-\frac{1}{2\lambda_R}}^{\text{1-loop}}\Pi_T^2 + \frac{1}{\lambda_R}\Pi_T\Pi_{full} + \frac{1}{4}\Pi_T^2 G_0 + \bullet$$

Let us now check the finiteness of this result.

$$-J_0(m^2+\Pi_T) = \frac{m^4+2m^2\Pi_T+\Pi_T^2}{64\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}}+\frac{3}{2}+\log\frac{\mu^2}{m^2+\Pi_T}\right)$$

$$\frac{1}{\lambda_R}\Pi_T\Pi_{full} = \frac{1}{2}\Pi_T\, iA_0(m^2) = -\frac{m^2\Pi_T}{32\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}}+1+\log\frac{\mu^2}{m^2}\right)$$

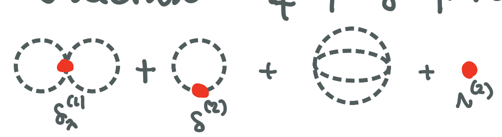
$$\frac{1}{4}\Pi_T^2 G_0 = \frac{1}{4}\Pi_T^2\, iB_0(0,m^2,m^2) = -\frac{\Pi_T^2}{64\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}}+\log\frac{\mu^2}{m^2}\right)$$

$\uparrow\ m^2+\Pi_T$

$$\Rightarrow\quad -J_0(m^2+\Pi_T) + \frac{1}{\lambda_R}\Pi_T\Pi_{full} + \frac{1}{4}\Pi_T^2 G_0 + \bullet$$

$$= \frac{1}{64\pi^2}\left\{-m^4\log\left(1+\frac{\Pi_T}{m^2}\right) + 2m^2\Pi_T\left(\frac{1}{2}-\log\left(1+\frac{\Pi_T}{m^2}\right)\right) + \Pi_T^2\left(\frac{3}{2}-\log\left(1+\frac{\Pi_T}{m^2}\right)\right)\right\}$$

This is finite and $\longrightarrow 0$ as $T\to 0$. So all is well? No, the term is again IR-divergent, eg blows up when $m^2\to 0$. The resummation failed.

The problem is with the mixed correction $-\frac{1}{4}\Pi_T^2 G_0$ from the 2-loop contribution coming from $SP_{(2b)} = \frac{T}{V}\Big\{$  $\Big\}$.

If we replace $G_0(m^2)$ here with $G_0(m^2+\pi_T)$, then the offending log-term just drops! This is weird though, because G_0 came from S_2, eg. the counter term.

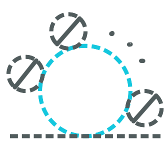
So, if we want to resum the whole thing, we should apparently somehow replace ------- by ======== everywhere (?) in the 2-loop terms.
Is there a systematic way to do this?
Yes! The 2PI-expansion. We come back to this later, if time permits.

Daisy resummation for self-energy

Leading IR-divergences are again $n=0$ loops $\quad \pi = \pi_T + \pi_{ring}$.
with

$-\pi_T =$  $+$  $=$  $= \pi_{T,0} + \pi_{T,n\neq 0}$

and

$-\pi_{ring} = \sum_{N=2}^{\infty} S_N'$  $\qquad S_N' = \frac{1}{N!} N! \, (S_\pi)^N$

$$\Rightarrow \quad \pi = \pi_T + \frac{\lambda}{2} T \int_{\vec{p}} \sum_{N=0}^{\infty} (-1)^N \pi_T^N \Delta_0^{N+1} - \pi_{T,0}$$

$$= \pi_T + \frac{\lambda}{2} T \left\{ \int_{\vec{p}} \frac{1}{\vec{p}^2 + m^2 + \pi_T} - \int_{\vec{p}} \frac{1}{\vec{p}^2 + m^2} \right\}$$

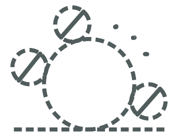
$$= \pi_T - \frac{\lambda T}{8\pi} \left(\sqrt{m^2 + \pi_T} - m \right) \xrightarrow{m \to 0} \frac{\lambda T^2}{24} \left(1 - \frac{\lambda^{1/2} \sqrt{24}}{8\pi} \right)$$

So the ring correction to mass is $\frac{\lambda^{3/2} T^2}{16\sqrt{6}\,\pi}$.

Full Daisy resummation of self-energy

To do full Daisy resummation we again need more ct's at 2-loop level

$-\Pi =$  $- \frac{1}{2}\lambda_R \Pi_T G_0$

$+ \sum_{N=2}^{\infty} S_N'$  $+ \dots$

leftover from
2-loop
renormalized 

$$\Rightarrow \Pi = \Pi_T + \frac{\lambda_R^2}{4} \sum\!\!\!\!\!\!\int \frac{1}{(p^2+m^2)^2}\overbrace{\left(\sum\!\!\!\!\!\!\int \frac{1}{p^2+m^2} - \int_p \frac{1}{p^2+m^2}\right)}^{\frac{2}{\lambda_R}\Pi_T} - \frac{1}{2}\lambda_R \Pi_T \int_p \frac{1}{(p^2+m^2)^2}$$

$$+ \frac{\lambda}{2} \sum\!\!\!\!\!\!\int \underbrace{\sum_{N=2}^{\infty} \Delta (-1)^{\overset{?}{N+1}} (\Pi_T \Delta)^N}$$

$$= \Delta\left(\frac{1}{1+\Pi_T\Delta} - \Pi_T\Delta - 1\right)$$

$$= \Pi_T + \frac{\lambda_R}{2}\Pi_T\Big(\underbrace{\cancel{\sum\!\!\!\!\!\!\int}}_{G_T} - \int_p\Big)\frac{1}{(p^2+m^2)^2} + \frac{\lambda}{2}\sum\!\!\!\!\!\!\int \frac{1}{p^2+m^2+\Pi_T} - \frac{\lambda_R}{2}\sum\!\!\!\!\!\!\int \Delta - \frac{\lambda_R}{2}\Pi_T \cancel{\sum\!\!\!\!\!\!\int \Delta^2}$$

$$= \frac{\lambda_R}{2}\sum\!\!\!\!\!\!\int \frac{1}{p^2+m^2+\Pi_T} - \frac{\lambda_R}{2}\int \frac{1}{p^2+m^2} - \frac{\lambda_R}{2}\Pi_T \int \frac{1}{(p^2+m^2)^2} .$$

$$= \frac{\lambda_R}{2}\left(I_0\left(\sqrt{m^2+\Pi_T}\right) + I_T\left(\sqrt{m^2+\Pi_T}, T\right)\right) - \frac{\lambda_R}{2} iA_0(m^2) - \frac{\lambda_R}{2}\Pi_T \underbrace{iB_0(0,m^2,m^2)}_{-\frac{1}{16\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}} + \log\frac{\mu^2}{m^2}\right)}$$

thermal resummed
I_T-function, OK

This is UV-finite:

$$I_0\left(\sqrt{m^2+\Pi_T}\right) - iA_0(m^2) - \Pi_T\, iB_0(m^2, m^2, 0)$$

$$= \frac{-1}{16\pi^2}\left((m^2+\Pi_T)\left(\cancel{\frac{2}{\epsilon_{\overline{MS}}}} + \cancel{1} + \log\frac{\mu^2}{m^2+\Pi_T}\right) - (m^2+\Pi_T)\left(\cancel{\frac{2}{\epsilon_{\overline{MS}}}} + \cancel{1} + \log\frac{\mu^2}{m^2}\right) + \Pi_T\right)$$

$$= \frac{1}{16\pi^2}\left((m^2+\Pi_T)\log\left(1+\frac{\Pi_T}{m^2}\right) + \Pi_T \right)$$ finite, but again $\xrightarrow{m\to 0} \infty$

