Assouad dimensions: characterizations and applications

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FinEst Math 2014, Helsinki 09.01.2014
This talk is based on parts of the following works (in chronological order):


1. Metric spaces
Let $X$ be a metric space.

How to describe the (local) dimension of a set $E \subset X$?

Take a piece of the set, i.e. $E \cap B(w, R)$, where $w \in E$, cover this with balls of radius $0 < r < R$, and count how many balls are needed.
(upper) Assouad dimension

Let $E \subset X$. We consider all exponents $\lambda \geq 0$ for which there is $C = C(E, \lambda) \geq 1$ s.t. $E \cap B(w, R)$ can be covered by at most $C(r/R)^{-\lambda}$ balls of radius $r$ for all $0 < r < R < \text{diam}(E)$ and $w \in E$.

The infimum of such exponents $\lambda$ is the (upper) Assouad dimension $\dim_A(E)$.

Recall that a metric space $(X, d)$ is doubling if there is $N = N(X) \in \mathbb{N}$ so that any closed ball $B(x, r) \subset X$ can be covered by at most $N$ balls of radius $r/2$. Iteration of this doubling condition shows that then $\dim_A(E) \leq \dim_A(X) \leq \log_2 N$ for all $E \subset X$. In particular:

**Lemma**

A metric space $X$ is doubling if and only if $\dim_A(X) < \infty$. 
lower Assouad dimension

Conversely: consider all $\lambda \geq 0$ for which there is $c > 0$ s.t. if $0 < r < R < \text{diam}(E)$, then for every $w \in E$ at least $c(r/R)^{-\lambda}$ balls of radius $r$ are needed to cover $E \cap B(w, R)$. The supremum of all such $\lambda$ is the lower Assouad dimension of $E$.

Recall that a set $E \subset X$ is uniformly perfect if $\#E \geq 2$ and there is $C \geq 1$ s.t. for every $w \in E$ and $r > 0$ we have $(B(w, r) \cap E) \setminus B(w, r/C) \neq \emptyset$ whenever $E \setminus B(w, r) \neq \emptyset$.

Lemma

A set $E$ is uniformly perfect if and only if $\text{dim}_A(E) > 0$. 
In our example

- \( \text{dim}_A(E) = \frac{\log 4}{\log 3} \) because of the snowflake part
- \( \text{dim}_A(E) = 0 \) because of the isolated point
- (without the isolated point would have \( \text{dim}_A(E) = 1 \))
(Upper) Assouad dimension was introduced by P. Assouad around 1980 in connection to bi-Lipschitz embedding problem between metric and Euclidean spaces. However, equivalent (or closely related) concepts have appeared under different names, e.g. *(uniform) metric dimension*, some dating back (at least) to [Bouligand 1928]. See [Luukkainen 1998] for a nice account on the basic properties of (upper) Assouad dimension as well as some historical comments.

Lower Assouad dimension has (essentially) appeared under names *lower dimension*, *minimal dimensional number*, and *uniformity dimension*. Some basic properties of this are recently established in [Fraser 2013].
Other concepts of dimension: Minkowski

So once again:
\[ \underline{\dim}_A(E) \] is the infimum of \( \lambda \geq 0 \) s.t. \( E \cap B(w, R) \) can (always) be covered by at most \( C(r/R)^{-\lambda} \) balls of radius \( 0 < r < R < \text{diam}(E) \)

\[ \overline{\dim}_A(E) \] is the supremum of \( \lambda \geq 0 \) s.t. (always) at least \( C(r/R)^{-\lambda} \) balls of radius \( 0 < r < R < \text{diam}(E) \) are needed to cover \( E \cap B(w, R) \)

For comparison, recall the upper and lower Minkowski dimensions of a compact \( E \subset X \):

\[ \underline{\dim}_M(E) \] is the infimum of \( \lambda \geq 0 \) s.t. \( E \) can be covered by at most \( Cr^{-\lambda} \) balls of radius \( 0 < r < \text{diam}(E) \)

\[ \overline{\dim}_M(E) \] is the supremum of \( \lambda \geq 0 \) s.t. at least \( Cr^{-\lambda} \) balls of radius \( 0 < r < \text{diam}(E) \) are needed to cover \( E \).

