[11]S. Coleman, Ref. 9, section 5.

[12]S. Weinberg, Phys. Rev. D 7, 1068 (1973).

[13]For an explanation of the notation, see S. Coleman, Ref. 9, section 5.

[14]We have normalized the field variable $[dA]$ so that $\int[dA]\exp[-\frac{1}{2}(A,DA)]=(\beta\text{-independent constant})\times(\det D)^{1/2}$. With the same normalization for $[df]$, we have

$$\int[df]\exp\left[-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x\, f^2(\vec{x},\tau)\right] = \beta\text{-independent constant}.$$

Note also that the δ function is normalized so that $\int[dA]\delta(A)=1$, so we have

$$\int[df]\exp\left(-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x\, f^2\right)\delta(\partial_\mu A^\mu - f) = \exp\left[-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x(\partial_\mu A^\mu)^2\right].$$

The presence or absence of β-dependent normalization factors is the trickiest part of the whole business. It is therefore comforting to recall that the normalization factors are irrelevant as long as we calculate Green's functions like (2.18) or (3.12) and stay away from calculating $\mathrm{Tr}\, e^{-\beta H}$ itself.

[15]R. P. Feynman and A. P. Hibbs, Ref. 5, Chap. 10. Equation (A5) comes from Feynman's Eq. (10-44) after the correction of a typographical error.

PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Symmetry behavior at finite temperature*

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(Received 18 March 1974)

Spontaneous symmetry breaking at finite temperature is studied. We show that for the class of theories discussed, symmetry is restored above a critical temperature β_c^{-1}. We determine β_c by a functional-diagrammatic evaluation of the effective potential and the effective mass. A formula for β_c is obtained in terms of the renormalized parameters of the theory. By examining a large subset of graphs, we show that the formula is accurate for weak coupling. An approximate gap equation is derived whose solutions describe the theory near the critical point. For gauge theories, special attention is given to ensure gauge invariance of physical quantities. When symmetry is violated dynamically, it is argued that no critical point exists.

I. INTRODUCTION

By drawing an analogy with the Meissner effect, Kirzhnits and Linde[1] have suggested that spontaneous symmetry violation in relativistic field theory will disappear above a critical temperature. They gave qualitative arguments to support this contention in a theory with global symmetry (*not* a gauge theory) and obtained an order-of-magnitude expression for the critical temperature in terms of the parameters of the theory. This problem was next examined by Weinberg, who, in a preliminary investigation,[2] derived a numerical value for the critical temperature in the Kirzhnits-Linde model. He then began to develop a complete analysis of spontaneous symmetry violation and/or persistence at finite temperature, with special emphasis on gauge theories with local symmetries.

It was Weinberg who suggested to us that the diagrammatic-functional methods for evaluating effective potentials in field theory, which had recently been developed,[3-5] might be profitably employed to study temperature effects. We report here the results of our investigation. Weinberg has also presented an analysis of the problem.[6] He uses diagrammatic methods to determine a temperature-dependent mass, as well as operator techniques to compute a temperature-dependent potential. We give a functional-diagrammatic evaluation of these quantities, from which the critical temperature can be deduced. All physical results are in agreement and confirm the qualitative observations of Kirzhnits and Linde.[1]

We examine a field theory at nonzero temperature, or equivalently the ensemble of finite-temperature Green's functions, defined by

$$G_\beta(x_1,\ldots,x_j)=\frac{\mathrm{Tr}\, e^{-\beta H}\, T\varphi(x_1)\cdots\varphi(x_j)}{\mathrm{Tr}\, e^{-\beta H}}. \quad (1.1)$$

Here H is the Hamiltonian governing the dynamics of the field $\varphi(x)$, and β^{-1} is proportional to the temperature. Spontaneous symmetry violation is conveniently studied with the help of the finite-temperature effective action $\Gamma^\beta(\bar\varphi)$—the generating functional for single-particle irreducible Green's functions. Alternatively, $\Gamma^\beta(\bar\varphi)$ may be defined by the following equations:

$$Z^\beta(J)=\frac{\mathrm{Tr}e^{-\beta H}T\exp[i\int d^4x\,\varphi(x)J(x)]}{\mathrm{Tr}e^{-\beta H}}, \qquad (1.2a)$$

$$W^\beta(J)=-i\ln Z^\beta(J), \qquad (1.2b)$$

$$\bar\varphi(x)=\frac{\delta W^\beta(J)}{\delta J(x)} \qquad (1.2c)$$

$$\Gamma^\beta(\bar\varphi)=W^\beta(J)-\int d^4x\,\bar\varphi(x)J(x). \qquad (1.2d)$$

In (1.2d) $J(x)$ is eliminated in favor of $\bar\varphi(x)$ by the definition (1.2c). It follows that

$$\frac{\delta\Gamma^\beta(\bar\varphi)}{\delta\bar\varphi(x)}=-J(x) \qquad (1.3)$$

and $\bar\varphi(x)$, evaluated at $J=0$, is the thermodynamic average of the field $\varphi(x)$:

$$\bar\varphi(x)|_{J=0}=\frac{\mathrm{Tr}e^{-\beta H}\varphi(x)}{\mathrm{Tr}e^{-\beta H}}. \qquad (1.4)$$

It is assumed that H possesses a symmetry which in the normal course of events would imply that $\bar\varphi=0$ at $J=0$. Alternatively, symmetry violation is signaled by a nonvanishing value of $\bar\varphi$, for which $\delta\Gamma^\beta(\bar\varphi)/\delta\bar\varphi(x)$ is zero. Since we do not expect translation invariance to be spontaneously violated, (1.4) should be independent of x. Hence it is sufficient to study $\Gamma^\beta(\bar\varphi)$ for constant $\bar\varphi(x)=\hat\varphi$. The effective potential $V^\beta(\hat\varphi)$ is then defined by

$$V^\beta(\hat\varphi)=-(\text{space-time volume})^{-1}\Gamma^\beta(\bar\varphi)|_{\bar\varphi=\hat\varphi}, \qquad (1.5)$$

and symmetry breaking occurs when $\partial V^\beta(\hat\varphi)/\partial\hat\varphi=0$ for $\hat\varphi\neq 0$. The effective potential is the generating function for single-particle irreducible Green's functions at zero momentum.

Weinberg[6] has given an operator method for calculating $V^\beta(\hat\varphi)$ to the one-loop approximation in nongauge theories, while we develop a functional-diagrammatic method for computing $V^\beta(\hat\varphi)$. Two advantages of the diagrammatic approach should be mentioned: (1) Operator techniques are extremely cumbersome beyond low orders of the perturbation. Progress in conventional (zero temperature) field theory in the last quarter century derives precisely from the supplanting of earlier operator methods with modern diagrammatic analysis. High orders of perturbation, to be sure, remain intractable even in the diagrammatic approach. Nevertheless, the existence of a systematic expansion and of a pictorial method allows one to survey large classes of graphs and to make summations of interesting subsets. (2) The operator method is based on the canonical theory and on the Hamiltonian. Diagrammatic analysis can be formulated purely in terms of an effective Lagrangian. In addition to the attendant simplification, this is important when one comes to discuss gauge theories. For such theories, operator evaluation of $V^\beta(\hat\varphi)$ becomes problematical.[6]

The diagrammatic method for field theory at finite temperature was invented by Martin and Schwinger and others.[7] The crucial observation which reduces this body of work to familiar concepts of zero-temperature field theory is the following. The differential equations satisfied by finite-temperature Green's functions are identical with those of the zero-temperature theory. The difference lies in the boundary conditions. Whereas the familiar causal boundary conditions at $t=\pm\infty$ are appropriate at zero temperature, periodic boundary conditions for imaginary time are relevant at finite temperature. The diagrammatic expansion gives a series solution of these differential equations, where each term in the series is composed of free 2-point functions and vertices. The identity of the differential equations then implies that diagrammatic analysis is the same at finite temperature as at zero temperature. The only difference lies in the type of free 2-point function employed.

The Feynman path integral provides an indefinite integral representation of the differential equations of field theory. However, the path integral does *not* contain a complete specification of the boundary conditions. Hence we may use the same path-integral representation in both cases, supplemented with appropriate boundary conditions. An explicit example will illustrate our remarks. Consider

$$Z(J)=\frac{\int d\varphi\,\exp[i(\tfrac{1}{2}\varphi iD^{-1}\varphi+J\varphi)]}{\int d\varphi\,\exp[i(\tfrac{1}{2}\varphi iD^{-1}\varphi)]}. \qquad (1.6)$$

Here D^{-1} is the inverse propagator for a free spinless field, $D^{-1}(x-y)=i(\Box+m^2)\delta^4(x-y)$. (We are using a compact notation where all summations and integrations are suppressed. Thus $J\varphi=\int d^4x\,J(x)\varphi(x)$,

$$\begin{aligned}\tfrac{1}{2}\varphi iD^{-1}\varphi&=\tfrac{1}{2}\int d^4x\,d^4y\,\varphi(x)iD^{-1}(x-y)\varphi(y)\\&=-\tfrac{1}{2}\int d^4x\,\varphi(x)(\Box+m^2)\varphi(x)\\&=\tfrac{1}{2}\int d^4x[\partial_\mu\varphi(x)\partial^\mu\varphi(x)-m^2\varphi^2(x)].)\end{aligned}$$

Elementary integration gives for (1.6)

$$Z(J)=\exp[-\tfrac{1}{2}JDJ]. \qquad (1.7)$$

The point is that D is not well defined until boundary conditions are given. For finite temperature, (1.7) remains valid, with D determined by the appropriate boundary conditions. [In momentum space, the boundary condition specifies the coefficient of $\delta(p^2-m^2)$; see (3.5) and (4.5) below.]

The above considerations imply that the diagrammatic series for the zero-temperature effective potential previously derived from the Feynman path integral[3] may also be used at finite temperature, with appropriate replacement of the free 2-point function of the conventional theory with the finite-temperature 2-point function. Let us recall the series representation for $V^\beta(\hat{\varphi})$.

Consider a theory described by the (effective) Lagrangian $\mathcal{L}\{\varphi_a(x)\}$. Here φ_a stands for all the fields of the theory labeled by a, not only scalar fields. For gauge theories $\mathcal{L}\{\varphi_a(x)\}$ contains the appropriate gauge-determining and gauge-compensating terms. To compute $V^\beta(\hat{\varphi})$, shift φ_a in $\mathcal{L}\{\varphi_a(x)\}$ by constant fields $\hat{\varphi}_a$, and drop all terms independent of and linear in φ_a. [If one is not interested in $V^\beta(\hat{\varphi})$ as a function of all the fields, but rather in a subset of the fields—e.g., the scalar fields—only those fields need be shifted in $\mathcal{L}\{\varphi_a(x)\}$.] The shifting and truncation procedure defines a new Lagrangian $\hat{\mathcal{L}}\{\hat{\varphi}_a;\varphi_a(x)\}$, which can be decomposed into a "free" term, bilinear in the fields φ_a, and an "interaction" term:

$$\hat{\mathcal{L}}\{\hat{\varphi}_a;\varphi_a(x)\}=\hat{\mathcal{L}}_0\{\hat{\varphi}_a;\varphi_a(x)\}+\hat{\mathcal{L}}_I\{\hat{\varphi}_a;\varphi_a(x)\}. \tag{1.8}$$

The effective potential is

$$V^\beta(\hat{\varphi})=V_0(\hat{\varphi})+V_1^\beta(\hat{\varphi})+i\left\langle \exp\left(i\int d^4x\,\hat{\mathcal{L}}_I\{\hat{\varphi}_a;\varphi_a(x)\}\right)\right\rangle . \tag{1.9}$$

Here $V_0(\hat{\varphi})$ is the classical potential—the tree approximation. $V_1^\beta(\hat{\varphi})$, defined by

$$V_1^\beta(\hat{\varphi})=(\text{space-time volume})^{-1}\times i\ln\int d\varphi_a\exp\left[i\int d^4x\,\hat{\mathcal{L}}_0\{\hat{\varphi}_a,\varphi_a(x)\}\right] \tag{1.10}$$

is the one-loop approximation. Higher loops are given by $\langle\exp[i\int d^4x\,\hat{\mathcal{L}}_I\{\hat{\varphi}_a;\varphi_a(x)\}]\rangle$ —the sum of all the single-particle irreducible vacuum graphs. The free, finite-temperature propagators used in these graphs are to be deduced from $\hat{\mathcal{L}}_0\{\hat{\varphi}_a;\varphi_a(x)\}$, which is quadratic in φ_a, and the vertices are determined by $\hat{\mathcal{L}}_I\{\hat{\varphi}_a;\varphi_a(x)\}$. Also an over-all factor of space-time volume is deleted. Note that the functional integral (1.10) is elementary, since $\hat{\mathcal{L}}_0\{\hat{\varphi}_a;\varphi_a(x)\}$ is quadratic in φ_a.[8]

In Sec. II, we discuss spontaneous symmetry breaking and symmetry persistence at finite temperature and define the critical temperature β_c from $V^\beta(\hat{\varphi})$. In Sec. III, a theory of self-interacting spinless fields is examined, $V^\beta(\hat{\varphi})$ is computed exactly on the one-loop level, and β_c is determined for weak coupling. Next, an approximate two-loop calculation is performed. This is of interest since it demonstrates that no difficulty is encountered with renormalization at finite temperature, and that two-loop effects are negligible for weak coupling. In an $O(N)$-invariant theory, all graphs that dominate for large N are summed, and a gap equation is obtained. The equation determines β_c and gives a parameter-independent description of the theory near the critical temperature. The effect of fermions is briefly considered in Sec. IV. Gauge theories are discussed at length in Sec. V. It is here that our methods are especially useful, since we can calculate in an arbitrary gauge, expose clearly the gauge dependence of the effective potential at finite temperature, and extract a gauge-invariant critical temperature. In the above examples, symmetry breaking is carried by a vacuum expectation value of a scalar field. In Sec. VI we examine an example of dynamical symmetry violation—the Schwinger model of two-dimensional spinor electrodynamics.[9] We show that no critical temperature exists; the "photon" retains its mass at all temperatures. We also argue that in four-dimensional models of dynamical symmetry violation,[10] the same phenomenon should also happen, and symmetry is never restored. Concluding remarks comprise Sec. VII, where we briefly show how our summation methods can be used at zero temperature to establish the occurrence of symmetry breaking.

Appendix A is devoted to a derivation of noninteracting finite-temperature 2-point functions for bosons and fermions. Both the imaginary-time and real-time representations are obtained. Calculations in the text are performed for the most part in the imaginary-time formalism. In Appendix B, some of them are redone in the real-time formalism and various ambiguities of this technique are exposed. Finally, various technical computations are presented in Appendix C.

II. DEFINITION OF THE CRITICAL TEMPERATURE

Consider a theory involving scalar fields φ_a such that the effective potential at finite temperature V^β is a function only of $\hat{\varphi}^2$. At zero temperature $V^\beta(\hat{\varphi}^2)=V^0(\hat{\varphi}^2)$ is assumed to possess a symmetry-breaking solution $\partial V^0(\hat{\varphi}^2)/\partial\hat{\varphi}_a=0$, $\hat{\varphi}_a\neq 0$. We inquire whether the finite-temperature contribution to $V^\beta(\hat{\varphi}^2)$ can eliminate the symmetry breaking so that the only solution to

$$\frac{\partial V^\beta(\hat\varphi^2)}{\partial \hat\varphi_a} = 2\hat\varphi_a \frac{\partial V^\beta(\hat\varphi^2)}{\partial \hat\varphi^2} = 0$$

is $\hat\varphi_a = 0$.