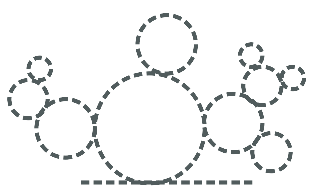
The culprit is the same as before: the $\frac{1}{2}\lambda_R \Pi_T G_0$-term. "Resumming" the vacuum mass in G_0 here would remove the log-term associated with $\Pi_T \Rightarrow$ IR-finite.

Superdaisy resummation & Gap equation

One can extend our previous results to "superdaisy" resummation by generalizing the ring-equation for Π to a gap-equation:

$$\Pi_{SD} \equiv \frac{\lambda_R}{2} \sum\!\!\!\!\!\!\!\int \frac{1}{p^2+m^2+\Pi_{SD}} - \frac{\lambda_R}{2}\int \frac{1}{p^2+m^2} - \frac{\lambda_R}{2}\Pi_{SD}\int\frac{1}{(p^2+m^2)^2}$$

Note that this is UV-finite by our previous argument and its IR-problem for $m=0$ is similar to full ring result. This equation captures diagrams of the type



etc.

Deriving the gap equation more precisely, including consistent renormalization ct's is not that easy. It is best done with 2PI-techniques.

As stated before these "full" resummations may be desirable for continuity in m/T. But they may be useful also in other contexts, due to other reasons.

Large N - expansion

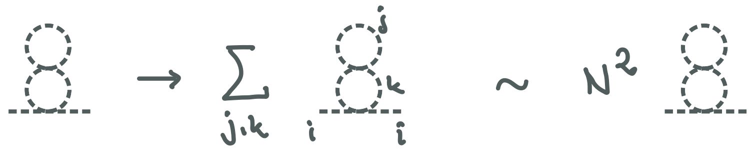
Consider a model, when $\phi \longrightarrow \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$, where $N \gg 1$. But still

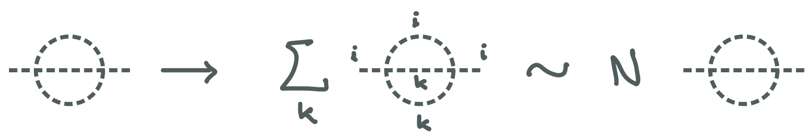
$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{4!}\vec{\phi}^4$$

This is $O(N)$ symmetric model, where $\vec{\phi}^2 = \sum_i \phi_i^2$ and

$$\vec{\phi}^4 = \Big(\sum_i \phi_i^2\Big)^2 = \sum_i \phi_i^4 + 2\sum_{i<j}\phi_i^2\phi_j^2 \qquad \sum_i \phi_i^2 \sum_j \phi_j^2$$

$\Rightarrow$ eg at two-loop level, due to simple combinatorics





So, at large-N limit the simple bubble diagrams dominate and

$$\frac{\Pi_{SD}}{\Pi_{exact}} \longrightarrow 1 \quad \text{when} \quad N \to \infty$$

That is, the SD-result from the Gap-equation is the exact result for $O(N)$-theory in $N \to \infty$ limit.

Other interactions (some simple 2-loop results)

Yukawa theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\phi\psi$$

$$\Rightarrow \quad \mathcal{L}_E = \frac{1}{2}(\partial_\mu\phi)^2_E + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - \bar{\psi}(i\tilde{\not{\partial}} - m_f)\psi + g_f\bar{\psi}\phi\psi$$

where $\tilde{\gamma}^\mu \equiv (i\gamma^0, \vec{\gamma}) \Rightarrow \{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = -2\delta^{\mu\nu}$ — Indices are raised & lowered by $-\delta^{\mu\nu}$ and $-\delta_{\mu\nu}$.

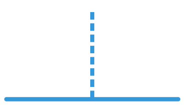
$\Rightarrow$

- $\tilde{\not{p}}\tilde{\not{p}} = \frac{1}{2}\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} p^\mu p^\nu = -p_E^2$
- $\mathrm{Tr}(\tilde{\not{k}}\tilde{\not{p}}) = \frac{1}{2} k_\mu p_\nu \, \mathrm{Tr}(\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\}) = -4\, k\cdot p \xrightarrow{\text{around } d=3} -(d+1)\, k\cdot p$
- $\tilde{\gamma}^\mu \tilde{\not{k}} \gamma_\mu \equiv \{\gamma^\mu, \gamma^\alpha\} k_\alpha \gamma_\mu - \underbrace{\tilde{\gamma}^\mu \tilde{\gamma}_\mu}_{} \tilde{\not{k}} = 2\tilde{\not{k}} - (d+1)\tilde{\not{k}} = -(d-1)\tilde{\not{k}}$

$$= +\delta_{\mu\nu}\tilde{\gamma}^\mu\tilde{\gamma}^\nu = +\delta_{\mu\nu}\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = -\delta_{\mu\nu}\delta^{\mu\nu} = -(d+1)$$

$$\hat{=} \quad \frac{1}{P_0^2 + \omega_p^2} \quad ; \; P_0 = \omega_{nB} + i\mu$$

$$\hat{=} \quad \frac{1}{\tilde{\not{p}} + m} = -\frac{\tilde{\not{p}} - m_f}{\omega_{nF}^2 + m_f^2} \quad ; \quad p^\mu = (\omega_{nF} - i\mu ; \vec{p})$$


$$\hat{=} \quad -g_f\, \delta^4(\Sigma p_i)$$


$$\hat{=} \quad -\frac{\lambda}{4!}\,\delta^4(\Sigma p_i)$$

Fermion line within a loop $\quad \displaystyle\oint_F \frac{1}{\not{p}+m}$

Boson line within a loop $\quad \displaystyle\oint_B \frac{1}{p^2+m^2}$

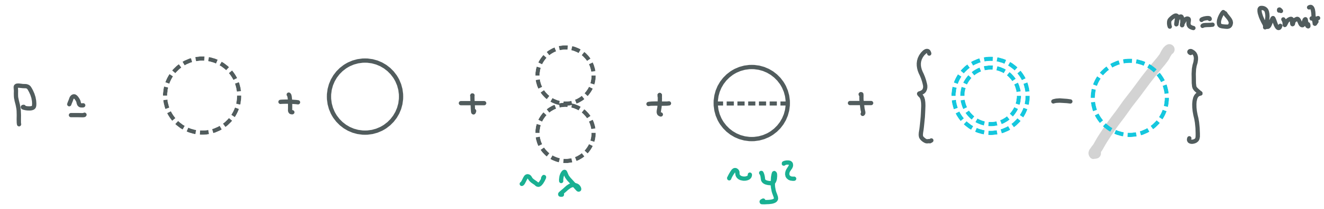
Closed fermion loop add − sign.