Thus \( \underline{\dim}_A(E) \leq \underline{\dim}_M(E) \leq \overline{\dim}_M(E) \leq \overline{\dim}_A(E) \).
More examples (1)

General idea: Assouad dimensions reflect the ‘extreme’ behavior of sets and take into account all scales $0 < r < d(E)$.

- If $E = \{0\} \cup [1, 2] \subset \mathbb{R}$, then $\dim_A(E) = 0$ and $\overline{\dim}_A(E) = 1$ ($\dim_M(E) = \dim_M(E) = 1$).
- $\dim_A(\mathbb{Z}) = 0$ and $\overline{\dim}_A(\mathbb{Z}) = 1$.
- If $E = \{(1/j, 0, \ldots, 0) : j \in \mathbb{N}\} \cup \{(0, 0, \ldots, 0)\} \subset \mathbb{R}^n$, then $\dim_A(E) = 0$ and $\overline{\dim}_A(E) = 1$ ($\dim_M(E) = \dim_M(E) = 1/2$).
More examples (2)

- If $S \subset \mathbb{R}^2$ is an unbounded, locally rectifiable von Koch snowflake type curve consisting of unit intervals, then $\overline{\dim}_A(S) = 1$ and $\underline{\dim}_A(E) = \frac{\log 4}{\log 3}$ (flat on small scales, fractal on large scales).

- If $S \subset \mathbb{R}^2$ consists of infinitely many copies of the usual (fractal) von Koch snowflake curve, laid side by side, then $\overline{\dim}_A(S) = 1$ and $\underline{\dim}_A(E) = \frac{\log 4}{\log 3}$ (fractal on small scales, flat on large scales).
More examples (2)

- If $S \subset \mathbb{R}^2$ is an unbounded, locally rectifiable von Koch snowflake-type curve consisting of unit intervals, then $\dim_A(S) = 1$ and $\dim_A(E) = \log 4 / \log 3$ (flat on small scales, fractal on large scales).

- If $S \subset \mathbb{R}^2$ consists of infinitely many copies of the usual (fractal) von Koch snowflake curve, laid side by side, then $\dim_A(S) = 1$ and $\dim_A(E) = \log 4 / \log 3$ (fractal on small scales, flat on large scales).
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- If $S \subset \mathbb{R}^2$ consists of infinitely many copies of the usual (fractal) von Koch snowflake curve, laid side by side, then $\dim_A(S) = 1$ and $\dim_A(E) = \log 4 / \log 3$ (fractal on small scales, flat on large scales).
Recall that the \textit{Hausdorff (r-)content} of dimension $\lambda$, for $E \subset X$, is

$$\mathcal{H}^\lambda_r(E) = \inf \left\{ \sum_k r_k^\lambda : E \subset \bigcup_k B(x_k, r_k), \ x_k \in E, \ 0 < r_k \leq r \right\}.$$ 

The $\lambda$-\textit{Hausdorff measure} of $E$ is $\mathcal{H}^\lambda(E) = \lim_{r \to 0} \mathcal{H}^\lambda_r(E)$ and the \textit{Hausdorff dimension} of $E$ is

$$\dim_H(A) = \inf \{ \lambda \geq 0 : \mathcal{H}^\lambda(A) = 0 \}.$$
Lemma

If $E \subset X$ is closed, then $\dim_A(E) \leq \dim_H(E \cap B)$ for all balls $B$ centered at $E$.

The proof is based on the fact (obtained by iteration), that for each $0 < t < \dim_A(E)$ it holds that

$$\mathcal{H}_R^t(E \cap B(w, R)) \geq c R^t$$

for all $w \in E$, $0 < r < R < \text{diam}(E)$ (1)

(see e.g. [L. 2009] for details). Therefore in particular $\dim_H(E \cap B(w, R)) \geq t$ and the claim follows.