Symmetry breaking will be absent when $\partial V^\beta(\hat\varphi^2)/\partial\hat\varphi^2 \neq 0$ for $\hat\varphi^2 \neq 0$. We shall assume that for large $\hat\varphi^2$, $\partial V^\beta(\hat\varphi^2)/\partial\hat\varphi^2$ is positive; hence persistence of symmetry requires

$$\left.\frac{\partial V^\beta(\hat\varphi^2)}{\partial \hat\varphi^2}\right|_{\hat\varphi\neq 0} > 0 . \tag{2.1}$$

Let us decompose $V^\beta(\hat\varphi^2)$ into its zero-temperature part $V^0(\hat\varphi^2)$ and the finite-temperature part $\overline{V}^\beta(\hat\varphi^2)$. From (2.1) it follows that a *necessary* condition for symmetry persistence is

$$\left.\frac{\partial V^0(\hat\varphi^2)}{\partial \hat\varphi^2}\right|_{\hat\varphi=0} + \left.\frac{\partial \overline{V}^\beta(\hat\varphi^2)}{\partial \hat\varphi^2}\right|_{\hat\varphi=0} \geq 0 . \tag{2.2a}$$

The first term in (2.2a) is recognized to be the renormalized mass parameter of the symmetric theory,

$$m^2\delta_{ab} = \left.\frac{\partial^2 V^0(\hat\varphi^2)}{\partial\hat\varphi_a\partial\hat\varphi_b}\right|_{\hat\varphi=0}$$

$$= 2\delta_{ab}\left.\frac{\partial V^0(\hat\varphi^2)}{\partial\hat\varphi^2}\right|_{\hat\varphi=0} . \tag{2.2b}$$

(The "mass parameter" is defined to be the inverse propagator, i.e., the single-particle irreducible 2-point function, at zero momentum.) Hence (2.2a) may be rewritten as

$$\left.\frac{\partial \overline{V}^\beta(\varphi^2)}{\partial \hat\varphi^2}\right|_{\hat\varphi=0} \geq -\frac{m^2}{2} . \tag{2.2c}$$

(Since symmetry breaking is assumed to take place at zero temperature, $-m^2$ is a positive quantity.) The critical temperature β_c is defined by

$$\left.\frac{\partial \overline{V}^{\beta_c}(\hat\varphi^2)}{\partial \hat\varphi^2}\right|_{\hat\varphi=0} = -\frac{m^2}{2} . \tag{2.3}$$

For weak coupling, in the examples considered by us, it shall be seen that there is only one value of β_c which satisfies (2.3). At a lower temperature $\beta^{-1} < \beta_c^{-1}$, (2.2c) is never satisfied, and symmetry breaking occurs; for high temperature $\beta^{-1} > \beta_c^{-1}$, the symmetry persists.

Equations (2.2c) and (2.3) have an obvious interpretation. The zero-temperature theory possesses symmetry breaking and is characterized by negative m^2. The mass correction due to finite temperature is $2\partial\overline{V}^\beta(\hat\varphi^2)/\partial\hat\varphi^2|_{\hat\varphi=0}$. When this exceeds m^2, the effective mass squared becomes positive and symmetry breaking disappears.

III. SELF-INTERACTING SPINLESS FIELDS

A. Temperature Green's function

The finite-temperature 2-point function of spinless fields is defined by

$$D_\beta(x-y) = \frac{\mathrm{Tr}\, e^{-\beta H} T\varphi(x)\varphi(y)}{\mathrm{Tr}\, e^{-\beta H}} . \tag{3.1}$$

Two diagonal representations for $D_\beta(x-y)$ can be given.

(i) Imaginary time. The time arguments of $D_\beta(x-y)$ are continued to the interval $0 \leq ix_0, iy_0 \leq \beta$, and

$$D_\beta(x) = \int_k e^{-ikx} D_\beta(k) . \tag{3.2}$$

Here $\int_k$ stands for

$$\frac{1}{(-i\beta)}\sum_n \int \frac{d^3k}{(2\pi)^3} ;$$

the summation is over $n = 0, \pm 1, \ldots$. The four-vector k has time component $\omega_n = 2\pi n/(-i\beta)$. For noninteracting fields,

$$D_\beta(k) = \frac{i}{k^2 - m^2}$$

$$= \frac{-i}{(4\pi^2 n^2/\beta^2) + \vec{k}^2 + m^2} . \tag{3.3}$$

Note that for positive m^2, $k^2 - m^2$ is never zero.

(ii) Real time. No continuation is performed and a Fourier representation is given:

$$D_\beta(x) = \int_k e^{-ikx} \overline{D}_\beta(k) . \tag{3.4}$$

Now k is a real Minkowski four-vector and $\int_k = \int d^4k/(2\pi)^4$. In the absence of interactions

$$\overline{D}_\beta(k) = \frac{i}{k^2 - m^2 + i\epsilon} + \frac{2\pi}{e^{\beta E} - 1}\delta(k^2 - m^2) ,$$
$$E = (\vec{k}^2 + m^2)^{1/2} . \tag{3.5}$$

In Appendix A, these formulas will be derived.

Evaluation of the effective potential in the one-loop approximation leads to expressions of the form

$$\ln \mathrm{Det}\, D_\beta(x-y) ,$$

where Det stands for a functional determinant. Since both representations diagonalize D, the above is given, both in the imaginary and real formalism, by[11]

$$(\text{space-time volume})\left\{\int_k \ln D_\beta(k) \text{ or } \int_k \ln \overline{D}_\beta(k)\right\} .$$

In the text we shall use the imaginary-time formalism for the most part; some sample calculations with the real-time formalism are presented in Appendix B, where it will also be seen that the real-time method is sometimes ambiguous.[12]

B. $V^\beta(\hat\varphi^2)$ in the one-loop approximation

We consider the simplest model of one self-interacting Bose field described by the Lagrangian (apart from counterterms)

$$\mathcal{L}\{\varphi(x)\} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 . \tag{3.6}$$

The quadratic part of the shifted Lagrangian is

$$\int d^4x\,\hat{\mathcal{L}}_0\{\hat\varphi;\varphi(x)\} = \int d^4x\,d^4y\,\tfrac{1}{2}\varphi(x) \times i\mathfrak{D}^{-1}\{\hat\varphi; x-y\}\varphi(y), \tag{3.7}$$

$$i\mathfrak{D}^{-1}\{\hat\varphi;k\} = k^2 - M^2, \quad M^2 = m^2 + \tfrac{1}{2}\lambda\hat\varphi^2 .$$

The zero-loop effective potential is temperature-independent:

$$V_0(\hat\varphi^2) = \tfrac{1}{2}m^2\hat\varphi^2 + \frac{\lambda}{4!}\hat\varphi^4 . \tag{3.8}$$

The one-loop approximation has been frequently computed[3-5]; it is

$$\begin{aligned} V_1^\beta(\hat\varphi^2) &= -\tfrac{1}{2}i\int_k \ln i\mathfrak{D}^{-1}\{\hat\varphi;k\} \\ &= \frac{1}{2\beta}\sum_n \int\frac{d^3k}{(2\pi)^3}\ln(k^2-M^2) \\ &= \frac{1}{2\beta}\sum_n \int\frac{d^3k}{(2\pi)^3}\ln\left(-\frac{4\pi^2n^2}{\beta^2}-E_M^{\,2}\right), \\ & \qquad E_M^{\,2} = \vec{k}^2 + M^2 . \end{aligned} \tag{3.9}$$

The sum on n diverges; it may be evaluated by the following trick. Define

$$\begin{aligned} v(E) &= \sum_n \ln\left(\frac{4\pi^2n^2}{\beta^2}+E^2\right), \\ \frac{\partial v(E)}{\partial E} &= \sum_n \frac{2E}{4\pi^2n^2/\beta^2+E^2} . \end{aligned} \tag{3.10}$$

From the fact that

$$\sum_{n=1}^{\infty}\frac{y}{y^2+n^2} = -\frac{1}{2y} + \tfrac{1}{2}\pi\coth\pi y , \tag{3.11}$$

we deduce that

$$\begin{aligned} \frac{\partial v(E)}{\partial E} &= 2\beta\left(\frac{1}{2}+\frac{1}{e^{\beta E}-1}\right), \\ v(E) &= 2\beta\left[\frac{E}{2}+\frac{1}{\beta}\ln(1-e^{-\beta E})\right] \\ &\quad + \text{terms independent of } E . \end{aligned} \tag{3.12}$$

Consequently we find, apart from unimportant constants,

$$\begin{aligned} V_1^\beta(\hat\varphi^2) &= \int\frac{d^3k}{(2\pi)^3}\left[\frac{E_M}{2}+\frac{1}{\beta}\ln(1-e^{-\beta E_M})\right] \\ &= V_1^0(\hat\varphi^2) + \overline{V}_1^\beta(\hat\varphi^2), \end{aligned} \tag{3.13a}$$

$$V_1^0(\hat\varphi^2) = \int\frac{d^3k}{(2\pi)^3}\frac{E_M}{2}, \tag{3.13b}$$

$$\overline{V}_1^\beta(\hat\varphi^2) = \frac{1}{2\pi^2\beta^4}\int_0^\infty dx\,x^2\ln(1-e^{-(x^2+\beta^2M^2)^{1/2}}) . \tag{3.13c}$$

The zero-temperature one-loop term, (3.13b), is to be compared to the usual expression[3-5]

$$V_1^0(\hat\varphi^2) = -\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln(-k_0^{\,2}+\vec{k}^2+M^2-i\epsilon) . \tag{3.14a}$$

That this agrees with (3.13b) follows from the fact that, apart from an infinite constant,

$$-\frac{i}{2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\ln(-k_0^{\,2}+E^2-i\epsilon) = \tfrac{1}{2}E . \tag{3.14b}$$

Thus, we know from previous calculations that[3-5]

$$V_1^0(\hat\varphi^2) = \frac{1}{64\pi^2}\left[M^4\ln\frac{M^2}{m^2} - \tfrac{3}{2}(M^2-\tfrac{2}{3}m^2)^2\right], \tag{3.15}$$

where the $\hat\varphi^2$ polynomial is determined by the usual renormalization conditions which are imposed at zero temperature.

The finite-temperature contribution $\overline{V}_1^\beta$, (3.13c), vanishes as it should at zero temperature, $\beta\to\infty$ (for $M^2\geq 0$). *We shall now show that the complete expression (3.13c) cannot be used to determine the critical temperature β_c for symmetry persistence.* The difficulty is that, according to (2.3), β_c is determined by $\overline{V}^\beta$ at $\hat\varphi=0$. But for small values of $\hat\varphi$, $M^2=m^2+\frac{1}{2}\lambda\hat\varphi^2$ is *negative* and $\overline{V}_1^\beta$ becomes complex. (Recall $m^2<0$.) This would lead to a physically unacceptable, complex β_c. The problem is that the higher-loop contributions are significant, if one wishes to compute β_c exactly.

However, if one wishes to compute β_c approximately for small β_c (high temperature), we may expand $\overline{V}_1^\beta$:

$$\begin{aligned} \overline{V}_1^\beta(\hat\varphi^2) &= -\frac{\pi^2}{90\beta^4} + \frac{M^2}{24\beta^2} - \frac{1}{12\pi}\frac{M^3}{\beta} \\ &\quad - \frac{1}{64\pi^2}M^4\ln M^2\beta^2 + \frac{c}{64\pi^2}M^4 + O(M^6\beta^2) . \end{aligned} \tag{3.16}$$

Here $c=\frac{3}{2}+2\ln 4\pi-2\gamma\approx 5.41$. The remaining terms are positive integer powers of $M^2\beta^2$ times an overall factor M^4. Note that the $M^4\ln M^2$ term is the negative of the corresponding zero-temperature contribution (3.15). This expansion is derived in

Appendix C.

The first two terms in (3.16) do not become complex for negative M^2, hence we may rely on them. Thus, according to (2.3) we find

$$\frac{1}{\beta_c{}^2}=\frac{-12m^2}{\frac{1}{2}\lambda}, \tag{3.17}$$

which is indeed large in the weak-coupling limit. This agrees with Kirzhnits, Linde,[1] and Weinberg.[6] In Sec. III C we show that two-loop corrections to (3.17) are insignificant for weak coupling, while in Sec. III D corrections to weak coupling are discussed.

Let us observe that it is possible to obtain (3.17) without first computing the effective potential. According to the general theory presented in Sec. II, especially in the last paragraph, all that is needed is the self-mass correction at finite temperature. In the one-loop approximation this is given by the graph of Fig. 1. Hence the entire temperature-dependent mass is

$$\begin{aligned} m_\beta{}^2 &= m^2+\delta m^2+\frac{\lambda}{2}\int_k \frac{i}{k^2-m^2} \\ &= m^2+\delta m^2+\frac{\lambda}{2}\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\frac{1}{4\pi^2n^2/\beta^2+E_m{}^2} \\ &= m^2+\delta m^2 \\ &\quad +\frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2E_m}+\frac{1}{E_m(e^{\beta E_m}-1)}\right]. \end{aligned} \tag{3.18a}$$

[The "temperature-dependent mass" is defined to be $2\partial V^\beta(\hat\varphi^2)/\partial\hat\varphi^2|_{\hat\varphi=0}$.] The mass counterterm δm^2 cancels the zero-temperature contribution—the first term in the integrand. Thus we are left with

$$\begin{aligned} m_\beta{}^2 &= m^2+\frac{\frac{1}{2}\lambda}{2\pi^2\beta^2}\int_0^\infty dx\,\frac{x^2}{(x^2+\beta^2m^2)^{1/2}} \\ &\qquad\times\frac{1}{e^{(x^2+\beta^2m^2)^{1/2}}-1} \\ &= m^2+\frac{\lambda}{2}\left[\frac{1}{12\beta^2}-\frac{m}{4\pi\beta}+O(m^2\ln m^2\beta^2)\right]. \end{aligned} \tag{3.18b}$$

The critical temperature is then given by

$$\begin{aligned} 0 &= m^2+\frac{\frac{1}{2}\lambda}{2\pi^2\beta_c{}^2}\int_0^\infty dx\,\frac{x^2}{(x^2+\beta_c{}^2m^2)^{1/2}} \\ &\qquad\times\frac{1}{e^{(x^2+\beta_c{}^2m^2)^{1/2}}-1}. \end{aligned} \tag{3.19a}$$

Again we see that the integral is complex for $m^2<0$; hence (3.18) cannot be correct. However, the $1/\beta^2$ part is real; therefore for small $\beta_c{}^2$ we find

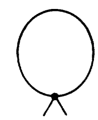


FIG. 1. Lowest-order mass correction.

$$0=m^2+\frac{\frac{1}{2}\lambda}{12\beta_c{}^2}. \tag{3.19b}$$

This agrees with (3.17).