Each vertex carries a δ^4-function

$$\delta^4(\Sigma p_i) = \delta(\omega_i)\,\delta^3(\Sigma \vec{p}_i)$$

$\uparrow$ Kronecker delta

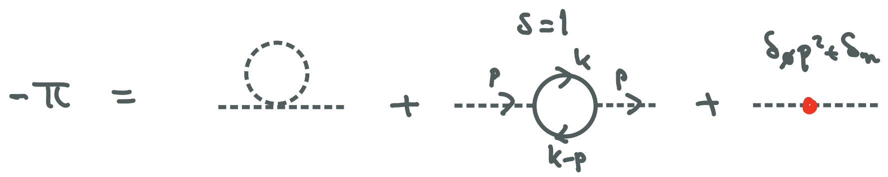
Pressure in Yukawa theory to order $\lambda^{3/2}$, y^3. (in massless case)



From p. 63, the scalar part is just

$$= \frac{\lambda_R}{8}\left(\frac{2}{\lambda_R}\right)^2 \pi_T^2 = \frac{2}{\lambda_R}\pi_T^2 = \frac{\lambda_R}{12\cdot 24} T^4 .$$

Renormalization in limit $p^2 \to 0$ & $m_f \to 0$.



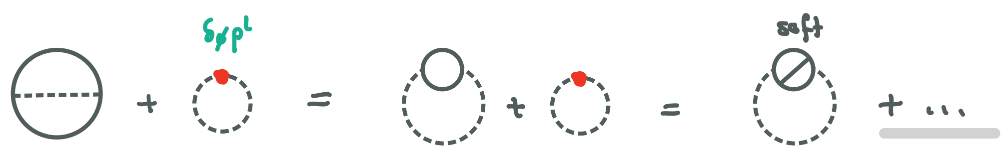
$$\underbrace{k^2}_{}-\tfrac{1}{2}\left((k-p)^2-k^2-p^2\right)$$

$$= -y^2 \int_k \frac{\mathrm{Tr}\left(\tilde{k}(\tilde{k}-\tilde{p})\right)}{(k^2-m_f^2)((k-p)^2-m_f^2)} + \delta_\phi p^2 = 4y^2 \int_k \frac{k^2 - k\cdot p}{(k^2-m_f^2)[(k-p)^2-m_f^2)} + \delta_\phi p^2$$

$$= 2y^2 p^2\, i B_0(0, m_f^2, m_f^2) = \underline{p^2 \cdot \frac{y^2}{8\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}} + \log\frac{\mu^2}{m_f^2}\right) + \delta_\phi p^2} \equiv 0$$

$$\Rightarrow \quad \delta_\phi = -\frac{d\pi}{dp^2} = -\frac{y^2}{8\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}} + \log\frac{\mu^2}{m_f^2}\right)$$

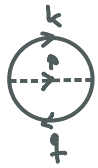
All that δ_ϕ does is cancelling the vacuum IR-div. part from the 2-loop diagram:



In fact, in dim reg one can just put $m_f \equiv 0$ and $p \equiv 0$ to begin with and then use the result $\boxed{\int_k \frac{1}{k^{2n}} \equiv 0}$ $\Rightarrow$ No divergence & no counter-term!

In this definition the IR-singularity ~ "$\log\left(\frac{\mu^2}{m_f^2}\right) \to -\frac{1}{\epsilon_{\overline{MS}}}$" and cancels the UV-divergence. This is standard result with the dim Reg in the massless limit. At any rate, we can compute the 2-loop diagram not worrying of either IR- or UV-divergences.

fermion loop $\qquad -(d+1)\,k\cdot q = \left(\frac{d+1}{2}\right)\left((k-q)^2-k^2-q^2\right)$

$\delta P_{f\phi} = \frac{T}{V}$ 

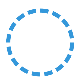
$$+ c.t. = -(-y_f)^2 \sum\!\!\!\!\!\!\int_{k,p,q} \frac{Tr(\slashed{k}\slashed{q})}{p^2k^2q^2}\,\delta^4(k+p-q) + c.t.$$

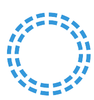
$$= \left(\frac{d+1}{2}\right) y_f^2 \sum\!\!\!\!\!\!\int_{k,p,q} \frac{k^2+q^2-p^2}{k^2q^2p^2}\,\delta^4(k+p-q) + c.t.$$

$$= \frac{d+1}{2}\, y_f^2 \left(\sum\!\!\!\!\!\!\int_{p,B} \sum\!\!\!\!\!\!\int_{k,F} \frac{2}{p^2k^2} - \sum\!\!\!\!\!\!\int_{k,F} \sum\!\!\!\!\!\!\int_{q,F} \frac{1}{k^2q^2} \right) + c.t.$$

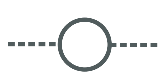
$$= 2y_f^2\left(2I_T^-(0)\,I_T^+(0) - \left(I_T^+(0)\right)^2 \right) = \frac{y_f^2}{96}\,T^4$$

Where we used the fact that in massive limit $\sum\!\!\!\!\!\int_{\pm} \frac{1}{k^2} = I_\pm(0,T) = I_T^\pm(0)$, where $-(+)$ refers to bosons (fermions) and finally $I_T^-(0) = T^2/12$ and $I_T^+(0) = T^2/24$.

Ring correction because  $= 0$, the ring correction now simple

$P_{ring} =$  $= \frac{m^3(T)\,T}{12\pi^2}$

where

$\underset{(=\Pi_\gamma)}{m^2(T)} =$  $+$ 

$$= \frac{\lambda_R T^2}{24} - y^2 \sum\!\!\!\!\!\!\int_{k,F} \frac{\overbrace{Tr(\slashed{k}\slashed{k})}^{\sim 4k^2}}{k^4} = (\lambda_R + 4y^2)\frac{T^2}{24}$$

hard

Combinin all, to order $y^3, \lambda^{3/2}$:

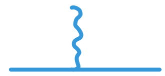
$$P_{yukawa} = \left\{ \frac{\pi^2}{90}\left(1+\frac{7}{2}\right) + \frac{\lambda_R + 3y^2}{288} + \frac{(\lambda_R+4y^2)^{3/2}}{12\pi(24)^{3/2}} \right\} T^4$$

QED The QED Lagrangian in Euclidean space is ($S_E \equiv \int \mathcal{L}_E$)

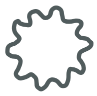
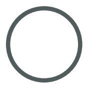
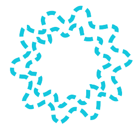
$$\mathcal{L}_E = \bar{\psi}\left(i(\tilde{\not\partial}+ie\tilde{\not A}) - m_f\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

which gives F-rules:

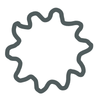
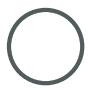