In fact, for closed $E \subset X$ we have $\dim_A(E) = \sup\{ t \geq 0 : (1) \text{ holds} \}$.

(Note however that e.g. $\dim_A(\mathbb{Q}) = 1$ but $\dim_H(\mathbb{Q}) = 0$)
Whitney covers

An open set $\Omega \subset X$ can be covered with a countable collection $\mathcal{W}(\Omega)$ of closed balls $B_i = B\left(x_i, \frac{1}{8} \text{dist}(x_i, X \setminus \Omega)\right)$, $x_i \in \Omega$, such that the overlap of these balls is uniformly bounded (the factor $\frac{1}{8}$ is not special).

For $k \in \mathbb{Z}$ and $A \subset X$ we set
$$\mathcal{W}_k(\Omega; A) = \{B(x_i, r_i) \in \mathcal{W}(\Omega) : 2^{-k-1} < r_i \leq 2^{-k} \text{ and } A \cap B(x_i, r_i) \neq \emptyset\}$$
and
$$\mathcal{W}_k(\Omega) = \mathcal{W}_k(\Omega; X).$$

In [Martio–Vuorinen 1987], the relation between upper Minkowski dimension and upper bounds for Whitney cube count was considered for compact $E \subset \mathbb{R}^n$. In particular, if $\mathcal{H}^n(E) = 0$, then
$$\dim_{\text{M}}(E) = \inf\{\lambda \geq 0 : \#\mathcal{W}_k(\mathbb{R}^n \setminus E) \leq C2^{\lambda k} \text{ for all } k \geq k_0\}.$$
A (blue) ball $B(x, r)$, $x \in E$, in the cover of $E$ intersects *always* at most a fixed number of Whitney balls of $\Omega = X \setminus E$ with a comparable radius.

Conversely, each $B(x, r)$, contains *usually* at least one Whitney ball of a comparable radius. (The latter is not true in general but under some geometric assumptions.)
Geometric conditions

A metric space $X$ is $q$-quasiconvex if there is $q \geq 1$ such that for every $x, y \in X$ there is a curve $\gamma : [0, 1] \to X$ so that $x = \gamma(0)$, $y = \gamma(1)$, and $\text{length}(\gamma) \leq qd(x, y)$.

We say that a set $E \subset X$ is (uniformly) $\rho$-porous (for $0 \leq \rho \leq 1$), if for every $x \in E$ and all $0 < r < d(E)$ there is $y \in X$ such that $B(y, \rho r) \subset B(x, r) \setminus E$.

Under these conditions balls covering $E$ always contain Whitney balls of comparable radius.

The porosity assumption is more or less crucial in this context, but quasiconvexity (as such) is not that essential; in particular, the existence of rectifiable curves is not necessary. However, without any local connectivity assumptions some generations $\mathcal{W}_k$ of Whitney balls might be empty.
Assouad dimensions and Whitney covers

The relation between Assouad dimensions and Whitney covers (from [KLV]) can be summarized as follows; here $E \subset X$ is closed and $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$:

If $\dim_A(E) < \lambda$, then $\mathcal{W}_k(X \setminus E; B_0) \leq C2^{\lambda k} R^\lambda$ for all $B_0$ and all $k > -\log_2 R$.

If we have for all $B_0$ and all $k \geq -\log_2 R + \ell$
\[ \mathcal{W}_k(X \setminus E; B_0) \geq c2^{\lambda k} R^\lambda, \text{ then } \dim_A(E) \geq \lambda. \]

If $X$ is q-convex and $E \subset X$ is porous, and $\dim_A(E) > \lambda$, then
\[ \mathcal{W}_k(X \setminus E; B_0) \geq c2^{\lambda k} R^\lambda \text{ for all } B_0 \text{ and all } k > -\log_2 R + \ell. \]

If $X$ is q-convex and $E \subset X$ is porous, and for all $B_0$ and all $k \geq -\log_2 R$
\[ \mathcal{W}_k(X \setminus E; B_0) \leq C2^{\lambda k} R^\lambda, \text{ then } \dim_A(E) \leq \lambda. \]