In (3.17) and (3.19), the critical temperature is expressed in terms of $-m^2$, an unphysical, renormalized parameter of the Lagrangian. We may rewrite this in terms of the physical mass of the non-Goldstone meson—the "σ meson." Recalling the lowest-order formula $m_\sigma{}^2=-2m^2$, we have

$$\beta_c{}^2=\frac{1}{6m_\sigma{}^2}\left(\frac{\lambda}{2}\right). \tag{3.20}$$

C. $V^\beta(\hat\varphi^2)$ in the two-loop approximation

We compute $V^\beta(\hat\varphi^2)$ to the two-loop level. Our purpose in this further approximation is to demonstrate explicitly the workings of renormalization at finite temperature and to show that higher orders do not modify the lowest-order calculation of β_c for weak coupling. Since the two-loop calculation is rather tedious, we shall perform it only approximately. We consider an N-component spinless field with an O(N)-invariant interaction.

$$\begin{aligned} &\mathcal{L}\{\varphi_a(x)\}=\tfrac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a-\tfrac{1}{2}m^2\varphi^2-\frac{\lambda}{4!}\varphi^4, \\ &\varphi^2=\varphi_a\varphi_a,\quad \varphi^4=(\varphi^2)^2,\quad a=1,\ldots,N. \end{aligned} \tag{3.21}$$

In each order we keep only the term dominant in N. This calculation has been already performed at zero temperature, and we shall refer to this work for details.[3,13]

The counterterms which must be added to (3.21) are

$$-\tfrac{1}{2}\delta m^2\varphi^2-\frac{\delta\lambda}{4!}\varphi^4. \tag{3.22}$$

To the order we are working it is unnecessary to consider wave-function renormalization.[3] The shifted "free" and "interacting" Lagrangians are

$$\begin{aligned} &\hat{\mathcal{L}}_0\{\hat\varphi_a;\varphi_a(x)\}=\tfrac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a-\tfrac{1}{2}\varphi_aM^2{}_{ab}\varphi_b, \\ &M^2{}_{ab}=[m^2+\delta m^2+\tfrac{1}{6}(\lambda+\delta\lambda)\hat\varphi^2]\delta_{ab}+\tfrac{1}{3}(\lambda+\delta\lambda)\hat\varphi_a\hat\varphi_b, \end{aligned} \tag{3.23a}$$

$$\hat{\mathcal{L}}_I\{\hat\varphi_a;\varphi_a(x)\}=-\tfrac{1}{6}(\lambda+\delta\lambda)\hat\varphi_a\varphi_a\varphi^2-\frac{\lambda+\delta\lambda}{4!}\varphi^4. \tag{3.23b}$$

$\hat{\mathcal{L}}_0$ determines the free propagator at finite temperature:

$$\mathfrak{D}_{ab}\{\hat{\varphi};k\}=\frac{i}{k^2-m_1{}^2}\frac{\hat{\varphi}_a\hat{\varphi}_b}{\hat{\varphi}^2}+\frac{i}{k^2-m_2{}^2}\left(\delta_{ab}-\frac{\hat{\varphi}_a\hat{\varphi}_b}{\hat{\varphi}^2}\right),$$
$$m_1{}^2=m^2+\delta m^2+\tfrac{1}{2}(\lambda+\delta\lambda)\hat{\varphi}^2,\qquad(3.24)$$
$$m_2{}^2=m^2+\delta m^2+\tfrac{1}{6}(\lambda+\delta\lambda)\hat{\varphi}^2,$$
$$k^2=-\frac{4\pi^2n^2}{\beta^2}-\vec{k}^2.$$

The lowest-order effective potential is the tree approximation:

$$V_0(\hat{\varphi}^2)=\tfrac{1}{2}(m^2+\delta m^2)\hat{\varphi}^2+\frac{1}{4!}(\lambda+\delta\lambda)\hat{\varphi}^4.\qquad(3.25)$$

The one-loop term

$$V_1^\beta(\hat{\varphi}^2)=-\tfrac{1}{2}i\int_k \ln\det i\mathfrak{D}^{-1}{}_{ab}\{\hat{\varphi};k\}$$
$$=-\tfrac{1}{2}i\int_k \ln(k^2-m_1{}^2)(k^2-m_2{}^2)^{N-1}\qquad(3.26\text{a})$$

has the dominant N contribution

$$V_1^\beta(\hat{\varphi}^1)=-\tfrac{1}{2}iN\int_k \ln(k^2-m_2{}^2).\qquad(3.26\text{b})$$

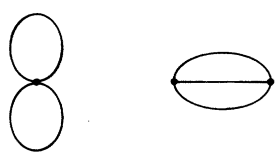
The two-loop contribution consists of the two graphs portrayed in Fig. 2. Previous calculations show that only the double bubble of Fig. 2(a) survives for large N: It is $O(N^2)$, while the graph of Fig. 2(b) is only $O(N)$.[3] Therefore the two-loop term dominant in N is

$$V_2^\beta(\hat{\varphi}^2)=\frac{\lambda}{4!}\left(N\int_k\frac{i}{k^2-m_2{}^2}\right)^2.\qquad(3.27)$$

The effective potential to this order becomes

$$V^\beta(\hat{\varphi}^2)=V_0(\hat{\varphi}^2)-\tfrac{1}{2}iN\int_k\ln(k^2-M^2-\delta M^2)+\tfrac{1}{6}\lambda\left(\tfrac{1}{2}N\int_k\frac{i}{k^2-M^2}\right)^2,\qquad(3.28\text{a})$$

where



(a) (b)

FIG. 2. Two-loop contributions to the effective potential; graph (a) is $O(N^2)$, graph (b) is $O(N)$.

$$M^2=m_2{}^2-\delta M^2$$
$$=m^2+\tfrac{1}{6}\lambda\hat{\varphi}^2,\qquad(3.28\text{b})$$
$$\delta M^2=\delta m^2+\tfrac{1}{6}\delta\lambda\hat{\varphi}^2.$$

The counterterms δM^2 are determined at zero temperature. Since (3.28a) can be expressed as

$$V^\beta(\hat{\varphi}^2)=V_0(\hat{\varphi}^2)-\tfrac{1}{2}iN\int_k\ln(k^2-M^2)+\tfrac{1}{2}\delta M^2N\int_k\frac{i}{k^2-M^2}+\tfrac{1}{6}\lambda\left(\tfrac{1}{2}N\int_k\frac{i}{k^2-M^2}\right)^2,\qquad(3.29)$$

we see that the one-loop term develops a temperature-dependent infinite part to second order, which happily is canceled by a similar temperature-dependent infinity in the two-loop term. The remaining temperature-independent infinities are removed in the usual manner.

A simple calculation gives for the renormalized effective potential

$$V^\beta(\hat{\varphi}^2)=V_0(\hat{\varphi}^2)+V_1^\beta(\hat{\varphi}^2)+\tfrac{1}{6}\lambda\left[\frac{\partial}{\partial M^2}V_1^\beta(\varphi^2)\right]^2,\qquad(3.30\text{a})$$

$$V_0(\hat{\varphi}^2)=\tfrac{1}{2}m^2\hat{\varphi}^2+\frac{\lambda}{4!}\hat{\varphi}^4,\qquad(3.30\text{b})$$

$$V_1^\beta(\hat{\varphi}^2)=\frac{N}{64\pi^2}\left[M^4\ln\frac{M^2}{m^2}-\tfrac{3}{2}(M^2-\tfrac{2}{3}m^2)^2\right]+\frac{N}{2\pi^2\beta^4}\int_0^\infty dx\,x^2\ln(1-e^{-(x^2+\beta^2M^2)^{1/2}})$$
$$=N\left[-\frac{\pi^2}{90\beta^4}+\frac{M^2}{24\beta^2}-\frac{1}{12\pi}\frac{M^3}{\beta}+O(M^4\ln M^2)\right].\qquad(3.30\text{c})$$

The critical temperature obtained from the one-loop potential is evaluated for large temperature as before [compare (3.17)],

$$\frac{1}{\beta_c{}^2}=-\frac{12m^2}{\frac{1}{6}N\lambda}.\qquad(3.31)$$

The two-loop part of V^β in the limit of small β is

$$V_2^\beta(\hat{\varphi}^2)=-N(\tfrac{1}{6}N\lambda)\left[\frac{M}{96\pi\beta^3}+O\left(\frac{M^2}{\beta^2}\right)\right].\qquad(3.32)$$

Note that the dominant, small-β term is imaginary for $M^2<0$. Consequently we cannot include it in a calculation of β_c. In Sec. III D we discuss the order of magnitude of the terms which we have dropped.

D. Discussion of higher-order effects

Our evaluation of the critical temperature to the one-loop level in (3.17), (3.19), or (3.31) was approximate in that the exact one-loop effective potential was expanded for small β and only terms of $O(M^2/\beta^2)$ were retained; see (3.16) and (3.18). We noted that the next term, $O(M^3/\beta)$, cannot be relied upon since it leads to a complex value for β_c. Observe that the contribution of this term to the temperature-dependent mass is [see (3.18)]

$$m_\beta{}^2 = m^2 + \frac{\frac{1}{2}\lambda}{12\beta^2} + O\left(\frac{\lambda m}{\beta}\right). \qquad (3.33)$$

Near the critical temperature $\beta \sim \beta_c = O(\sqrt{\lambda}/|m|)$, hence $\frac{1}{2}\lambda/12\beta^2$ is $O(m^2)$ and $\lambda m/\beta$ is $O(\sqrt{\lambda}\, m^2)$. Thus, the terms we ignore in the one-loop calculation are depressed by a factor $\sqrt{\lambda}$ relative to the terms we keep, and the result for β_c is valid up to terms of order $\sqrt{\lambda}\,\beta_c$.[14]

The two-loop calculation confirms the above estimates. We found the dominant, nonconstant term in the effective potential to be $O(\lambda M/\beta^3)$; it contributes to the effective mass $O(\lambda^2/|m|\beta^3)$; see (3.32). Again, this must be ignored since it is complex. Fortunately, near $\beta=\beta_c$, $O(\lambda^2/|m|\beta^3) = O(\sqrt{\lambda}\, m^2)$; and the error made in our lowest-order formula for β_c is again $O(\sqrt{\lambda}\,\beta_c)$.

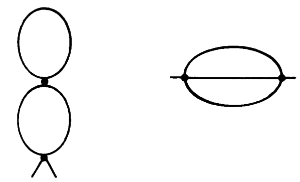
Clearly to improve our calculation it is necessary to survey all multiloop graphs. The next-to-leading terms have a distinguishing property: They are not analytic in the mass parameter near $m^2 = 0$. From the explicit evaluation of the relevant integrals presented in Appendix C, it is seen that these terms arise from the infrared region of integration: The $n=0$ mode in the discrete sum and the $\vec{k}=0$ region in the integration are infrared-singular when the mass vanishes. Thus, a more exact determination of β_c requires an analysis of the infrared behavior of the field theory.

We shall now show that in the O(N)-invariant theory discussed in Sec. III C, the next-to-leading terms can be easily summed, in the limit of large N. It is more convenient to concentrate on the temperature-dependent mass, rather than on the effective potential.

Let us begin by recalling the one- and two-loop calculations of $m_\beta{}^2$. The one-loop term depicted in Fig. 1 contributes to $m_\beta{}^2$ the amount

$$\frac{N\lambda}{6}\left(\frac{1}{12\beta^2} - \frac{m}{4\pi\beta} + \cdots\right).$$

This quantity is the dominant small-β expansion of the "one-vertex bubble" $\int_k i/(k^2 - m^2)$. [It is also $2\partial V_1^\beta(\hat{\varphi}^2)/\partial\hat{\varphi}^2|_{\hat{\varphi}=0}$, where $V_1^\beta(\hat{\varphi}^2)$ is given in (3.30c).] The two-loop contributions to $m_\beta{}^2$ are depicted in Fig. 3. They are just the derivative



(a) (b)

FIG. 3. Two-loop contribution to mass correction; graph (a) is $O(N^2)$, graph (b) is $O(N)$.

of Fig. 2 with respect to $\hat{\varphi}^2$ at $\hat{\varphi}=0$. In the large-N limit, of course, only Fig. 3(a) is relevant. Its value at small β is, according to (3.32)

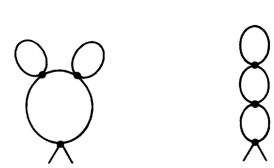
$$2\,\frac{\partial V_2^\beta(\hat{\varphi}^2)}{\partial\hat{\varphi}^2}\bigg|_{\hat{\varphi}=0} = -\left(\frac{N\lambda}{6}\right)^2 \frac{1}{96\pi\beta^3 m}. \qquad (3.34)$$

For purposes of subsequent analysis it is instructive to deduce this number directly from Fig. 3(a). First, the two vertices give a factor $i(-\frac{1}{6}iN\lambda)^2$. Then the upper "one-vertex bubble" gives $1/12\beta^2 - m/4\pi\beta$. The lower "two-vertex bubble" is

$$\int_k \left(\frac{i}{k^2-m^2}\right)^2 = i\,\frac{\partial}{\partial m^2}\int_k \frac{i}{k^2-m^2}$$
$$= i\,\frac{\partial}{\partial m^2}\left(\frac{1}{12\beta^2} - \frac{m}{4\pi\beta}\right)$$
$$= \frac{-i}{8\pi\beta m}.$$

Therefore the entire contribution, which dominates at small β, is $-(\frac{1}{6}\lambda N)^2(1/96\pi\beta^3 m)$. This of course agrees with(3.34).

As we consider higher loops in the large-N limit, it remains true that only iterated bubbles need be considered in each order. Other graphs always involve a lower power of N.[3] Thus in third order, only two graphs are $O(N^3)$. These are drawn in Fig. 4. For small β, the graph of Fig. 4(a) dominates over that of Fig. 4(b). The former involves two "one-vertex bubbles," which give $O((1/\beta^2)^2)$; and one "three-vertex bubble"



(a) (b)

FIG. 4. Three-loop, $O(N^3)$ contribution to mass correction; graph (a) dominates over graph (b) at high temperature.

$$\int_k \left(\frac{i}{k^2-m^2}\right)^3 \propto \left(\frac{\partial}{\partial m^2}\right)^2 \int_k \frac{i}{k^2-m^2}$$
$$\propto \left(\frac{\partial}{\partial m^2}\right)^2 \left(\frac{m}{\beta}\right)$$
$$\propto \frac{1}{m^3\beta}.$$

Therefore, the graph is $O(\lambda^3/m^3\beta^5)$, which for β near β_c is $O(\sqrt{\lambda}\, m^2)$—the same magnitude as the next-to-leading one-loop term and the dominant two-loop term. The latter graph of Fig. 4(b) has two "two-vertex bubbles" giving $O(1/\beta^2 m^2)$, and one "one-vertex bubble" $O(1/\beta^2)$. Thus it is $O(\lambda^3/m^2\beta^4)$, a factor $m\beta = O(\sqrt{\lambda})$ smaller. Similar reasoning in any order yields the conclusion that for large N and small β only one graph is important in each perturbative order: the "daisy" of Fig. 5. (It is clear that we are selecting those graphs which, for zero mass, would be most infrared-divergent.)