$\frac{1}{p^2}\left(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$


$-e\tilde{\gamma}^\mu$

The QED pressure up to e^3:

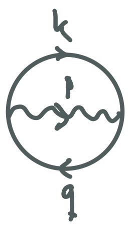
$\beta V P \;\hat{=}$  +  +  + 

1) Free gas limit:

 +  $= \frac{\pi^2}{90}T^4\left(2 + \frac{7}{2}N_f\right)$

Renormalization. $p^2=0$ limit $m_f \equiv 0$, there is no correction in dim. reg.

ii) e^2- correction

$\delta P_{fA} = \frac{T}{V}$ 

$$= -(-e)^2 N_f \oint_{p,k,q} \frac{\mathrm{Tr}(\tilde{\not k}\tilde{\gamma}^\mu \tilde{\not q}\tilde{\gamma}^\nu)}{k^2p^2q^2}\left(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p_2}\right)\delta^4(k+p-q)$$

$$= e^2 N_f \oint_{p,k,q} \frac{\mathrm{Tr}[\tilde{\not k}\tilde{\gamma}^\mu\tilde{\not q}\tilde{\gamma}_\mu]}{k^2p^2q^2}\delta^4(k+p-q) + (1-\xi)\text{-part}$$

$$= (d-1)N_f e^2 \oint_{p,k,q}\frac{k\cdot q}{p^2k^2q^2}\delta(k+p-q) + ec$$

$$= 4e^2N_f\left(\oint_{p_F}\oint_{k_B}\frac{1}{k^2p^2} - \oint_{k_F}\oint_{q_F}\frac{1}{k^2q^2}\right) = \frac{e^2}{48}N_fT^4$$

Gauge invariance

$$\propto \sum\!\!\!\!\!\!\int_{p,k,q} \frac{1}{p^4k^2q^2}\,\mathrm{Tr}\left[\tilde{\not k}\tilde{\not p}\tilde{\not q}\tilde{\not p}\right]\delta^4(k+p-q)$$

$$= \sum\!\!\!\!\!\!\int_{p,k,q} \frac{4}{p^4k^2q^2}\left(2k\cdot p\, q\cdot p - p^2\, k\cdot q\right)\delta^4(k+p-q)$$

$$4\,k\cdot p\; q\cdot p = \left((k+p)^2-k^2-p^2\right)\left(q^2+p^2-(q-p)^2\right) = (q^2-k^2-p^2)(q^2-k^2-p^2)^2$$

$$2p^2\, k\cdot q = p^2\left((q-k)^2-k^2-q^2\right) = p^2(p^2-k^2-q^2)$$

$$\Rightarrow \quad 4\,k\cdot p\; q\cdot p - 2p^2\, k\cdot p = (q^2-k^2)^2 - p^2(q^2+k^2)$$

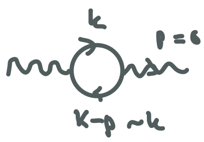
$$= \sum\!\!\!\!\!\!\int_{p,k,q} \frac{2}{p^4k^2q^2}\left(q^4+k^4-2k^2q^2 - p^2(q^2+k^2)\right)\delta^4(k+p-q)$$

$$= 4\sum\!\!\!\!\!\!\int_{q,k,p}\left(\frac{q^2}{p^4k^2} - \overset{\text{d-reg}}{\cancel{\frac{1}{p^4}}} - \frac{1}{p^2k^2}\right)\delta^4(k+p-q) = 4\sum\!\!\!\!\!\!\int_{k,p}\frac{\overbrace{(p+k)^2-p^2}^{=k^2+2k\cdot p}}{p^4k^2}$$

$$= 4\sum\!\!\!\!\!\!\int_{p,k}\left(\overset{\text{dreg}}{\cancel{\frac{1}{p^4}}} + \overset{\text{symm}}{\cancel{\frac{2p\cdot k}{p^4k^2}}}\right) = 0.$$

Ring correction

First we need the photon mass

$$\Pi_{\mu\nu} = -$$ 

$$= -(-e)^2 N_f \sum\!\!\!\!\!\!\int_K \frac{\mathrm{Tr}\left(\tilde{\not K}\tilde{\gamma}^\mu\tilde{\not K}\tilde{\gamma}^\nu\right)}{K^4} = -4e^2N_f\sum\!\!\!\!\!\!\int \frac{2K^\mu K^\nu - K^2\delta^{\mu\nu}}{K^4}$$

- $$\sum\!\!\!\!\!\!\int \frac{K^\mu K^\nu}{K^4} = \delta_{\mu 0}\delta_{\nu 0}\sum\!\!\!\!\!\!\int \frac{\omega_n^2}{K^4} + \delta_{\mu i}\delta_{\nu i}\overbrace{\sum\!\!\!\!\!\!\int \frac{k_i^2}{K^4}}^{=\frac{1}{d}\sum\!\!\!\!\!\!\int \frac{k^2}{K^4} = \frac{1}{d}\sum\!\!\!\!\!\!\int \frac{K^2-\omega_n^2}{K^4}}$$

Bosonic or Fermionic

$$= \delta_{\mu\nu}\delta_{\nu 0}\, I_T'(0) - \frac{1}{3}\delta_{\mu i}\delta_{\nu i}\left(I_T'(0) - I_T(0)\right)$$

$$\frac{\partial}{\partial T} I_T(0) = \frac{\partial}{\partial T} \sum\!\!\!\!\!\!\int_k \frac{1}{k^2} = \sum_n \int_{\vec{k}} \frac{\partial}{\partial T} \frac{T}{\omega_n^2 + \vec{k}^2}$$

$$= \sum_n \int_{\vec{k}} \left(\frac{1}{\omega_n^2 + \vec{k}^2} - \frac{2\omega_n^2}{(\omega_n^2 + \vec{k}^2)^2} \right) = \frac{1}{T} I_T(0) - \frac{2}{T} I_T'(0)$$

$$\Rightarrow I_T'(0) = \frac{1}{2}\left(I_T(0) - T \frac{\partial}{\partial T} I_T(0) \right) = -\frac{1}{2} I_T(0)$$

$$\Rightarrow \sum\!\!\!\!\!\!\int_k \frac{k^\mu k^\nu}{k^4} = \left(\delta_{\mu i}\delta_{\nu i} - \delta_{\mu 0}\delta_{\nu 0} \right) \frac{1}{2} I_T(0)$$

valid for both bosonic & fermionic sum.

$$\Rightarrow \Pi_{\mu\nu} = 4e^2 N_f \left(1 - \left(\delta_{\mu i}\delta_{\nu i} - \delta_{\mu 0}\delta_{\nu 0} \right) \right) I_T^+(0) = 8 e^2 N_f \delta_{\mu 0}\delta_{\nu 0} I_T^+(0)$$

$$= \frac{e^2}{3} T^2 N_f \delta_{\mu 0}\delta_{\nu 0} = m_D^2(T)\, \delta_{\mu 0}\delta_{\nu 0}$$

Debye mass

Thus, only the longitudinal polarization mode of the photon gets a thermal mass. This means that only the electric field is screened, but magnetic field, which comes from spatial components only: $B_i = \epsilon_{ijk} F_{jk}$ is not screened at this level. The ring sum then picks only one component

$$\delta P_{\text{ring}}^{QED} = \frac{m_D^3(T)\, T}{12\pi} = \frac{e^3 N_f^{3/2}}{36\sqrt{3}\,\pi} T^4$$

All together, to order e^3

$$P = \left\{ \frac{\pi^2}{90}\left(2 + \frac{7}{2} N_f \right) + \frac{e^2}{48} N_f T^4 + \frac{e^3 N_f^{3/2}}{36\sqrt{3}\,\pi} \right\} T^4$$

Non-Abelian gauge symmetry.