The contribution of p-order perturbation theory will now be evaluated. The over-all factor is $i(-\frac{1}{6}iN\lambda)^p$. There are $p-1$ "one-vertex bubbles" giving

$$\left(\int_k \frac{i}{k^2-m^2}\right)^{p-1} = \left(\frac{1}{12\beta^2}\right)^{p-1},$$

and one "p-vertex bubble"

$$\int \left(\frac{i}{k^2-m^2}\right)^p = \frac{1}{(p-1)!}\left(i\frac{\partial}{\partial m^2}\right)^{p-1} \int_k \frac{i}{k^2-m^2}$$
$$= \frac{1}{(p-1)!}\left(i\frac{\partial}{\partial m^2}\right)^{p-1}\left(\frac{1}{12\beta^2} - \frac{m}{4\pi\beta}\right).$$

(Then correctness of the combinatorial factors is established by a tedious study of Wick's theorem.) Therefore the temperature-dependent mass is

$$m_\beta{}^2 = m^2 + \sum_{p=1}^{\infty} i\left(-\frac{iN\lambda}{6}\right)^p \left(\frac{1}{12\beta^2}\right)^{p-1} \frac{1}{(p-1)!} \times \left(i\frac{\partial}{\partial m^2}\right)^{p-1}\left(\frac{1}{12\beta^2} - \frac{m}{4\pi\beta}\right)$$
$$= m^2 + \frac{N\lambda}{6}\sum_{p=0}^{\infty}\frac{1}{p!}\left(\frac{N\lambda}{6}\frac{1}{12\beta^2}\right)^p \times \left(\frac{\partial}{\partial m^2}\right)^p \left(\frac{1}{12\beta^2} - \frac{m}{4\pi\beta}\right)$$
$$= m^2 + \frac{N\lambda}{6}\exp\left(\frac{\frac{1}{6}N\lambda}{12\beta^2}\right)\frac{\partial}{\partial m^2}\left(\frac{1}{12\beta^2} - \frac{(m^2)^{1/2}}{4\pi\beta}\right)$$
$$= m^2 + \frac{N\lambda}{6}\left[\frac{1}{12\beta^2} - \frac{1}{4\pi\beta}\left(m^2 + \frac{\frac{1}{6}N\lambda}{12\beta^2}\right)^{1/2}\right]. \qquad (3.35)$$

The value of the critical temperature is not affected by the inclusion of the next order terms. It is seen from (3.35) that

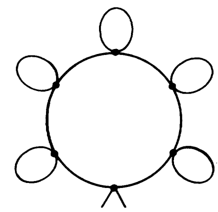


FIG. 5. An example of "daisy" graph contribution to mass correction.

$$m_{\beta_c}{}^2 = 0 \quad \text{for } \frac{1}{\beta_c{}^2} = -\frac{12m^2}{\frac{1}{6}N\lambda}, \qquad (3.36)$$

which reproduces the lowest-order result. Thus, in the large-N limit, the $O(\sqrt{\lambda}\,\beta_c)$ terms give no correction. The summation of daisies has achieved the marvelous result of removing the imaginary, unphysical terms in $m_\beta{}^2$. Clearly (3.35) is real for $\frac{1}{6}N\lambda/12\beta^2 \geq -m^2$, i.e., above the critical temperature.

Nevertheless, the improved expression for $m_\beta{}^2$ is still not satisfactory. The difficulty is that as the critical temperature is approached from above, $m^2 + \frac{1}{6}N\lambda/12\beta^2$ vanishes, and the last term in the square brackets in (3.35) dominates. This gives a *negative* value for $m_\beta{}^2$. Therefore we seek a further refinement of the approximation.

In order to develop the theory further, let us examine our previous formulas for $m_\beta{}^2$. In lowest order one has

$$m_\beta{}^2 = m^2 + \tfrac{1}{6}N\lambda i \int_k (k^2-m^2)^{-1}. \qquad (3.37a)$$

The infinities and counterterms have been removed by renormalization. Hence the integral $i\int_k 1/(k^2-m^2)$ is defined to be just the finite-temperature part. The daisy sum replaces this by

$$m_\beta{}^2 = m^2 + \tfrac{1}{6}N\lambda i \int_k \left[k^2 - m^2 - \tfrac{1}{6}N\lambda i \int_l (l^2-m^2)^{-1}\right]^{-1}. \qquad (3.37b)$$

It is natural to continue the iteration, and we are led to a "gap equation" for $m_\beta{}^2$:

$$m_\beta{}^2 = m^2 + \tfrac{1}{6}N\lambda i \int_k (k^2 - m_\beta{}^2)^{-1}. \qquad (3.38)$$

The graphs summarized by the gap equation (3.38) are the "superdaisies"; an example is in Fig. 6. These graphs exhaust *all* the dominant N contributions, and (3.38) is exact for large N. This is most easily seen by recalling the exact Schwinger-Dyson equation for the propagator in our theory. It is given in Fig. 7. The heavy lines represent

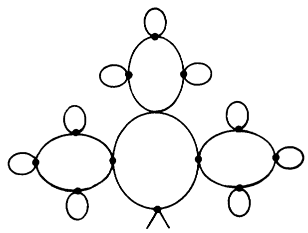


FIG. 6. An example of "superdaisy" graph contribution to mass correction.

the complete propagator. T is the scattering amplitude. The equation is not renormalized, hence m_0^2 is the bare mass and the dot is the bare interaction strength. The large-N limit instructs us to drop the term involving T.[15] The equation is then trivial to solve. We find

$$m_\beta^2 = m_0^2 + \tfrac{1}{6}N\lambda_0 i \int_k (k^2 - m_\beta^2)^{-1}. \tag{3.39}$$

Finally we renormalize and obtain (3.38), which for high temperature and weak coupling becomes

$$m_\beta^2 = m^2 + \tfrac{1}{6}N\lambda\left(\frac{1}{12\beta^2} - \frac{m_\beta}{4\pi\beta}\right). \tag{3.40}$$

The critical temperature which follows from (3.38) or (3.40) is still given by (3.31). In addition, the gap equation can be solved for m_β. When $\beta \sim \beta_c$, (3.40) implies

$$m_\beta = \frac{2\pi}{3}\left(\frac{1}{\beta} - \frac{1}{\beta_c}\right). \tag{3.41}$$

The approach to the critical temperature (from above) is linear in temperature.[15] It is remarkable that all reference to the parameters of the theory has disappeared, and the critical exponent is found to be unity. For high temperature above the critical temperature, the mass is again proportional to the temperature[15]:

$$\begin{aligned} m_\beta &= \left(\frac{\frac{1}{6}N\lambda}{12\beta^2}\right)^{1/2} \\ &= \frac{|m|\beta_c}{\beta}. \end{aligned} \tag{3.42}$$

Our gap equation gives an entirely consistent description of the behavior of m_β above the critical point. Below the critical point our theory is not applicable since a phase transition occurs.

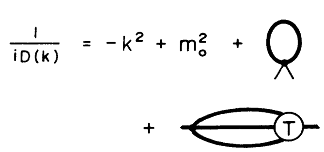


FIG. 7. Schwinger-Dyson equation for the propagator.

It is of great interest to extend the validity of our gap equation so that graphs, subdominant in N, are also included. This entails analyzing the infrared structure of the full Schwinger-Dyson equation, Fig. 7, including the last term. (Only the infrared properties of field theory are important for our calculations of finite-temperature effects.) Also, the procedure of renormalizing temperature-dependent quantities with zero-temperature renormalization conditions must be fully developed. Another interesting line of development can be the study of theories with symmetry groups other than O(N), to see whether exact solutions, in some limit, can again be obtained.

IV. FERMION FIELDS

A. Temperature Green's function

For Fermi fields, the finite-temperature Green's function is

$$S_\beta(x-y) = \frac{\mathrm{Tr}\, e^{-\beta H} T\psi(x)\bar\psi(y)}{\mathrm{Tr}\, e^{-\beta H}}. \tag{4.1}$$

S_β can be represented in two ways.

(i) Imaginary time:

$$S_\beta(x) = \int_k e^{-ikx} S_\beta(k). \tag{4.2}$$

The symbol $\int_k$ has the same meaning as in the Bose case, but the time component of k is given by $\omega_n = (2n+1)\pi/(-i\beta)$. For free fermions one has

$$S_\beta(k) = \frac{i}{\not k - m}. \tag{4.3}$$

(ii) Real time:

$$S_\beta(x) = \int_k e^{-ikx}\bar S_\beta(k). \tag{4.4}$$

Here $\int_k = \int d^4k/(2\pi)^4$. The free, momentum-space propagator is

$$\begin{aligned} \bar S_\beta(k) &= \frac{i}{\not k - m} - \frac{2\pi}{e^{\beta E}+1}(\not k + m)\delta(k^2 - m^2), \\ E &= (\vec k^2 + m^2)^{1/2}. \end{aligned} \tag{4.5}$$

These formulas are derived in Appendix A.[12]

B. $V^\beta(\hat\varphi)$ in the one-loop approximation

If the theory is described by the Lagrangian

$$\mathcal{L}\{\varphi_a(x), \psi(x)\} = i\bar\psi\not\partial\psi - m\bar\psi\psi - \bar\psi G^a\psi\varphi_a + \text{boson Lagrangian}, \tag{4.6}$$

where φ_a is a multiplet of Bose fields and the G^a's are matrices, then the shifted "free" Lagrangian is

$$\hat{\mathcal{L}}_0\{\hat{\varphi};\varphi(x),\psi(x)\}=i\bar{\psi}\not{\partial}\psi-\bar{\psi}M\psi+\text{boson terms}\,,$$
$$M=m+G^a\hat{\varphi}_a\,. \tag{4.7}$$

The one-loop effective potential, apart from the boson contribution, has been previously computed for zero temperature[3,5]:

$$V_1^\beta(\hat{\varphi})=i\int_k \ln\det(\not{k}-M)$$
$$=2i\int_k \ln\det(k^2-M^2)$$
$$=-\frac{2}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\sum_i\ln\left(\frac{(2n+1)^2\pi^2}{\beta^2}+E_{M_i}{}^2\right). \tag{4.8}$$

The first determinant is over Dirac and internal indices, while the second is only over internal indices. M_i is the ith eigenvalue of the matrix M, and the sum in i is over these eigenvalues. The n summation is evaluated by the same differentiation trick employed in (3.10). We find

$$V_1^\beta(\hat{\varphi})=-4\int\frac{d^3k}{(2\pi)^3}\sum_i\left[\tfrac{1}{2}E_{M_i}+\frac{1}{\beta}\ln(1+e^{-\beta E_{M_i}})\right], \tag{4.9a}$$

$$V_1^0(\hat{\varphi})=-4\int\frac{d^3k}{(2\pi)^3}\sum_i\tfrac{1}{2}E_{M_i}\,, \tag{4.9b}$$

$$V_1^\beta(\hat{\varphi})=-4\,\frac{1}{2\pi^2\beta^4}\int_0^\infty dx\,x^2\times\sum_i\ln(1+e^{-(x^2+\beta^2M_i{}^2)^{1/2}})\,. \tag{4.9c}$$

Various aspects of the above have a simple physical significance. The minus sign is a consequence of Fermi-Dirac statistics; the factor 4 reflects the four degrees of freedom present in a fermion field: particle, antiparticle, spin up, spin down. The first term in brackets in (4.9a) is the zero-temperature result; the second arises from finite-temperature effects. For small β, (4.9c) can be expanded; see Appendix C:

$$\overline{V}_1^\beta(\hat{\varphi})=\sum_i\left[\frac{-7\pi^2}{180\beta^4}+\frac{M_i{}^2}{12\beta^2}+\frac{M_i{}^4}{16\pi^2}\ln M_i{}^2\beta^2+\frac{M_i{}^4}{16\pi^2}c+O(M_i{}^6\beta^2)\right], \tag{4.10}$$
$$c=2\gamma-\tfrac{3}{2}-2\ln\pi\approx-2.84\,.$$

The first two terms agree with the formula obtained by Weinberg's operator method.[6]

V. VECTOR-MESON GAUGE THEORIES

A. Preliminary remarks

The development of statistical mechanics for gauge theories raises the following question. Quantization of these theories requires a choice of gauge, which frequently introduces unphysical states in the spectrum. It is not clear whether it is permissible to make the statistical hypothesis for such states. Can one do statistical mechanics in any gauge, or must one select a gauge in which only physical states are present? Even if one decides to calculate in a physical gauge, one does not know which gauge is physical: If symmetry breaking occurs, only the unitary Lagrangian is physical; if symmetry persists, there are several gauges which are physical, for example the Coulomb gauge or the axial gauge in which one component of the vector field is set to zero.

We shall compute the critical temperature in the simplest gauge theory, scalar quantum electrodynamics, and show that it is gauge-invariant on the one-loop level. Thus there is *no* preferred gauge for statistical calculations of the critical temperature in this model. However, two conditions must be met: (1) The critical temperature is to be computed only from the $1/\beta^2$ term in the effective potential. (2) The gauge must *not* be such that higher orders of the perturbation are emphasized. (This will be explained in detail below.) The calculation will be performed for an arbitrary translation-invariant gauge which does not require gauge-compensating terms, as well as in class of gauges requiring gauge-compensating ghosts.

It is for these calculations that our diagrammatic technique becomes especially useful. Not only is a survey of various gauges quite easily performed, but also the operator method meets with difficulties which have been explained by Weinberg.[6]

The theory which we study is described by the Lagrangian (counterterms are suppressed)

$$\mathcal{L}\{\varphi_a(x),A^\mu(x)\}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi^a-\tfrac{1}{2}m^2\varphi^2-\frac{\lambda}{4!}\varphi^4-e\epsilon_{ab}\partial_\mu\varphi_a\varphi_bA^\mu+\tfrac{1}{2}e^2\varphi^2A^2+\text{gauge terms}\,, \tag{5.1}$$
$$\varphi^2=\varphi_a\varphi_a\,,\quad \varphi^4=(\varphi^2)^2\,,\quad a=1,2\,,$$
$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\,.$$

The quadratic part of the shifted Lagrangian is

$$\hat{\mathcal{L}}_0\{\hat{\varphi}_a;\varphi_a(x),A^\mu(x)\}$$
$$=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a-\tfrac{1}{2}\varphi_aM_{ab}\varphi_b-e\epsilon_{ab}\partial_\mu\varphi_a\hat{\varphi}_bA^\mu+\tfrac{1}{2}e^2\hat{\varphi}^2A^2+\text{gauge terms}\,,$$
$$M^2{}_{ab}=(m^2+\tfrac{1}{6}\lambda\hat{\varphi}^2)\delta_{ab}+\tfrac{1}{3}\lambda\hat{\varphi}_a\hat{\varphi}_b\,. \tag{5.2}$$

B. Ghost-free gauges

It is easy to show that the most general inverse propagator for a free photon field in a translation-invariant gauge which does not require ghost-compensating terms is

$$i\Delta^{-1}{}_{\mu\nu}(k) = -k^2 g_{\mu\nu} + k_\mu k_\nu + d_\mu(k) d_\nu(-k), \tag{5.3}$$

where $d^\mu(k)$ is an arbitrary vector which satisfies $k^\mu d_\mu(k) \neq 0$. Hence, for these gauges, (5.2) implies that

$$\begin{aligned} I &= \int d^4x \hat{\mathcal{L}}_0\{\hat{\varphi}_a; \varphi_a(x), A^\mu(x)\} \\ &= \int d^4x d^4y \left[\tfrac{1}{2} A^\mu(x) i\overline{\Delta}^{-1}{}_{\mu\nu}\{\hat{\varphi}; x-y\} A^\nu(y) \right. \\ &\quad + \tfrac{1}{2}\varphi_a(x) i\mathfrak{D}^{-1}{}_{ab}\{\hat{\varphi}; x-y\}\varphi_b(y) \\ &\quad \left. + A^\mu(x) M_{\mu a}\{\hat{\varphi}; x-y\}\varphi_a(y)\right]. \end{aligned} \tag{5.4}$$

In momentum space, the propagators are

$$\begin{aligned} i\overline{\Delta}^{-1}{}_{\mu\nu}\{\hat{\varphi}; k\} &= (-k^2 + e^2\hat{\varphi}^2) g_{\mu\nu} + k_\mu k_\nu \\ &\quad + d_\mu(k) d_\nu(-k), \end{aligned} \tag{5.5a}$$

$$i\mathfrak{D}^{-1}{}_{ab}\{\hat{\varphi}; k\} = k^2 \delta_{ab} - M^2{}_{ab}, \tag{5.5b}$$

$$M^\mu_a\{\hat{\varphi}; k\} = iek^\mu \epsilon_{ab}\hat{\varphi}_b. \tag{5.5c}$$

According to the general procedure, $V^\beta(\hat{\varphi}^2)$ in the one-loop approximation is given by

$$V^\beta(\hat{\varphi}^2) = V_0(\hat{\varphi}^2) + V_1^\beta(\hat{\varphi}^2), \tag{5.6}$$

$$V_0(\hat{\varphi}^2) = \tfrac{1}{2}m^2\hat{\varphi}^2 + \frac{\lambda}{4!}\hat{\varphi}^4. \tag{5.7}$$

To calculate $V_1^\beta(\hat{\varphi})$, the functional integral $\int d\varphi_a dA^\mu e^{iI}$ must be evaluated. We see from (5.4) that I is quadratic and the integration is elementary:

$$\begin{aligned} V_1^\beta(\hat{\varphi}^2) &= -\tfrac{1}{2}i\int_k \ln\det i\mathfrak{D}^{-1}{}_{ab}\{\hat{\varphi}; k\} \\ &\quad - \tfrac{1}{2}i\int_k \ln\det[i\overline{\Delta}_{\mu\nu}\{\hat{\varphi}; k\} + iN_{\mu\nu}\{\hat{\varphi}; k\}], \end{aligned}$$

$$N^{\mu\nu}\{\hat{\varphi}; k\} = M^\mu_a\{\hat{\varphi}; k\}\mathfrak{D}_{ab}\{\hat{\varphi}; k\} M^\nu_b\{\hat{\varphi}; -k\}. \tag{5.8a}$$

Evaluation of the determinants finally gives

$$\begin{aligned} V_1^\beta(\hat{\varphi}^2) &= -\tfrac{1}{2}i\int_k \ln(k^2 - m_1{}^2) - i\int_k \ln(k^2 - \mu^2) \\ &\quad - \tfrac{1}{2}i\int_k \ln\left[(k^2-\mu^2)\left(\mu^2 m_2{}^2 - \frac{(k\cdot d)^2}{k^2}(k^2 - m_2{}^2)\right) - \mu^2 m_2{}^2\left(d^2 - \frac{(k\cdot d)^2}{k^2}\right)\right], \\ m_1{}^2 &= m^2 + \tfrac{1}{2}\lambda\hat{\varphi}^2, \quad m_2{}^2 = m^2 + \tfrac{1}{6}\lambda\hat{\varphi}^2, \quad \mu^2 = e^2\hat{\varphi}^2, \quad (k\cdot d)^2 = k_\mu d^\mu(k) k_\nu d^\nu(-k), \quad d^2 = d_\mu(k) d^\mu(-k). \end{aligned} \tag{5.8b}$$

(i) Lorentz gauges. The evaluation of the remaining k integration in (5.8b) is simple for Lorentz gauges where $d^\mu(k) = (1/\sqrt{\alpha})k^\mu$. With this choice, we have, apart from unimportant constants,

$$\begin{aligned} V_1^\beta(\hat{\varphi}^2) &= -\tfrac{1}{2}i\int_k \ln(k^2 - m_1{}^2) \\ &\quad - \tfrac{3}{2}i\int_k \ln(k^2 - \mu^2) \\ &\quad - \tfrac{1}{2}i\int_k \ln(k^4 - m_2{}^2k^2 + \alpha\mu^2 m_2{}^2) \\ &= V_1^0(\hat{\varphi}^2) + \overline{V}_1^\beta(\hat{\varphi}^2), \end{aligned} \tag{5.9}$$

$$\begin{aligned} V_1^0(\hat{\varphi}^2) &= \frac{1}{64\pi^2}\left[m_1{}^4 \ln\frac{m_1{}^2}{m^2} + 3\mu^4 \ln\frac{\mu^2}{m^2} + R_1{}^4 \ln\frac{R_1{}^2}{m^2}\right. \\ &\quad + R_2{}^4\ln\frac{R_2{}^2}{m^2} - \tfrac{2}{3}\lambda m^2\hat{\varphi}^2 - \tfrac{5}{12}\lambda^2\hat{\varphi}^4 \\ &\quad \left. + \alpha m_2{}^2\mu^2 + a\hat{\varphi}^4\right], \end{aligned} \tag{5.10}$$

$$\begin{aligned} \overline{V}_1^\beta(\hat{\varphi}^2) &= \frac{1}{2\pi^2\beta^4}\int_0^\infty dx\, x^2 [3\ln(1 - e^{-(x^2+\beta^2\mu^2)^{1/2}}) \\ &\quad + \ln(1 - e^{-(x^2+\beta^2 m_1{}^2)^{1/2}}) \\ &\quad + \ln(1 - e^{-(x^2+\beta^2 R_1{}^2)^{1/2}}) \\ &\quad + \ln(1 - e^{-(x^2+\beta^2 R_2{}^2)^{1/2}})]. \end{aligned} \tag{5.11}$$

The $R_i{}^2$'s are roots of $x^2 - m_2{}^2 x + \alpha\mu^2 m_2{}^2$:

$$R_{1,2}{}^2 = \tfrac{1}{2}m_2{}^2\{1 \pm [1 - (4\alpha\mu^2/m_2{}^2)]^{1/2}\}. \tag{5.12}$$

The zero-temperature contribution (5.10) has been conventionally mass-renormalized. Coupling-constant renormalization cannot be determined because of infrared divergences. Hence the quantity a is arbitrary; however, we have arranged it to be of order e^2.

Observe that the effective potential is gauge-dependent—α-dependent. Even if we compute the critical temperature from (2.3), we find a gauge-dependent expression

$$-\frac{m^2}{2}=\frac{\partial \overline{V}_1^{\beta_c}(\hat{\varphi}^2)}{\partial\hat{\varphi}^2}\bigg|_{\hat{\varphi}=0}$$
$$=\frac{1}{4\pi^2\beta_c{}^2}\int_0^\infty dx\,x^2\left[\frac{3e^2}{x(e^x-1)}+\frac{\frac{1}{2}\lambda+\frac{1}{6}\lambda}{(x^2+\beta_c{}^2m^2)^{1/2}(e^{(x^2+\beta_c{}^2m^2)^{1/2}}-1)}\right]$$
$$+\frac{\alpha e^2}{4\pi^2\beta_c{}^2}\int_0^\infty dx\,x^2\left[\frac{1}{x(e^x-1)}-\frac{1}{(x^2+\beta_c{}^2m^2)^{1/2}(e^{(x^2+\beta_c{}^2m^2)^{1/2}}-1)}\right]. \tag{5.13}$$

Clearly this gives an unacceptable, gauge-dependent (α-dependent) answer for $\beta_c{}^2$. However, we have already shown that the one-loop calculation is reliable only for the $1/\beta_c{}^2$ term in (5.13). If this part is extracted, we get a result which is α-independent, hence gauge-invariant, for the class of Lorentz gauges under consideration:

$$-m^2=\frac{1}{12\beta_c{}^2}\left[3e^2+\tfrac{1}{2}\lambda+\tfrac{1}{6}\lambda\right]. \tag{5.14}$$

The numerical coefficients in (5.14) are related to the masses induced by the shift and to the available degrees of freedom. Recall that after the shift there appears a photon "mass" term $e^2\hat{\varphi}^2$, while the two scalar particles acquire additional masses $\frac{1}{2}\lambda\hat{\varphi}^2$ and $\frac{1}{6}\lambda\hat{\varphi}^2$. Finally, the factor 3 arises from the three degrees of freedom of a massive vector meson.

(ii) Other gauges. For arbitrary $d^\mu(k)$ we cannot evaluate (5.8b), since we do not know the k dependence of $d^\mu(k)$. Nevertheless, it is possible to compute the critical temperature. First we determine $\partial V_1^\beta(\hat{\varphi}^2)/\partial\hat{\varphi}^2|_{\hat{\varphi}=0}$:

$$\frac{\partial V_1^\beta(\hat{\varphi}^2)}{\partial\hat{\varphi}^2}\bigg|_{\hat{\varphi}=0}=\frac{i}{2}\int_k\left(\frac{3e^2}{k^2}+\frac{\frac{1}{2}\lambda+\frac{1}{6}\lambda}{k^2-m^2}\right)+\frac{i}{2}e^2m^2\int_k\left(\frac{k^2}{(k\cdot d)^2}-\frac{d^2}{(k\cdot d)^2}+\frac{1}{k^2}\right). \tag{5.15}$$

The first integral in (5.15) corresponds to the first integral in (5.13) and leads to (5.14); the second integral is gauge-dependent. As always, only the $1/\beta^2$ term, for small β, is significant in (5.15). It is not difficult to see that the $1/\beta^2$ term arises only from that part of the integral which in the zero-temperature limit is quadratically divergent. The last two terms in the second integral are of $O(1/k^4)$ for large k; hence no quadratic divergence arises, and they do not contribute to the $O(1/\beta^2)$ term. Only

$$\frac{i}{2}\int_k\frac{e^2m^2}{k^2-m^2}\frac{k^2}{(k\cdot d)^2}$$

can possibly give a $1/\beta^2$ contribution.

For axial gauges $d^\mu(k)=n^\mu/\sqrt{\alpha}$, where n^μ is an arbitrary 4-vector and α is set to zero at the end of the calculation. Similarly, for Coulomb gauges $d^\mu(k)=(1/\sqrt{\alpha})(k^\mu-n^\mu n\cdot k)$; n^μ is a timelike unit vector and α tends to zero. For both cases $k^2/(k\cdot d)^2\propto\alpha$, and the dangerous term vanishes identically. Thus in the physical gauges, the critical temperature is determined by the first integral in (5.15) and agrees with (5.14).

It may appear that there are gauges for which the critical temperature is gauge-dependent, provided $(k\cdot d)^2$ goes as $(k^2)^{-\epsilon}$, $\epsilon\geq -1$ for large k^2. This leads to a quadratic (or stronger) divergence in $\int_k[k^2/(k^2-m^2)]1/(k\cdot d)^2$ at zero temperature, and can contribute to $1/\beta^2$ terms at finite temperature. [For example if we choose $d^\mu(k)=k^\mu/(\alpha k^2)^{1/2}$ then

$$\tfrac{1}{2}i\,e^2m^2\int_k\frac{k^2}{(k^2-m^2)(k\cdot d)^2}=-\tfrac{1}{2}i\alpha e^2m^2\times\int_k\frac{1}{k^2-m^2}$$

and this has a $1/\beta^2$ part, for small β^2.] However, we shall now argue that gauges for which $(k\cdot d)^2$ behaves as above are unacceptable. The point is that the free vector-meson propagator corresponding to (5.3) is

$$i\Delta^{\mu\nu}(k)=\frac{g^{\mu\nu}}{k^2}+\frac{d^2}{(k\cdot d)^2}\frac{k^\mu k^\nu}{k^2}-\frac{d^\mu(k)k^\nu}{k^2k\cdot d(k)}-\frac{k^\mu d^\nu(-k)}{k^2k\cdot d(-k)}-\frac{k^\mu k^\nu}{(k\cdot d)^2}. \tag{5.16}$$

All terms but the last are $O(1/k^2)$ for large k. The last term, however, has an asymptotic behavior determined by $(k\cdot d)^2$, and for the above gauges it is $O((k^2)^{1+\epsilon})$ at large k. This corresponds to a nonrenormalizable theory, and we must expect that higher orders are important. Consequently, it is not surprising that nonsensical results are obtained when the higher orders are ignored.

Thus we conclude that the critical temperature is gauge-invariant on the one-loop level, for ghost-free, translation-invariant gauges which do not emphasize higher orders, provided only the leading high-temperature form is computed.

C. R_ξ gauge

Another popular gauge is the R_ξ gauge,[16] which may also be viewed as a regularization of the unitary Lagrangian. The computation of an effective potential in this gauge requires a modification of the conventional definition of the R_ξ gauge, which has been already given by us.[4] The gauge-fixing and -compensating terms in (5.1) are

$$-\frac{1}{2\alpha}(\partial_\mu A^\mu + v_a\varphi_a)^2 + \partial_\mu\psi^*\partial^\mu\psi - e\psi^*\psi\epsilon_{ab}v_a\varphi_b, \tag{5.17}$$

where v_a is an arbitrary 2-vector and ψ is a ghost field. The gauge terms lead to a shifted, quadratic Lagrangian contributing to (5.2):

$$-\frac{1}{2\alpha}(\partial_\mu A^\mu + v_a\varphi_a)^2 + \partial_\mu\psi^*\partial^\mu\psi - e\psi^*\psi\epsilon_{ab}v_a\hat{\varphi}_b. \tag{5.18}$$

The effective potential is determined by the functional integral $\int d\varphi_a d\psi^* d\psi dA^\mu e^{iI}$, where I is given by

$$\begin{aligned} I &= \int d^4x\,\hat{\mathcal{L}}_0\{\hat{\varphi}_a; \varphi_a(x), A^\mu(x), \psi(x)\} \\ &= \int d^4x\,d^4y\,[\tfrac{1}{2}\varphi_a(x) i\mathfrak{D}^{-1}{}_{ab}\{\hat{\varphi}; x-y\}\varphi_b(y) \\ &\quad + \tfrac{1}{2}A^\mu(x) i\bar{\Delta}^{-1}{}_{\mu\nu}\{\hat{\varphi}; x-y\}A^\nu(y) \\ &\quad + A^\mu(x) M_{\mu a}\{\hat{\varphi}; x-y\}\varphi_a(y) \\ &\quad + \psi^*(x) iS^{-1}\{\hat{\varphi}; x-y\}\psi(y)]. \end{aligned} \tag{5.19}$$

In momentum space, the propagators are

$$\begin{aligned} i\mathfrak{D}^{-1}{}_{ab}\{\hat{\varphi}; k\} &= (k^2 - m^2 - \tfrac{1}{6}\lambda\hat{\varphi}^2)\delta_{ab} - \tfrac{1}{3}\lambda\hat{\varphi}_a\hat{\varphi}_b \\ &\quad - \frac{1}{\alpha}v_a v_b, \end{aligned} \tag{5.20}$$

$$i\bar{\Delta}^{-1}{}_{\mu\nu}\{\hat{\varphi}; k\} = (-k^2 + e^2\hat{\varphi}^2)g_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right)k_\mu k_\nu,$$

$$M_{\mu a}\{\hat{\varphi}; k\} = i\epsilon_{ab}\hat{\varphi}_b k_\mu - \frac{i}{\alpha}k_\mu v_a,$$

$$iS^{-1}\{\hat{\varphi}; k\} = k^2 - e\epsilon_{ab}v_a\hat{\varphi}_b.$$

The functional integral has been evaluated previously.[4] We find[17]

$$\begin{aligned} V_1^\beta(\hat{\varphi}) = -\tfrac{1}{2}i\int_k [&-2\ln iS^{-1}\{\hat{\varphi}; k\} \\ &+ \ln\det i\bar{\Delta}^{-1}{}_{\mu\nu}\{\hat{\varphi}; k\} \\ &+ \ln\det(i\mathfrak{D}^{-1}{}_{ab}\{\hat{\varphi}; k\} + iN_{ab}\{\hat{\varphi}; k\})], \end{aligned} \tag{5.21}$$

$$N_{ab}\{\hat{\varphi}; k\} = M_a^\mu\{\hat{\varphi}; k\}\bar{\Delta}_{\mu\nu}\{\hat{\varphi}; k\}M_b^\nu\{\hat{\varphi}; -k\}.$$

We shall not evaluate the remaining integration completely because of the tedium involved. The $1/\beta^2$ term is easy to extract. Previously we found the quadratically divergent part of the zero-temperature potential[4]:

$$V_1^0(\hat{\varphi}) = \frac{1}{32\pi^2}(3e^2 + \tfrac{1}{2}\lambda + \tfrac{1}{6}\lambda)\hat{\varphi}^2\Lambda^2 + \text{less-divergent terms}. \tag{5.22}$$

This is gauge-independent—α- and v_a-independent. Since the quadratically divergent term at *zero* temperature determines the $1/\beta^2$ term at *high* temperature, we conclude that the critical temperature, computed from (5.21), agrees with the previous result (5.14).