The PI-Quantization very similar to QED. The difference is that Gauge-transf. is non-linear & hence one gets ghosts even in simple gauges.

Introduction. Consider a model, where $\mathcal{L} = \bar{\psi}_i (i\not{\partial} - m)\psi_i$ is invariant under global SU(N)-transform

$$\psi_i \to U_{ij}\psi_j, \quad \text{where} \quad U \equiv e^{iT^a\theta^a} \quad \text{and} \quad [T^a, T^b] = if^{abc}T^c$$

structure functions

To make this theory locally invariant we write

$$\mathcal{L} = \bar{\psi}(i\not{D} - m\psi) - \frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

sum implied

where the covariant derivative $D_\mu \equiv \delta_{ij}\partial_\mu - igT^a_{ij}A^a_\mu$ and the gauge invariant Yang-Mills field strength tensor is

$$F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T^a - ig[T^b, T^c]A^b_\mu A^c_\nu$$

$$= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)T^a = F^a_{\mu\nu}T^a$$

Invariance of $\bar{\psi}i\not{D}\psi$ implies that when $\psi \to U_\theta\psi = e^{iT^a\theta^a}\psi$

$$T\cdot A \longrightarrow U_\theta\, T\cdot A\, U_\theta^\dagger + \frac{i}{g}U(\partial_\mu U^\dagger)$$

Infinitesimally

$$\delta\psi_\theta \equiv i\theta^a T^a\psi$$

$$\delta A^a_{\theta\mu} = \frac{1}{g}\partial_\mu\theta^a + f^{abc}\theta^b A^c$$

Ex: show these

The obviously GI - YM Lagrangian is

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}^2) = -\frac{1}{2}\overbrace{\mathrm{Tr}(T^aT^b)}^{=\frac{1}{2}\delta^{ab}\ \text{(fundamental reps.)}} F^a_{\mu\nu}F^{b,\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}$$

The quadratic part of $\mathcal{L}_{YM}$ is just sum over QED-like fields, in addition to which we pick cubic and quadratic self-interactions.

Gauge fixing

Again, we introduce covariant gauge condition: $G_\omega[A] \equiv \partial_\mu A^{a\mu} - \omega^a = 0$

$$\Rightarrow Z = \int[\mathcal{D}A^a_\mu]_\beta \Delta^{YM}_{FP}[A^a_\mu]\int[\mathcal{D}\theta^a]\,\delta(G_\omega[A_\theta])\,e^{\int_{x^\beta_E}\mathcal{L}_{YM}}$$

$$= \left(\int[\mathcal{D}\theta]\right)\int[\mathcal{D}A^a_\mu]_\beta\,\Delta^{YM}_{FP}[A^a_\mu]\,\delta(G_\omega[A^{a\mu}])\,e^{\int_{x^\beta_E} -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}}$$

$$\Rightarrow \underline{Z_{phys} = N(\xi)\int[\mathcal{D}A^a_\mu]_\beta\,\Delta^{YM}_{FP}[A^a_\mu]\exp\left(\int_{x^\beta_E}\left[-\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2\right]\right)}$$

with $\log N(\xi) = -\log\prod_a\int[\mathcal{D}\theta_a]\exp\left(-\frac{1}{2\xi}\int_{x^\beta_E}\omega^2\right) \simeq -\frac{N_c^2-1}{2}\mathrm{Tr}\log\xi.$

Ghosts

These appear due to nonlinearity. In R_ξ-gauge

$$\Delta^{YM}_{FP}[A] = \det\left(\frac{\delta\,\partial^\mu\left(\frac{1}{g}\partial_\mu\theta^a_x + f^{adc}\theta^d_x A^c\right)}{\delta\theta^b_y}\right) = \det\left(\partial^\mu\left[\frac{1}{g}\delta^{ab}\partial_\mu + f^{abc}A^c_\mu\right]\right)$$

Bosonic determinant

As usual, we express this as a fermionic integral over ghost fields c and $\bar{c}$

$$\Delta^{YM}_{FP}[A] = \int[\mathcal{D}\bar{c}\,\mathcal{D}c]_\beta\, e^{-\int_{x^\beta_E}\bar{c}_a\left[\partial^\mu(\delta^{ab}\partial_\mu + g f^{abc}A^c_\mu)\right]c_b}$$

periodic

Note: despite their Grassmann nature, ghosts should obey the same (periodic) boundary conditions as gauge fields. ⇒ bosonic Matsubara frequencies

So the full G-fixed YM-theory Generating function is

$$Z[\beta; j^\mu_a, \eta_a, \bar\eta_a] = N(\xi)\exp\left[\int_{x^\beta_E} \mathcal{L}_I\left(\frac{\delta}{\delta j^\mu_a}, \frac{\delta}{\delta \eta_a}\frac{\delta}{\delta \bar\eta_a}\right)\int [\mathcal{D}A^a_\mu]_\beta [\mathcal{D}\bar c\,\mathcal{D}c]_\beta \times\right.$$

$$\times \exp\left[\int_{x^\beta_E} \mathcal{L}_{free} + j^\mu_a A^a_\mu + \bar c_a \eta_a + \bar\eta_a c_a\right]$$

where

$$\mathcal{L}_{free} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar c_a(\delta^{ab}\Box)c_b$$

$$= -\frac{1}{2}A^a_\mu\left(\delta^{\mu\nu}\Box + (1-\tfrac{1}{\xi})\partial^\mu\partial^\nu\right)A^a_\nu - \bar c_a(\delta^{ab}\Box)c_b$$

$$\Box = \partial_\mu\partial^\mu = -\delta_{\mu\nu}\partial^\mu\partial^\nu$$

Quadratic part is, for each a, the same as for QED, so we get

μ, a ~~~~~~ ν, b $= \delta_{ab}\frac{1}{p^2}\left(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$; $p^2 = \omega_n^2 + \omega_p^2$

and

.... $= \delta_{ab}\frac{1}{p^2}$; bosonic matsubara frequencies

The interaction Lagrangian is $F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)$

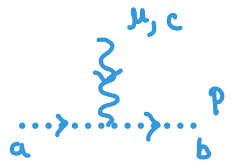
$$\mathcal{L}_I[A_\mu, \bar c_a, c_b] \overset{?}{=} + g f^{abc}(\partial_\mu \bar c_a)A^c_\mu c_b \quad \text{after a partial integration}$$

$$- \frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^b_\alpha A^c_\delta\, \delta^{\alpha\mu}\delta^{\beta\nu}$$

$$- \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\alpha A^e_\beta\, \delta^{\alpha\mu}\delta^{\beta\nu}$$

From these we can infer the F-rules. Eg the ghost rule, using $A = \int(\hat d\epsilon e^{-ikx} + \hat d^+ \epsilon^* e^{ikx})$, $c = \int(\hat a\, e^{-ikx} + \hat b^+_c e^{+ikx})$ and $\bar c = \int(\hat a^+_c e^{ikx} + b_c e^{-ikx})$

$$= \langle 0| a_b \left(-g f^{def} (\partial^\mu \hat{\bar{c}}_d) \hat{A}^e_\mu \hat{c}_f\right) a_a^\dagger d_c^\dagger |0\rangle_{amp}$$

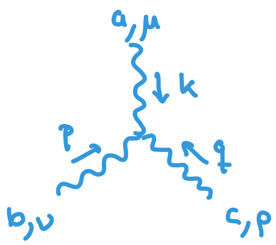


$$\hat{=} -g f^{bca}\; i g p_\mu = i g f^{abc} p_\mu$$

$$\langle 0| -\frac{g}{2} f^{def} (\partial_\alpha \hat{A}^d_\beta - \partial_\beta \hat{A}^d_\alpha) \hat{A}^e_\gamma \hat{A}^f_\delta \; \delta^{\alpha\gamma} \delta^{\beta\delta} \; d^\dagger_{a\mu k} d^\dagger_{b\nu p} d^\dagger_{c\rho q} |0\rangle$$

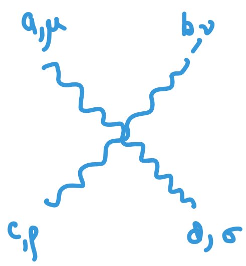
$$\begin{aligned} \hat{=} +ig\Big(& f^{abc} (k_\alpha \delta^{\beta\mu} - k_\beta \delta^{\alpha\mu}) \delta^{\alpha\nu} \delta^{\beta\rho} \\ & + f^{bca} (p_\alpha \delta^{\beta\nu} - p_\beta \delta^{\alpha\nu}) \delta^{\alpha\rho} \delta^{\beta\mu} \\ & + f^{cab} (q_\alpha \delta^{\beta\rho} - q_\beta \delta^{\alpha\rho}) \delta^{\alpha\mu} \delta^{\beta\nu} \Big) \end{aligned}$$

three different contractions. Always take ∂ from $\hat{A}$.



$$= i g f^{abc} \left[\delta_{\mu\nu} (p-k)_\rho + \delta_{\rho\mu} (k-q)_\nu + \delta_{\nu\rho} (q-p)_\mu \right]$$

$$\equiv i g V^{abc}_{\mu\nu\rho}(p,k,q)$$

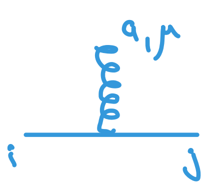


$$\begin{aligned} \hat{=} -g^2 \Big[& f^{eab} f^{ecd} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) \\ & + f^{eac} f^{ebd} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) \\ & + f^{ead} f^{ebc} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma}) \Big] \end{aligned}$$

(Exercise)

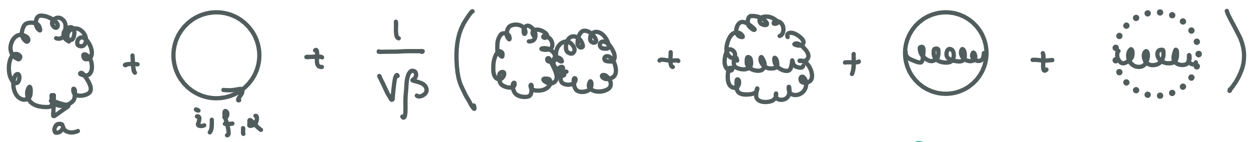
$$= g^2 C^{abcd}_{\mu\nu\rho\sigma}$$

Finally, there is the coupling to fermions $\bar{\Psi}_i \, g T^a_{ij} A^a_\mu \gamma^\mu \Psi_j$

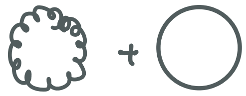


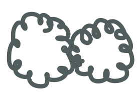
$$= + g T^a_{ij} \tilde{\gamma}^\mu$$

Pressure in QCD to order g^2.

$P =$ 

fermions in fund. representation

-  $= \left\{2(N_c^2-1) + \frac{7}{8}\cdot 4\, N_f N_c\right\}\frac{\pi^2}{90}T^4 = \underbrace{\left(16+\frac{7}{2}\cdot 6\cdot 3\right)}_{=79}\frac{\pi^2}{90}T^4$

- $\frac{1}{\beta V}$  $\hat{=} -\frac{1}{8}g^2 \sum_{ab,\mu\rho} C^{aabb}_{\mu\mu\rho\rho}\left(\oint \frac{1}{k^2}\right)^2 =$

$$= -\frac{1}{8}g^2\, 2 \underbrace{\sum_{a,b,c} f^{abc}f^{abc}}\sum_{\mu,\rho}\left(\delta_{\mu\mu}\delta_{\rho\rho}-\delta_{\mu\rho}\delta_{\mu\rho}\right)\left(\oint\frac{1}{k^2}\right)^2$$

using $f^{acd}f^{bcd} = C_2(G)\delta_{ab} = N_c\delta_{ab} = \sum_{ab} N_c\delta_{ab} = N_c\sum_a = N_c(N_c^2-1)$

$$= -\frac{1}{4}g^2 N_c(N_c^2-1)\, d(d+1)\left[I_T(0)\right]^2 = -72 g^2\left(\frac{T^2}{12}\right)^2 = -\frac{1}{2}g^2T^4$$

This can be also computed contracting directly the vertex:

$\langle\Delta(x)A(x)\rangle = I_T(0)$

$= \oint \frac{1}{(\ )}\frac{d^4k}{(2\pi)^4}$

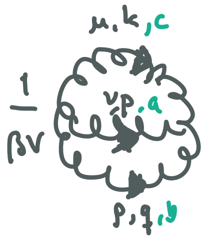
$$\langle 0| -\frac{g^2}{4} f^{abc}f^{ade}\, \hat{A}^b_\mu \hat{A}^c_\nu \hat{A}^d_\alpha \hat{A}^e_\beta\, \delta^{\alpha\mu}\delta^{\beta\nu}|0\rangle$$

$$= -\frac{g^4}{4}f^{abc}f^{ade}\left(\cancel{\delta^{bc}\delta^{de}\delta_{\mu\nu}\delta_{\alpha\beta}} + \delta^{bd}\delta^{ce}\delta_{\mu\alpha}\delta_{\nu\beta} + \delta^{be}\delta^{cd}\delta_{\mu\beta}\delta_{\nu\alpha}\right)\delta^{\alpha\mu}\delta^{\beta\nu}\left(\langle \bar{A}\bar{A}\rangle^2\right)$$

$$= -\frac{g^2}{4}\Big(\underbrace{f^{abc}f^{abc}\delta_{\mu\alpha}\delta^{\mu\alpha}\delta_{\nu\beta}\delta^{\nu\beta}}_{(d+1)^2} + \underbrace{\overbrace{f^{abc}f^{acb}}^{=-(f^{abc})^2}\delta_{\mu\nu}\delta^{\mu\nu}}_{-(d+1)}\Big) I(0)^2$$

$$= -\frac{g^2}{4}N_c(N_c^2-1)\, d(d+1)\, I_T(0)^2$$

For the rest, let us use the Feynman gauge:

$\frac{1}{\beta V}$ 

$= -\frac{1}{2!}\left(\frac{1}{3!}\right)^2 3!\; g^2 \sum_{\substack{a,b,c\\ \mu,\nu,\rho}} \sum\!\!\!\!\!\!\!\int_{k,p,q} \frac{V^{abc}_{\mu\nu\rho}(p,k,q)\,V^{abc}_{\mu\nu\rho}(-p,-k,-q)}{k^2p^2q^2}\,\delta^4(p+k+q)$

$$V^{abc}_{\mu\nu\rho}(p,k,q)\,V^{abc}_{\mu\nu\rho}(-p,-k,-q) = -V^{abc}_{\mu\nu\rho}(p,k,q)\,V^{abc}_{\mu\nu\rho}(p,k,q)$$

$$= -f^{abc}f^{abc}\left[\delta_{\mu\nu}(p-k)_\rho + \delta_{\rho\mu}(k-q)_\nu + \delta_{\nu\rho}(q-p)_\mu\right]\left[\delta_{\mu\nu}(p-k)_\rho + \delta_{\rho\mu}(k-q)_\nu + \delta_{\nu\rho}(q-p)_\mu\right]$$

$$= -N_c(N_c^2-1)\left\{(d+1)\Big(\underbrace{(p-k)^2+(k-q)^2+(q-p)^2}\Big) + \underbrace{2(p-k)\cdot(k-q) + 2(p-k)\cdot(q-p)} + 2(k-q)\cdot(q-p)\right\}$$

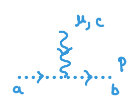
$$= 2(p^2+k^2+q^2) - 2(k\cdot p + k\cdot q + q\cdot p)$$
$$= 2(p^2+k^2+q^2)$$
$$- \left[(k+p)^2+(k+q)^2+(q+p)^2 - 2(k^2+q^2+p^2)\right]$$
$$= 3(p^2+k^2+q^2)$$

$$= 2(p\cdot k - \cancel{p\cdot q} - k^2 + \cancel{k\cdot q} + \cancel{p\cdot q} - p^2 - \cancel{k\cdot q} + \cancel{k\cdot p}$$
$$+ k\cdot q - \cancel{k\cdot p} - q^2 + q\cdot p)$$
$$= 2(-k^2-p^2-q^2 + p\cdot k + q\cdot p + k\cdot q)$$
$$= -3(k^2+p^2+q^2)$$

$$= -N_f(N_f^2-1)\,3d\,(p^2+k^2+q^2)$$

$\Rightarrow \frac{1}{\beta V}$ 
$= \frac{1}{4}N_c(N_c^2-1)(d^2-1)\,g^2 \sum\!\!\!\!\!\!\!\int_{pkq} \frac{\overbrace{p^2+k^2+q^2}^{=3p^2}}{k^2p^2q^2}\,\delta^4(p+k+q)$

$$= \frac{1}{4}N_c(N_c^2-1)\,3d\,g^2\,(I_T(0))^2 = 54\,g^2\,\frac{T^4}{12^2} = \frac{9}{24}\,g^2T^4$$



$\hat{=}\ -g f^{bca}\, i g p_\mu \overset{?}{=} i g f^{abc} p_\mu$

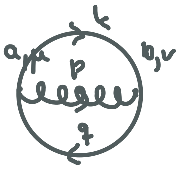
$\frac{1}{\beta V}$ 

$= -\frac{1}{2!}\,g^2 \sum\!\!\!\!\!\!\!\int_{k,p,q} \frac{f^{abc}q_\mu f^{bac}k_\nu\,\delta^{\mu\nu}}{k^2p^2q^2}\,\delta^4(k+p+q)$

$$= \frac{1}{2}g^2 N_c(N_c^2-1) \sum\!\!\!\!\!\!\!\int \frac{k\cdot q}{\underbrace{k^2p^2q^2}}\,\delta^4(k+p+q) = -\frac{1}{4}g^2 N_c(N_c^2-1)\,(I_T(0))^2$$

$$= \frac{1}{2}\frac{(k+q)^2-k^2-q^2}{k^2p^2q^2} = \frac{1}{2}\frac{p^2-k^2-q^2}{k^2p^2q^2} \;\hat{=}\; -\frac{1}{2}\frac{1}{k^2q^2} \qquad = -\frac{g^2}{24}T^4$$

Finally, the fermionic contribution is

$$\frac{1}{\beta V}$$ 
$$= -\frac{g^2}{4} T^a_{ij} T^a_{ji} \sum\!\!\!\!\!\!\int_{p,k,q} \frac{\mathrm{Tr}(\tilde{\not k}\tilde\gamma^\mu \not q \tilde\gamma^\nu)}{k^2p^2q^2}\,\delta_{\mu\nu}\,\delta^4(k+p-q)$$

$$= \frac{g^2}{4} N_f N_c C_2(N) \sum\!\!\!\!\!\!\int_{p,k,q} \frac{\overbrace{k\cdot q}^{=\frac12(k^2+q^2-(k-q)^2)=\frac12(k^2+q^2-p^2)}}{p^2k^2q^2}\,\delta(k+p-q)$$

$$= \frac{g^2}{4} N_f (N_c^2-1)\frac{(d^2-1)}{2}\Big(\underbrace{2 I_T^-(0) I_T^+(0) + I_T^+(0)^2}_{=\frac54 I_T^-(0)^2}\Big) = 24 g^2 (4+1)\frac{T^4}{(24)^2}$$

Alltogether then, to order g^2.

$$P_{QCD} = \frac{\pi^2}{45}\Big(N_c^2-1+\frac74 N_c N_f\Big)T^4 - g^2(N_c^2-1)\Big(\Big(3-\frac94+\frac14\Big)N_c+\frac54 N_f\Big)\frac{T^4}{(12)^2}$$

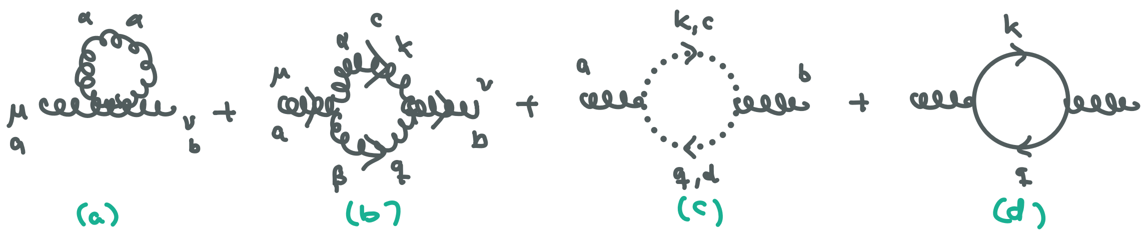
$$= \Big(\frac{79}{90}\pi^2 - \frac{g^2}{144}\Big(N_c+\frac54 N_f\Big)\Big)(N_c^2-1)T^4 = \Big(\frac{316}{45}\pi^2 - \frac{7g^2}{8}\Big)T^4$$

$$= \frac{316}{45}\pi^2\Big(1-\frac{45}{316}\frac{7}{2\pi}\alpha_s\Big)T^4$$

We still miss the ring correction. To compute this, we again need to work out the gluon mass.