D. The unitary Lagrangian

Yet another way to compute $\beta_c{}^2$ is to work with the unitary Lagrangian $\mathcal{L}_U$, which describes the charge-zero sector of the theory. This Lagrangian is obtained by removing the gauge degrees of freedom, or by taking a limit of the R_ξ Lagrangian. It is

$$\begin{aligned} \mathcal{L}_U\{\rho(x), A^\mu(x), \psi(x)\} &= -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\partial_\mu\rho\partial^\mu\rho - \tfrac{1}{2}m^2\rho^2 \\ &\quad - \frac{\lambda}{4!}\rho^4 + \tfrac{1}{2}e^2A^2\rho^2 + \psi^*\rho\psi, \end{aligned} \tag{5.23}$$

where the ψ fields are ghost fields. The shifted quadratic Lagrangian

$$\begin{aligned} \hat{\mathcal{L}}_{U0}\{\hat{\rho}; \rho(x), A^\mu(x), \psi(x)\} &= -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\partial_\mu\rho\partial^\mu\rho \\ &\quad - \tfrac{1}{2}(m^2 + \tfrac{1}{2}\lambda\hat{\rho}^2)\rho^2 \\ &\quad + \tfrac{1}{2}e^2\hat{\rho}^2A^2 + \psi^*\psi\hat{\rho} \end{aligned} \tag{5.24}$$

leads to the one-loop effective potential[4]

$$\begin{aligned} V_{U1}^\beta(\hat{\rho}^2) &= -\tfrac{1}{2}i\int_k [-2\ln\hat{\rho} + \ln\det((-k^2 + e^2\hat{\rho}^2)g_{\mu\nu} + k_\mu k_\nu) \\ &\qquad + \ln(k^2 - m^2 - \tfrac{1}{2}\lambda\hat{\rho}^2)] \\ &= -\tfrac{1}{2}i\int_k [3\ln(k^2 - e^2\hat{\rho}^2) + \ln(k^2 - m^2 - \tfrac{1}{2}\lambda\hat{\rho}^2)] \\ &\quad + \text{constant}. \end{aligned} \tag{5.25}$$

The zero-temperature limit has been evaluated previously, and it is renormalizable.[4] The $1/\beta^2$ part of the finite-temperature term is, apart from constants, $(1/24\beta^2)(3e^2 + \frac{1}{2}\lambda)\hat{\rho}^2$. Thus, the critical temperature computed by this method *differs* from (5.14):

$$-m^2 = \frac{1}{12\beta_c{}^2}(3e^2 + \tfrac{1}{2}\lambda). \tag{5.26}$$

Equation (5.26) disagrees with (5.14) in that the latter has two contributions proportional to $\frac{1}{2}\lambda$ and $\frac{1}{6}\lambda$ arising from the two scalar degrees of freedom. In the unitary theory there is only one scalar degree of freedom and the $\frac{1}{6}\lambda$ portion is absent.

On the other hand, if one calculates $\beta_c{}^2$ in the regulated unitary theory, i.e., from the R_ξ Lagrangian, and passes to the appropriate limit, one obtains (5.14). This follows from the considerations of Sec. V C, where it is shown that $\beta_c{}^2$ in the R_ξ gauge is gauge-invariant.

We attribute this inconsistency to the fact that the unitary Lagrangian does not correspond to a renormalizable theory (even though the one-loop effective potential is renormalizable). Consequently we must expect that higher-order, multiloop contributions modify the one-loop result (5.26). It is an open and interesting question how one might extract the missing $\frac{1}{6}\lambda$ term from the higher-order contributions to V_U^β. In any case, we believe that the correct answer is (5.14) and not (5.26).

The formula (5.14) for $\beta_c{}^2$ is written in terms of the unphysical parameter $-m^2$. We may use lowest-order expressions which relate $-m^2$ to the physical masses to rewrite (5.14). Recall that the Higgs particle, the "σ meson," has the mass given by $m_\sigma{}^2 = -2m^2$. Also the vector-meson mass satisfies $m_A{}^2 = (3e^2/\lambda)m_\sigma{}^2$. Hence from (5.14) we have

$$\beta_c{}^2 = e^2\left(\frac{1}{2m_\sigma{}^2} + \frac{1}{3m_A{}^2}\right). \tag{5.27}$$

An extension of our computations to non-Abelian gauge theories can be given. A most interesting further investigation would examine whether exact results can be obtained for an O(N) gauge theory in the limit of large N, analogous to our treatment of the O(N) scalar theory in Sec. III.

VI. CRITICAL TEMPERATURE IN THE SCHWINGER MODEL

There exists an explicitly solvable model field theory with dynamical symmetry breaking: the Schwinger model of two-dimensional, massless spinor electrodynamics,[9] governed by the Lagrangian

$$\begin{aligned} \mathcal{L} &= i\bar\psi\partial\!\!\!/\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + eJ^\mu A_\mu\,,\\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu\,,\\ J^\mu &= \bar\psi\gamma^\mu\psi\,. \end{aligned} \tag{6.1}$$

As is well known, because the vacuum-polarization tensor

$$\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2) \tag{6.2}$$

has a pole at $q^2 = 0$,

$$\Pi(q^2) = e^2/\Pi q^2\,, \tag{6.3}$$

the vector meson acquires a mass $\mu = e/\sqrt{\pi}$, and gauge invariance of (6.1) is spontaneously broken.

We show that at finite temperature the gauge symmetry remains broken, in sharp distinction with the state of affairs that transpires when symmetry breaking is carried by a scalar field.

The easiest way to understand spontaneous symmetry violation in (6.1) is through the anomaly of the axial-vector current, $J_5^\mu = i\bar\psi\gamma^\mu\gamma^5\psi$. It is known that in spite of the apparent chiral symmetry of (6.1), the current is not conserved.[18] Rather, one has

$$\partial_\mu J_5^\mu = c\epsilon_{\mu\nu}F^{\mu\nu}, \tag{6.4}$$

where $c = e/2\pi$ at zero temperature. Equation (6.4) together with the equation of motion

$$\partial_\nu F^{\mu\nu} = eJ^\mu \tag{6.5}$$

implies that the vector meson has a mass $(2ce)^{1/2}$. To see this, recall that in two dimensions $J_5^\mu = \epsilon^{\mu\nu}J_\nu$ and $F^{\mu\nu} = \epsilon^{\mu\nu}F$. Hence

$$\begin{aligned} &\partial^\nu\epsilon^{\alpha\mu}F_{\mu\nu} = eJ_5^\alpha\,,\\ &\partial_\alpha\partial^\nu\epsilon^{\alpha\mu}F_{\mu\nu} = ce\epsilon_{\mu\nu}F^{\mu\nu}\,,\\ &-\Box F = 2ceF\,. \end{aligned} \tag{6.6}$$

Consequently, symmetry can reassert itself only if $c = 0$ at some finite temperature.

However, c is not modified by temperature effects. A nonvanishing c is a consequence of the singular short-distance behavior of the zero-temperature theory, while the theory at finite temperature has the same short-distance behavior. This is best seen from the finite-temperature propagator, in the real-time formalism:

$$\begin{aligned} S_\beta(k) &= S(k) + \bar S_\beta(k)\,,\\ S(k) &= \frac{i}{k\!\!\!/}\,,\\ \bar S_\beta(k) &= -\frac{2\pi}{e^{\beta|k|}+1}k\!\!\!/\delta(k^2)\,. \end{aligned} \tag{6.7}$$

Temperature modifies the theory near the mass shell, and not for large k. Hence short-distance behavior is not affected. We conclude therefore, that c remains nonzero at all temperatures, and the symmetry is always broken.

The above general argument may be explicitly verified by computing $\Pi_\beta^{\mu\nu}$ at finite temperature, and showing that it retains the pole:

$$\Pi_\beta^{\mu\nu}(q) = e^2\,\mathrm{Tr}\int\frac{d^2k}{(2\pi)^2}\gamma^\mu S_\beta(q+k)\gamma^\nu S_\beta(k)\,. \tag{6.8}$$

Decomposing $S_\beta(k)$ as in (6.7), we find

$$\begin{aligned} \Pi_\beta^{\mu\nu}(q) &= \Pi_0^{\mu\nu}(q) + \bar\Pi_\beta^{\mu\nu}(q)\,,\\ \bar\Pi_\beta^{\mu\nu}(q) &= A^{\mu\nu}(q) + A^{\nu\mu}(-q) + B^{\mu\nu}(q)\,. \end{aligned} \tag{6.9}$$

$\Pi_0^{\mu\nu}(q)$ is the familiar zero-temperature contribution with a pole at $q^2 = 0$,

$$A^{\mu\nu}(q)=e^2\,\mathrm{Tr}\int\frac{d^2k}{(2\pi)^2}\gamma^\mu S(q+k)\gamma^\nu \overline{S}_\beta(k)\,,$$
$$B^{\mu\nu}(q)=e^2\,\mathrm{Tr}\int\frac{d^2k}{(2\pi)^2}\gamma^\mu \overline{S}_\beta(q+k)\gamma^\nu \overline{S}_\beta(k)\,. \tag{6.10}$$

Upon inserting (6.7) into (6.10), it is easy to show that $A^{\mu\nu}$ and $B^{\mu\nu}$ vanish identically. As an additional check, we have also calculated with the imaginary-time formalism. The result is the same—*the vacuum-polarization tensor at finite temperature coincides with the one at zero temperature.*

A dynamical model of symmetry violation in four dimensions has also been given.[10] For that explicit example we argue that finite-temperature effects do not restore the symmetry. In the dynamical model, symmetry violation is carried by a bound state. The existance of the bound state is a consequence of the non-Fredholm kernel in the Bethe-Salpeter equation. The kernel fails to be Fredholm because of singular high-energy behavior. As high energies are not affected by finite temperature, the bound state will remain even at high temperature and symmetry violation persists.[19] (The model of dynamical symmetry violation considered by Nambu and Jona-Lasinio uses a cutoff field theory.[20] The influence of finite temperatures in this context has not been examined by us.)

VII. CONCLUSION

The restoration of a spontaneously broken symmetry above a critical temperature is a phenomenon whose aspects in field theory have been exhibited in this investigation. An interesting distinction emerges between symmetries broken dynamically and those broken explicitly by scalar fields: The former remain broken at high temperature, the latter can be restored.

We have computed the critical temperature in terms of the renormalized parameters of the theory for a variety of models. Also in the large-N limit of an O(N)-invariant spinless theory, we obtained a parameter-independent description of the behavior near the critical point. Renormalization, which defined the parameters, was performed at zero temperature. This is by no means necessary—an alternate procedure is to renormalize at finite temperature; a convenient point is the critical temperature. In that case β_c is no longer calculable, but other parameters of the theory are expressed in terms of β_c, and no information is lost.[21] Finite-temperature field theories are examples of dynamical systems with long-range (infrared) modifications. It is possible that they present an analog to models with infrared trapping of various excitations.

The leading N summation can also be used at zero temperature to obtain information about the effective potential and about symmetry violation in the O(N)-invariant Bose field theory, (3.21). The zero-temperature version of our gap equation (3.39) is

$$m^2=m_0^{\,2}+\tfrac{1}{6}N\lambda_0\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}, \tag{7.1}$$

where $m^2=2\partial V^0(\hat\varphi^2)/\partial\hat\varphi^2|_{\hat\varphi=0}$. This is not especially interesting, since it merely sums the loops of the mass renormalization. However, it is easy to see that

$$\mathfrak{M}^2=2\frac{\partial V^0(\hat\varphi^2)}{\partial\hat\varphi^2} \tag{7.2}$$

is given, in the leading N approximation, by exactly the same series of loops. Hence

$$\begin{aligned}\mathfrak{M}^2&=m_0^{\,2}+\lambda_0\tfrac{1}{6}\hat\varphi^2+\tfrac{1}{6}N\lambda_0\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mathfrak{M}^2}\\&=m_0^{\,2}+\lambda_0\tfrac{1}{6}\hat\varphi^2+\tfrac{1}{6}N\lambda_0\frac{1}{16\pi^2}[\Lambda^2-\mathfrak{M}^2\ln(\Lambda^2/\mathfrak{M}^2)]\,.\end{aligned} \tag{7.3a}$$

Renormalization is trivial; we find

$$\mathfrak{M}^2=m^2+g\hat\varphi^2+g\mathfrak{M}^2\ln\frac{\mathfrak{M}^2}{|m^2|}, \tag{7.3b}$$

where we have set

$$\begin{aligned}m_0^{\,2}&=m^2-\tfrac{1}{6}N\lambda_0\frac{1}{16\pi^2}\left(\Lambda^2-m^2\ln\frac{\Lambda^2}{|m^2|}\right),\\ \frac{1}{\lambda_0}&=\frac{1}{\lambda}-\tfrac{1}{6}N\frac{1}{16\pi^2}\left(\ln\frac{\Lambda^2}{|m^2|}-1\right),\\ g&=\frac{\tfrac{1}{6}N\lambda(1/16\pi^2)}{1+\tfrac{1}{6}N\lambda(1/16\pi^2)},\end{aligned} \tag{7.4}$$

and have rescaled the field $\hat\varphi$ by $\hat\varphi\to(\sqrt{N}/4\pi)\hat\varphi$. Clearly (7.3b) has a symmetry-breaking solution for $m^2<0$:

$$\mathfrak{M}^2=\,,\quad \hat\varphi^2=-\frac{m^2}{g}. \tag{7.5}$$

A detailed study of (7.3b) is planned.[22]

ACKNOWLEDGMENTS

This research was suggested by S. Weinberg; he also shared with us unpublished results of his investigations of these questions and gave us much useful advice. We thank him. Profitable conversations with S. Colemen, K. Johnson, and P. Martin are acknowledged.

While our paper was readied for publication, we received a report from H. Schnitzer, who also

obtained the zero-temperature equation (7.3) for $\partial V^0(\hat{\varphi}^2)/\partial\hat{\varphi}^2$.

APPENDIX A: FINITE-TEMPERATURE GREEN'S FUNCTIONS

In this appendix we derive the finite-temperature boson and fermion Green's functions for noninteracting fields that have been used in the text. The results are well known[7]; however, we present them here for the convenience of those practitioners of conventional field theory who may be unfamiliar with this formalism.