Thermal gluon mass. We work in Feynman gauge

a)

$$-\Pi =$$ 

$$\Pi_a = \frac12 g^2 C^{abcd}_{\mu\nu\alpha\beta}\sum\!\!\!\!\!\!\int \frac{\delta_{cd}\delta_{\alpha\beta}}{k^2} = \frac12 g^2 \sum_{c,\alpha} C^{abcc}_{\mu\nu\alpha\alpha}\, I_T(0)$$

$$= g^2 \sum_{c,\alpha} f^{eac} f^{ebc} \left(\delta_{\mu\nu}\delta_{\alpha\alpha} - \delta_{\mu\alpha}\delta_{\nu\alpha} \right) I_T(0)$$

$$= g^2 N_c \delta_{ab}\, d\, \delta_{\mu\nu} \bar{I}_T(0)$$

b)

$$\pi_b = -\frac{g^2}{2} \sum_{\substack{c,d \\ \alpha,\beta}} \oint_{k,q} \frac{V^{acd}_{\mu\alpha\beta}(p,-k,-q)\, V^{bcd}_{\nu\alpha\beta}(-p,k,q)}{k^2 q^2} \delta^4(p-k-q) \quad ; \; p \to 0$$

$$\xrightarrow{p=0} \frac{g^2}{2} \sum_{\substack{c,d \\ \alpha\beta}} \oint_k \frac{V^{acd}_{\mu\alpha\beta}(0,-k,k)\, V^{bcd}_{\mu\alpha\beta}(0,k,-k)}{k^4}$$

$$\sum_{\alpha,\beta,c,d} V^{acd}_{\mu\alpha\beta}(0,-k,k)\, V^{bcd}_{\mu\alpha\beta}(0,-k,k)$$

$$= \sum_{\alpha,\beta,c,d} - f^{acd} f^{bcd} \left[\delta_{\mu\alpha} k_\beta - 2\delta_{\alpha\beta} k_\mu + \delta_{\beta\mu} k_\alpha \right] \left[\delta_{\nu\alpha} k_\beta - 2\delta_{\alpha\beta} k_\nu + \delta_{\beta\nu} k_\alpha \right]$$

$$= -N_c \delta^{ab} \left\{ 2\delta_{\mu\nu} k^2 + \left(-2+1-2+4(d+1)-2+1-2 \right) k_\mu k_\nu \right\}$$

$$= -2N_c \delta^{ab} \left\{ \delta_{\mu\nu} k^2 + (2d-1) k_\mu k_\nu \right\}$$

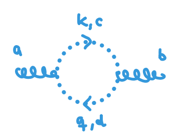
$$\Rightarrow \pi_b = -g^2 N_c \delta^{ab} \left(\delta_{\mu\nu} \oint \frac{1}{k^2} + (2d-1) \oint \frac{k_\mu k_\nu}{k^4} \right)$$

$$= -g^2 N_c \delta^{ab} \left(\delta_{\mu\nu} + \left(d - \tfrac{1}{2}\right)\left(\delta_{\mu i}\delta_{\nu i} - \delta_{\mu 0}\delta_{\nu 0} \right) \right) \bar{I}_T(0)$$

$$\oint \frac{k^\mu k^\nu}{k^4} = \frac{1}{2}\left(\delta_{\mu i}\delta_{\nu i} - \delta_{\mu 0}\delta_{\nu 0} \right) I_T(0)$$

c)

$$\pi_c = -g^2 \oint_{k,q} \frac{f^{acd} q_\mu f^{bcd} k_\nu}{k^2 q^2} \delta^4(k+p-q)$$



$$\xrightarrow{p\to 0} g^2 N_c \delta^{ab} \oint_k \frac{k_\mu k_\nu}{k^4} = \frac{g^2}{2} \delta^{ab} N_c \left(\delta_{\mu i}\delta_{\nu i} - \delta_{\mu 0}\delta_{\nu 0} \right) \bar{I}_T(0)$$

Then

$$\Pi_a + \Pi_b + \Pi_c = g^2 N_c \delta^{ab} (d-1) \left(\delta_{\mu\nu} - (\delta_{\mu i}\delta_{\nu i} - \delta_{\mu 0}\delta_{\nu 0}) \right) I_T(0)$$

$$= 4 g^2 N_c \delta^{ab} \delta_{\mu 0}\delta_{\nu 0} I_T(0) = \frac{g^2}{3} N_c T^2 \delta^{ab} \delta_{\mu 0}\delta_{\nu 0}$$

Finally

$$\Pi_f = g^2 N_f T^a_{ij} T^b_{ji} \SumInt_{k,q} \frac{\mathrm{Tr}(\tilde{\not{k}} \tilde{\gamma}^\mu \not{q} \tilde{\gamma}^\nu)}{k^2 q^2} \delta^4(k-q)$$

$k_0 = 0$

$\Pi^T_{\mu\nu} \sim \Pi^T_{ij} \sim \boxed{\delta_{ij} k^2 - k_i k_j}$

$$= g^2 N_f \frac{1}{2} \delta^{ab} \, 4 \SumInt_{k,F} \frac{2k^\mu k^\nu - k^2}{k^4} = 4 g^2 \delta^{ab} \delta_{\mu 0}\delta_{\nu 0} I^+(0)$$

$$= N_f \delta^{ab} \delta_{\mu 0}\delta_{\nu 0} \frac{T^2}{6}$$

SLAVNOV-TAYLOR

$(g_{\mu\nu} k^2 - k_\mu k_\nu)$

$k_\mu = (0, \vec{k})$

P. 184 Laine & Vuorinen

$$\Rightarrow \quad \Pi_{\mu\nu} = \frac{g^2}{6} (2N_c + N_f) T^2 \delta^{ab} \delta_{\mu 0}\delta_{\nu 0} = m_D^2(T) \delta^{ab} \delta_{\mu 0}\delta_{\nu 0}$$

The ring contribution then becomes

$$\delta P_{ring} = \frac{2}{3\pi} \left(\frac{N_c}{3} + \frac{N_f}{6} \right)^{3/2} g^3 T^4$$

and the full QCD pressure to order g^3 is

$$P_{QCD} = \frac{316}{45} \pi^2 \left(1 - \frac{45}{316} \frac{7}{8\pi^2} g_s^2 + \frac{4\sqrt{2}}{3\pi} g_s^3 \right) T^4$$