1. Spinless Bose fields

The finite-temperature 2-point function

$$D_\beta(x-y)=\frac{\mathrm{Tr}\, e^{-\beta H}T\varphi(x)\varphi(y)}{\mathrm{Tr}\, e^{-\beta H}} = \langle T\varphi(x)\varphi(y)\rangle \tag{A1}$$

for noninteracting fields, satisfies

$$(\Box_x+m^2)D_\beta(x-y)=-i\delta^4(x-y)\,. \tag{A2}$$

In order to solve this equation, boundary conditions must be specified. These are given for imaginary time. The time arguments of D_β are continued to the interval $0\leq ix_0, iy_0\leq\beta$, and "time ordering for imaginary time" is defined by

$$\begin{aligned}\langle T\varphi(x)\varphi(y)\rangle &= \langle\varphi(x)\varphi(y)\rangle\\ &= D_\beta^{>}(x-y)\,,\quad ix_0>iy_0\\ &=\langle\varphi(y)\varphi(x)\rangle\\ &= D_\beta^{<}(x-y)\,,\quad iy_0>ix_0\,.\end{aligned} \tag{A3}$$

Note that for imaginary times in the interval $[0,-i\beta]$ we have

$$\begin{aligned}D_\beta(x-y)|_{x_0=0}&=D_\beta^{<}(x-y)|_{x_0=0}\,,\\ D_\beta(x-y)|_{x_0=-i\beta}&=D_\beta^{>}(x-y)|_{x_0=-i\beta}\,.\end{aligned} \tag{A4}$$

The desired boundary condition now follows from (A4):

$$\begin{aligned}(\mathrm{Tr}\, e^{-\beta H})D_\beta^{<}(x-y)|_{x_0=0}&=\mathrm{Tr}\, e^{-\beta H}\varphi(y_0,\vec{y})\varphi(0,\vec{x})\\ &=\mathrm{Tr}\, e^{-\beta H}e^{\beta H}\varphi(0,\vec{x})e^{-\beta H}\\ &\quad\times\varphi(y_0,\vec{y})\\ &=\mathrm{Tr}\, e^{-\beta H}\varphi(-i\beta,\vec{x})\varphi(y_0,\vec{y})\\ &=(\mathrm{Tr}\, e^{-\beta H})D_\beta^{>}(x-y)|_{x_0=-i\beta}\,.\end{aligned} \tag{A5a}$$

Thus we have from (A3) a periodicity condition

$$D_\beta(x-y)|_{x_0=0}=D_\beta(x-y)|_{x_0=-i\beta}\,. \tag{A5b}$$

In the imaginary-time domain, D_β may be represented by Fourier series and integrals, which incorporate (A5b):

$$D_\beta(x-y)=\frac{1}{(-i\beta)}\sum_n e^{-i\omega_n x_0}\int\frac{d^3p}{(2\pi)^3}\frac{1}{(-i\beta)}\sum_{n'} e^{i\omega_{n'}y_0}\int\frac{d^3p'}{(2\pi)^3}D_\beta(\omega_n\vec{p},\,\omega_{n'}\vec{p}')\,,\quad \omega_n=\frac{2\pi n}{-i\beta}\,. \tag{A6a}$$

The summation extends over $n=0,\pm1,\ldots$. The inverse formula is

$$D_\beta(\omega_n\vec{p},\,\omega_{n'}\vec{p}')=\int_0^{-i\beta}dx_0 e^{i\omega_n x_0}\int d^3x e^{-i\vec{p}\cdot\vec{x}}\int_0^{-i\beta}dy_0 e^{-i\omega_{n'}y_0}\int d^3y e^{i\vec{p}'\cdot\vec{y}}D_\beta(x-y)\,. \tag{A6b}$$

Since $D_\beta(x-y)$ depends only on coordinate differences, the above transformation diagonalizes it:

$$D_\beta(\omega_n\vec{p},\,\omega_{n'}\vec{p}')=-i\beta\delta_{nn'}(2\pi)^3\delta^3(\vec{p}-\vec{p}')\,D_\beta(\omega_n,\vec{p})\,. \tag{A7}$$

A compact notation for these transformations is

$$\begin{aligned}D_\beta(x-y)&=\int_p e^{-ip(x-y)}D_\beta(p)\,,\\ D_\beta(p)&=\int_x e^{ipx}D_\beta(x)\,,\end{aligned} \tag{A8a}$$

where

$$\begin{aligned}\int_p&=\frac{1}{-i\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\,,\\ \int_x&=\int_0^{-i\beta}dx_0\int d^3x\,,\end{aligned} \tag{A8b}$$

and p is the 4-vector $(\omega_n,\vec{p})$. This vector is never timelike $p^2=\omega_n{}^2-\vec{p}^2=-(4\pi^2n^2/\beta^2+\vec{p}^2)\leq 0$.

From (A2) and (A8) it follows that $D_\beta(p)$ satisfies

$$\begin{aligned}(-p^2+m^2)D_\beta(p)&=-i\,,\\ D_\beta(p)&=\frac{i}{p^2-m^2}\,.\end{aligned} \tag{A9}$$

There is no ambiguity in the division process since

$p^2 - m^2 \neq 0$ (for $m^2 > 0$). This completes the derivation of the imaginary-time representation for the Green's function.

It is also possible to represent $D_\beta(x-y)$ for real time by Fourier integrals. We define first

$$\overline{D}_\beta^{\gtrless}(k) = \int d^4x\, e^{ikx} D_\beta^{\gtrless}(x)\,. \tag{A10}$$

(The bar indicates a Fourier *integral* transform.) The following manipulations are now performed:

$$\begin{aligned}\overline{D}_\beta^{<}(k_0, \vec{k}) &= \int d^4x\, e^{i(k_0x_0 - \vec{k}\cdot\vec{x})} D_\beta^{<}(x_0, \vec{x})\\ &= \int d^4x\, e^{i(k_0x_0 - \vec{k}\cdot\vec{x})} D_\beta^{>}(x_0 - i\beta, \vec{x})\\ &= e^{-\beta k_0}\int d^4x\, e^{ikx} D_\beta^{>}(x)\\ &= e^{-\beta k_0}\overline{D}_\beta^{>}(k)\,.\end{aligned} \tag{A11}$$

In passing from the first to the second equality, (A5a) was used. Equation (A11) may be summarized by

$$\begin{aligned}&\overline{D}_\beta^{>}(k) = [1 + f(k_0)]\rho(k)\,,\\ &\overline{D}_\beta^{<}(k) = f(k_0)\rho(k)\,,\\ &f(E) = \frac{1}{e^{\beta E} - 1}\,,\\ &\rho(k) = \overline{D}_\beta^{>}(k) - \overline{D}_\beta^{<}(k)\,.\end{aligned} \tag{A12}$$

Knowledge of $\rho(k)$ determines $\overline{D}_\beta(k)$, the Fourier integral transform of $D_\beta(x)$. To see this, observe that

$$\begin{aligned}\overline{D}_\beta(k) &= \int d^4x\, e^{ikx}[\theta(x_0)D_\beta^{>}(x) + \theta(-x_0)D_\beta^{<}(x)]\\ &= i\int_{-\infty}^{\infty}\frac{dk_0'}{2\pi}\left[\frac{\overline{D}_\beta^{>}(k_0', \vec{k})}{k_0 - k_0' + i\epsilon} - \frac{\overline{D}_\beta^{<}(k_0', \vec{k})}{k_0 - k_0' - i\epsilon}\right]\\ &= i\int_{-\infty}^{\infty}\frac{dk_0'}{2\pi}\rho(k_0', \vec{k})\left[\frac{1 + f(k_0')}{k_0 - k_0' + i\epsilon} - \frac{f(k_0')}{k_0 - k_0' - i\epsilon}\right]\\ &= i\int_{-\infty}^{\infty}\frac{dk_0'}{2\pi}\frac{\rho(k_0', \vec{k})}{k_0 - k_0' + i\epsilon} + f(k_0)\rho(k)\,.\end{aligned} \tag{A13}$$

The spectral function $\rho(k)$ can be obtained from the imaginary-time representation for $D_\beta(x)$,

$$D_\beta(\omega_n, \vec{k}) = \int_0^{-i\beta} dx_0 e^{-(2\pi n/\beta)x_0}\int d^3x e^{-i\vec{k}\cdot\vec{x}} D_\beta(x)\,. \tag{A14a}$$

Since $D_\beta^{<}(x-y)|_{y=0} = D_\beta^{>}(x-y)|_{y=0}$ for x_0 in the imaginary interval $[0, -i\beta]$, the above is also given by

$$\begin{aligned}D_\beta(\omega_n, \vec{k}) &= \int_0^{-i\beta} dx_0 e^{-(2\pi n/\beta)x_0}\int d^3x e^{-i\vec{k}\cdot\vec{x}} D_\beta^{>}(x)\\ &= \int_0^{-i\beta} dx_0 e^{-(2\pi n/\beta)x_0}\\ &\quad\times\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}e^{-ik_0x_0}\overline{D}_\beta^{>}(k)\\ &= i\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{e^{-\beta k_0} - 1}{k_0 - 2\pi n/(-i\beta)}[1 + f(k_0)]\rho(k)\\ &= i\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\rho(k_0, \vec{k})}{\omega_n - k_0}\,.\end{aligned} \tag{A14b}$$

Hence to determine $\rho(k)$, we extend $D_\beta(\omega_n, \vec{k})$ to a continuous function $D_\beta(k_0, \vec{k})$, and

$$\rho(k) = D_\beta(k_0 + i\epsilon, \vec{k}) - D_\beta(k_0 - i\epsilon, \vec{k})\,. \tag{A15}$$

In the free-field case where $D_\beta(k)$ is given by (A9), we find

$$\rho(k) = 2\pi\epsilon(k_0)\delta(k^2 - m^2)\,. \tag{A16}$$

Therefore the real-time Green's function which follows from (A13) is

$$\begin{aligned}&\overline{D}_\beta(k) = \frac{i}{k^2 - m^2 + i\epsilon} + \frac{2\pi}{e^{\beta E} - 1}\delta(k^2 - m^2)\,,\\ &E = (\vec{k}^2 + m^2)^{1/2}\,.\end{aligned} \tag{A17}$$

2. Fermion fields

The formalism for Fermi fields is developed analogously, except that anticommutativity must be taken into account. We record, with little comment, the relevant equations:

$$\begin{aligned}S_\beta(x-y) &= \frac{\mathrm{Tr}\, e^{-\beta H} T\psi(x)\overline{\psi}(y)}{\mathrm{Tr}\, e^{-\beta H}}\\ &= \langle T\psi(x)\overline{\psi}(y)\rangle\,,\end{aligned} \tag{A18}$$

$$(i\partial\!\!\!/_x - m)S_\beta(x-y) = i\delta^4(x-y)\,. \tag{A19}$$

For complex time in the interval $[0, -i\beta]$, we have

$$\begin{aligned}\langle T\psi(x)\overline{\psi}(y)\rangle &= \langle\psi(x)\overline{\psi}(y)\rangle\\ &= S_\beta^{>}(x-y)\,,\quad ix_0 > iy_0\\ &= -\langle\overline{\psi}(y)\psi(x)\rangle\\ &= S_\beta^{<}(x-y)\,,\quad iy_0 > ix_0\end{aligned} \tag{A20}$$

$$\begin{aligned}&S_\beta(x-y)|_{x_0=0} = S_\beta^{<}(x-y)|_{x_0=0}\,,\\ &S_\beta(x-y)|_{x_0=-i\beta} = S_\beta^{>}(x-y)|_{x_0=-i\beta}\,.\end{aligned} \tag{A21}$$

The boundary condition is antiperiodic:

$$S_\beta(x-y)|_{x_0=0} = -S_\beta(x-y)|_{x_0=-i\beta} \tag{A22}$$

The imaginary-time formalism leads to

$$S_\beta(p)=\int_x e^{ipx}S_\beta(x)\,, \tag{A23}$$

where $p^\mu=(\omega_n,\vec{p})$ and $\omega_n=(2n+1)\pi/(-i\beta)$. Combining (A23) with the equation of motion (A19), we have

$$S_\beta(p)=\frac{i}{\not p-m}\,. \tag{A24}$$

For real time, $\bar{S}^<_\beta(k_0,\vec{k})=e^{-\beta k_0}\bar{S}^>_\beta(k_0,\vec{k})$. This relation can be expressed by the following equations:

$$\begin{aligned}&\bar{S}^>_\beta(k)=[1-f(k_0)]\rho(k)\,,\\&\bar{S}^<_\beta(k)=f(k_0)\rho(k)\,,\\&f(E)=\frac{1}{e^{\beta E}+1}\,,\\&\rho(k)=\bar{S}^>_\beta(k)+\bar{S}^<_\beta(k)\,.\end{aligned} \tag{A25}$$

The spectral function $\rho(k)$ is obtained from the imaginary-time propagator:

$$\begin{aligned}S_\beta(\omega_n,\vec{k})&=\int_0^{-i\beta}dx_0e^{-[(2n+1)\pi/\beta]x_0}\\&\qquad\times\int d^3x e^{-i\vec{k}\cdot\vec{x}}S^>_\beta(x)\\&=i\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\rho(k_0,\vec{k})}{\omega_n-k_0}\,,\\ \rho(k_0,\vec{k})&=S_\beta(k_0+i\epsilon,\vec{k})-S_\beta(k_0-i\epsilon,\vec{k})\,,\end{aligned} \tag{A26}$$

and the real-time propagator is determined from the spectral function:

$$\begin{aligned}\bar{S}_\beta(k)&=\int d^4x\, e^{ikx}[\theta(x_0)S^>_\beta(x)-\theta(-x_0)S^<_\beta(x)]\\&=i\int_{-\infty}^{\infty}\frac{dk_0'}{2\pi}\frac{\rho(k_0',\vec{k})}{k_0-k_0'+i\epsilon}-f(k_0)\rho(k)\,.\end{aligned} \tag{A27}$$

For noninteracting fields,

$$\begin{aligned}&\rho(k)=2\pi\epsilon(k_0)(\not k+m)\delta(k^2-m^2)\,,\\&\bar{S}_\beta(k)=\frac{i}{\not k-m+i\epsilon}-\frac{2\pi(\not k+m)}{e^{\beta E}+1}\delta(k^2-m^2)\,.\end{aligned} \tag{A28}$$

APPENDIX B

The advantage of the real-time formalism is that it immediately splits computations into a zero-temperature and a temperature-dependent part, by virtue of that split in the propagator. However, the expressions encountered in this formalism are often ambiguous. We show a way to circumvent the problem for the calculations found in Secs. III B and IV B.

For a Bose field, the one-loop effective potential involves integrals of the form

$$-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln i\bar{D}^{-1}\,. \tag{B1}$$

This expression has no explicit dependence on temperature in the real-time formalism, since $i\bar{D}^{-1}=k^2-m^2$. But if we rewrite (B1) as

$$\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\frac{\bar{D}}{i}\,, \tag{B2}$$

then the β dependence appears, since

$$\bar{D}=\frac{i}{k^2-m^2}+\frac{2\pi\delta(k^2-m^2)}{e^{\beta E}-1}\,.$$

[Apparently the temperature dependence of (B1) is hidden in the choice of a particular Riemann sheet used to evaluate the logarithm.] To facilitate the evaluation of (B2) we differentiate with respect to m^2:

$$-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\bar{D}\frac{\partial}{\partial m^2}\bar{D}^{-1}=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\times\left[\frac{1}{2E}+\frac{1}{E(\rho^{\beta E}-1)}\right].$$

This gives the same result for the one-loop effective potential obtained by the imaginary-time formalism.

The two-loop calculation involves $\int[d^4k/(2\pi)^4]\bar{D}$, which obviously gives the same answer in the real-time formalism as in the imaginary-time formalism. However, in higher-order loop calculations, one encounters integrals of the form $\int[d^4k/(2\pi)^4](\bar{D})^n$. Clearly this is undefined since it appears to possess products of δ functions at the same point. For imaginary time, the analogous objects are also given by

$$\frac{1}{(n-1)!}\left(i\frac{\partial}{\partial m^2}\right)^{n-1}\int_k D\,.$$

Hence we define the real-time calculation in this manner. The Fermi case is argued similarly.

APPENDIX C: EVALUATION OF INTEGRALS

Various integrals which were encountered in the text will be evaluated here.

1. Derivation of Eq. (3.16)

It is easy to obtain the first two terms in the expansion. From (3.13c) we have

$$\bar{V}_1^\beta(\hat\varphi^2)=\frac{1}{2\pi^2\beta^4}\int_0^\infty dx\,x^2\ln[1-\exp(-(x^2+a^2)^{1/2})]\,,$$

$$a^2=\beta^2M^2 \tag{C1a}$$

$$\frac{\partial}{\partial a^2}\bar{V}_1^\beta(\hat\varphi^2)=\frac{1}{4\pi^2\beta^4}\int_0^\infty dx\frac{x^2}{(x^2+a^2)^{1/2}}\frac{1}{\exp((x^2+a^2)^{1/2})-1}\,, \tag{C1b}$$

$$\frac{\partial^2}{\partial a^4}\bar{V}_1^\beta(\hat{\varphi}^2)=\frac{1}{4\pi^2\beta^4}\times\int_0^\infty dx\,x^2\frac{\partial}{\partial x^2}\times\frac{1}{(x^2+a^2)^{1/2}[\exp((x^2+a^2)^{1/2})-1]}$$
$$=-\frac{1}{8\pi^2\beta^4}\times\int_0^\infty dx\frac{1}{(x^2+a^2)^{1/2}[\exp((x^2+a^2)^{1/2})-1]}\,. \quad \text{(C1c)}$$

The second equality in (C1c) follows from the first by an integration by parts. Thus we find

$$\bar{V}_1^\beta(\hat{\varphi}^2)|_{a^2=0}=\frac{1}{2\pi^2\beta^4}\int_0^\infty dx\,x^2\ln(1-e^{-x})=-\frac{\pi^2}{90\beta^4}, \quad \text{(C2a)}$$

$$\left.\frac{\partial\bar{V}_1^\beta(\hat{\varphi}^2)}{\partial a^2}\right|_{a^2=0}=\frac{1}{4\pi^2\beta^4}\int_0^\infty dx\frac{x}{e^x-1}=\frac{1}{24\beta^2}. \quad \text{(C2b)}$$

The remaining terms in (3.16) are more difficult. What is needed is an expansion of

$$I(a)=\int_0^\infty\frac{dx}{(x^2+a^2)^{1/2}}\frac{1}{\exp((x^2+a^2)^{1/2})-1}. \quad \text{(C3)}$$

It is convenient to study the regulated quantity

$$I_\epsilon(a)=\int_0^\infty dx\frac{x^{-\epsilon}}{(x^2+a^2)^{1/2}}\frac{1}{\exp((x^2+a^2)^{1/2})-1},\quad \epsilon<1 \quad \text{(C4)}$$

By use of the series (3.11), $I_\epsilon(a)$ may also be represented by

$$I_\epsilon(a)=I_\epsilon^{(1)}(a)+I_\epsilon^{(2)}(a), \quad \text{(C5a)}$$

$$I_\epsilon^{(1)}(a)=\int_0^\infty dx\,x^{-\epsilon}\sum_n\frac{1}{x^2+a^2+4\pi^2n^2},\quad n=0,\pm1,\dots \quad \text{(C5b)}$$

$$I_\epsilon^{(2)}(a)=-\tfrac{1}{2}\int_0^\infty dx\,x^{-\epsilon}\frac{1}{(x^2+a^2)^{1/2}}. \quad \text{(C5c)}$$

For convergence of the separate integrals $I_\epsilon^{(i)}$, we must assume that $1>\epsilon>0$. They will be estimated for small a^2 in this region, and then their sum will be continued to $\epsilon=0$, which is a regular point for $I_\epsilon(a)$.

By a change of variable $I_\epsilon^{(1)}(a)$ may be cast in the form

$$I_\epsilon^{(1)}(a)=\sum_n\frac{1}{(a^2+4\pi^2n^2)^{(1+\epsilon)/2}}\int_0^\infty dx\frac{x^{-\epsilon}}{1+x^2}$$
$$=\left[\frac{1}{a^{1+\epsilon}}+2\sum_{n=1}^\infty\frac{1}{(a^2+4\pi^2n^2)^{(1+\epsilon)/2}}\right]\frac{\frac{1}{2}\pi}{\cos\frac{1}{2}\pi\epsilon}$$
$$=\left[\frac{1}{a^{1+\epsilon}}+2\sum_{n=1}^\infty\frac{1}{(2\pi n)^{1+\epsilon}}+2\sum_{n=1}^\infty\frac{1}{(2\pi n)^{1+\epsilon}}\left(\frac{1}{(1+a^2/4\pi^2n^2)^{(1+\epsilon)/2}}-1\right)\right]\times\frac{\frac{1}{2}\pi}{\cos\frac{1}{2}\pi\epsilon}. \quad \text{(C6a)}$$

The second sum in (C6a) is $O(a^2)$. Also it posesses a limit as $\epsilon\to0$. Hence we get

$$I_\epsilon^{(1)}(a)=\frac{\pi}{2a}+\pi^{-\epsilon}2^{-1-\epsilon}\zeta(1+\epsilon)+\tilde{I}(a)+O(\epsilon), \quad \text{(C6b)}$$

$$\tilde{I}(a)=\frac{1}{2}\sum_{n=1}^\infty\frac{1}{n}\left[\left(1+\frac{a^2}{4\pi^2n^2}\right)^{-1/2}-1\right]=O(a^2). \quad \text{(C6c)}$$

Here $\zeta(1+\epsilon)$ is the Riemann zeta function which satisfies

$$\zeta(1+\epsilon)=-\frac{2^\epsilon\pi^{1+\epsilon}\zeta(-\epsilon)}{\sin\frac{1}{2}\pi\epsilon\;\Gamma(1+\epsilon)}=2^\epsilon\pi^\epsilon\left[\frac{1}{\epsilon}-\ln2\pi+\gamma+O(\epsilon)\right],\quad \gamma=0.577\dots. \quad \text{(C7)}$$

The estimate of $I_\epsilon^{(1)}(a)$ is

$$I_\epsilon^{(1)}(a)=\frac{1}{2\epsilon}+\frac{\pi}{2a}+\tfrac{1}{2}(\gamma-\ln2\pi)+\tilde{I}(a)+O(\epsilon). \quad \text{(C8)}$$

$I_\epsilon^{(2)}(a)$ is given by

$$I_\epsilon^{(2)}(a)=-\tfrac{1}{2}a^{-\epsilon}\int_0^\infty dx\frac{x^{-\epsilon}}{(1+x^2)^{1/2}}$$
$$=-\tfrac{1}{2}a^{-\epsilon}B(\tfrac{1}{2}-\tfrac{1}{2}\epsilon,\tfrac{1}{2}\epsilon)$$
$$=-\tfrac{1}{2}a^{-\epsilon}+2^\epsilon\left[\frac{1}{\epsilon}+O(\epsilon^2)\right]$$
$$=-\frac{1}{2\epsilon}+\tfrac{1}{2}\ln\tfrac{1}{2}a+O(\epsilon). \quad \text{(C9)}$$

Upon combining (C8) and (C9), we finally get

$$I(a)=\frac{\pi}{2a}+\tfrac{1}{2}\ln\frac{a}{4\pi}+\tfrac{1}{2}\gamma+\tilde{I}(a). \quad \text{(C10)}$$

Since

$$\frac{\partial^2\bar{V}_1^\beta(\hat{\rho}^2)}{\partial a^4}=-\frac{1}{8\pi^2\beta^4}I(a),$$

we can regain $\bar{V}_1^\beta(\hat{\varphi}^2)$ from (C10) when the boundary conditions (C2) are taken into account, and thus (3.16) is verified. Note that the $\pi/2a$ part of $I(a)$ which leads to the M^3/β term in $\overline{V}_1^\beta(\hat{\varphi}^2)$ in (3.16), and the m/β term in $m_\beta{}^2$, (3.18), comes from the $n=0$ term in the series (C5b). This verifies the assertion in the text that the infrared region determines the next-to-leading contributions.

The order $M^4(M^2\beta^2)$ part of (3.16) which arises from the $O(a^2)$ part of $I(a)$, can be obtained from (C6c). Integrating that expression twice, and recalling the boundary conditions, we find this contribution to $\overline{V}_1^\beta(\hat{\varphi})$:

$$\frac{32\pi^2}{3\beta^4}\sum_{n=1}^{\infty} n^3\left[\left(1+\frac{a^2}{4\pi^2 n^2}\right)^{3/2}-1-\frac{3}{8}\frac{a^2}{\pi^2 n^2}-\frac{3}{128}\frac{a^4}{\pi^4 n^4}\right]. \tag{C11}$$

2. Derivation of Eq. (4.10)

To estimate the fermion contribution to the effective potential

$$\overline{V}_1^\beta(\hat{\varphi})=-\frac{2}{\pi^2\beta^4}\int_0^\infty dx\,x^2\ln[1+\exp(-(x^2+a^2)^{1/2})], \tag{C12}$$

we proceed in a fashion entirely analogous to the Bose case, discussed above. Hence, we only sketch the relevant steps. The first two terms in an expansion in a^2 are easy to get. The hard part is an estimate of

$$\frac{\partial^2\overline{V}_1^\beta(\hat{\varphi})}{\partial a^4}=-\frac{1}{2\pi^2\beta^4}I(a),$$
$$I(a)=\int_0^\infty dx\frac{1}{(x^2+a^2)^{1/2}}\frac{1}{\exp((x^2+a^2)^{1/2})+1}. \tag{C13}$$

We regulate as before with $x^{-\epsilon}$ and express the integrand as a series with the help of (3.11):

$$I_\epsilon(a)=I_\epsilon^{(1)}(a)+I_\epsilon^{(2)}(a),$$
$$I_\epsilon^{(1)}(a)=-\int_0^\infty dx\,x^{-\epsilon}\sum_n\frac{1}{x^2+a^2+(2n+1)^2\pi^2},\quad n=0,\pm1,\ldots \tag{C14b}$$
$$I_\epsilon^{(2)}(a)=\tfrac{1}{2}\int_0^\infty dx\,x^{-\epsilon}\frac{1}{(x^2+a^2)^{1/2}},\quad 1>\epsilon>0. \tag{C14c}$$

Following our previous method, we find

$$I_\epsilon^{(1)}(a)=-\frac{1}{2\epsilon}+\tfrac{1}{2}(\ln\tfrac{1}{2}\pi-\gamma)+\bar{I}(a)+O(\epsilon), \tag{C15a}$$
$$\bar{I}(a)=-\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\left(1+\frac{a^2}{(2n+1)^2\pi^2}\right)^{-1/2}-1\right]=O(a^2), \tag{C15b}$$
$$I_\epsilon^{(2)}(a)=\frac{1}{2\epsilon}-\tfrac{1}{2}\ln\tfrac{1}{2}a+O(\epsilon). \tag{C15c}$$

Thus

$$I(a)=-\tfrac{1}{2}\ln\frac{a}{\pi}-\tfrac{1}{2}\gamma+\bar{I}(a). \tag{C16}$$

A double integration of (C16) establishes (4.10).

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*Work supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(11-1) 3069.

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[3]R. Jackiw, Phys. Rev. D 9, 1686 (1974).

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[5]Other relevant investigations of the effective potential include: B. DeWitt, in *Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honor of His 60th Birthday*, edited by John R. Klauder (Freeman, San Francisco, 1972); B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); S. Coleman and E. Weinberg, *ibid.* 7, 1888 (1973); S. Weinberg, *ibid.* 7, 2887 (1973).

[6]S. Weinberg, this issue, Phys. Rev. D 9, 3357 (1974).

[7]For a review and references to the original literature, see L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Menlo Park, 1962).

[8]C. Bernard [this issue, Phys. Rev. D 9, 3312 (1974)] has given a detailed analysis of the functional integral representation of field theory at finite temperature. His results validate the point of view which we express.

[9]J. Schwinger, Phys. Rev. 128, 2425 (1962).

[10]We have in mind dynamical models of symmetry violation based on a *renormalizable* field theory. See R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973); J. M. Cornwall and R. E. Norton, *ibid.* 8, 3338 (1973). Our remarks do not apply to the Nambu–Jona-Lasinio model [Phys. Rev. 122, 345 (1961)], which uses a nonrenormalizable cutoff theory.

[11]For imaginary-time calculations the "time volume" is $-i\beta$ since all time integrals are over the region $[0,-i\beta]$; see also Ref. 8.

[12]In the subsequent work, the subscript β is frequently omitted from the propagators. No confusion will arise since we are concerned with finite-temperature propagators exclusively.

[13]The large-N approximation is familiar in many-body theories, where it goes under the name "spherical model," and was invented by M. Kac. For a review, see M. Kac, Phys. Today 17, 40 (1964), and H. E.

Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford Univ. Press, New York, 1971). In field theory this approximation was first used by K. Wilson, Phys. Rev. D 7, 2911 (1973).

[14]Here we disagree with S. Weinberg, Ref. 6, who quotes an error $O(\lambda\beta_c)$. However, we find below, in an approximate summation of graphs, that all $O(\sqrt{\lambda}\beta_c)$ corrections cancel. We have not established if this is generally true.

[15]The approximation we make is equivalent to the Hartee approximation in many-body theory. Recall the exact equation for the propagator:

$$(\Box_x + m_0^2)\langle T\varphi_a(x)\varphi_b(0)\rangle = -i\delta_{ab}\delta^4(x) - \tfrac{1}{6}\lambda\langle T\varphi_a(x)\varphi^2(x)\varphi_b(0)\rangle .$$

Replacing the correlation function on the right-hand side by $\langle T\varphi_a(x)\varphi_b(0)\rangle\langle\varphi^2(x)\rangle$ produces the same result as dropping the terms involving T in the Schwinger-Dyson equation of Fig. 7. Moreover, it is known that the Hartee approximation is exact in the many-body version of our large-N limit. Correspondingly, our Eqs. (3.41) and (3.42) have well-known analogs in many-body theory. See S.-k. Ma, Phys. Rev. A 7, 2172 (1973); E. Brêzin and D. J. Wallace, Phys. Rev. B 7, 1967 (1973). We thank Professor P. Martin for explaining this to us. The Hartree approximation has a history of field-theoretic applications, e.g., E. H. Lieb, Proc. R. Soc. A241, 339 (1957); K. Johnson, in Proceedings of Seminar on Unified Theories of Elementary Particles, Rochester, 1963 (unpublished).

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[17]Note that even though the ghosts obey Fermi statistics, their temperature Green's function $S\{\hat{\varphi};k\}$ is that of Bose particles, i.e., the time component of k is $2n\pi/(-i\beta)$. That this is correct follows from the fact that S must cancel against parts of the remaining expression. For further discussion, see Ref. 8.

[18]K. Johnson, Phys. Lett. 5, 253 (1963). The analysis of the Schwinger model, with the help of the axial-vector-current anomaly, is due to D. J. Gross and R. Jackiw (unpublished); see also R. Jackiw, in proceedings of the International School of Subnuclear Physics, Erice, Italy, 1973 (unpublished).

[19]This observation is due to K. Johnson.

[20]Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).

[21]This point was developed in conversations with S. Coleman.

[22]S. Coleman, R. Jackiw, and H. Politzer (unpublished).

PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Quantum theory of dual relativistic parastring models*

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We paraquantize the classical massless relativistic-string action and find that the resulting theory is Poincaré-invariant in four space-time dimensions if we use para-Bose commutation relations of order 12. More generally, we find that if the dimension D of the space-time and the order q of parabosons are related by the expression $D = 2 + 24/q$, then the quantized theory is Poincaré-invariant. We also construct a fermionic parastring model which is the analog of the Ramond-Neveu-Schwarz model and find that it is invariant in D dimensions if $D = 2 + 8/q$, both the fermions and the bosons being of order q. We show by explicit Klein transformations that these theories are equivalent to "color"-endowed canonically quantized strings with SO$(q-1)$ "color" symmetry. We obtain dual tree amplitudes by suitable choice of vertices. Finally, we consider second-quantized parastring theories and show, by an explicit example, that they can be Poincaré-invariant in four space-time dimensions.

I. INTRODUCTION

The search for the understanding of the fundamental structure that underlies the dual resonance models has been the subject of many interesting investigations in recent years.[1-13] From among various approaches, the one which has reached the status of a bona fide theory is the gauge theory of the relativistic string,[14] which is based on a geometrical description initiated by Nambu.[5] In this case the fundamental structure is a massless relativistic string.

An important feature of the string model is that all of its properties follow from a single action:

$$S = \frac{1}{2\pi\alpha'}\int_0^\pi d^2\eta\sqrt{-g} , \tag{1.1}$$