Thus Assouad dimensions (of porous sets $E \subset X$) can be characterized in terms of $\mathcal{W}_k(X \setminus E)$.
Assouad dimensions and $r$-boundaries in $\mathbb{R}^n$

Let us also mention the following Euclidean results from [KLV]; here $E \subset \mathbb{R}^n$ is closed, $E_r = \{x \in \mathbb{R}^n : d(x, E) < r\}$, and $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$.

\[
\dim_{A}(E) < \lambda \quad \implies \quad \mathcal{H}^{n-1}(\partial E_r \cap B_0) \leq Cr^{n-1}(r/R)^{-\lambda} \quad \text{for all } B_0 \text{ and } 0 < r < R.
\]

\[
\mathcal{H}^{n-1}(\partial E_r \cap B_0) \geq cr^{n-1}(r/R)^{-\lambda} \quad \text{for all } B_0 \text{ and } 0 < r < \delta R \quad \implies \quad \dim_{A}(E) \geq \lambda.
\]

If $E$ is porous, then $\dim_{A}(E) > \lambda$

\[
\implies \quad \mathcal{H}^{n-1}(\partial E_r \cap B_0) \geq cr^{n-1}(r/R)^{-\lambda} \quad \text{for all } B_0 \text{ and } 0 < r < \delta R.
\]

If $\mathcal{H}^n(E) = 0$ (weaker than porosity), then

\[
\mathcal{H}^{n-1}(\partial E_r \cap B_0) \leq Cr^{n-1}(r/R)^{-\lambda} \quad \text{for all } B_0 \text{ and } 0 < r < R \quad \implies \quad \dim_{A}(E) \leq \lambda.
\]

Thus Assouad dimensions (of porous sets $E \subset \mathbb{R}^n$) can be characterized in terms of $\mathcal{H}^{n-1}(\partial E_r)$.
2. Metric measure spaces
A measure $\mu$ on $X$ is **doubling** if there is $C \geq 1$ so that
$0 < \mu(2B) \leq C\mu(B)$ for all closed balls $B \subset X$.

Iterating, we find $C > 0$ and $s > 0$ s.t.

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C \left( \frac{r}{R} \right)^s \quad (2)$$

for all $y \in B(x, R)$ and $0 < r < R < d(X)$. The infimum of $s$ satisfying (2) is called the **upper regularity dimension** of $\mu$, $\dim_{\text{reg}}(\mu)$.

It is easy to see that $\dim_A(X) \leq \dim_{\text{reg}}(\mu)$ whenever $\mu$ is doubling on $X$. In particular, if $X$ has a doubling measure, then $X$ is doubling.

On the other hand, if $X$ is doubling and complete, then there is a doubling measure $\mu$ on $X$ [Luukkainen–Saksman 1998; Vol’berg–Konyagin 1987 (for compact sets)].
Conversely, if $X$ is uniformly perfect and $\mu$ is doubling then there are $t > 0$ and $C \geq 1$ s.t.

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \leq C \left( \frac{r}{R} \right)^t$$

whenever $0 < r < R < d(X)$ and $y \in B(x, R)$. The supremum of $t$ satisfying (3) is called the \textit{lower regularity dimension} of $\mu$, $\dim_{\text{reg}}(\mu)$.

Thus $\dim_{\text{reg}}(\mu) > 0$ if $\mu$ is doubling and $X$ is uniformly perfect, and in fact $\dim_{\text{reg}}(\mu) \leq \dim_A(X)$.

Measure $\mu$ (and the space $X$) is called \textit{(Ahlfors) s-regular}, if there is $C > 0$ so that

$$\frac{1}{C} r^s \leq \mu(B(x, r)) \leq Cr^s$$

for every $x \in X$ and all $0 < r < d(X)$. Then $\dim_{\text{reg}}(\mu) = \dim_{\text{reg}}(\mu) = s$. 
Let $X = (X, \mu, d)$ be a metric measure space and let $E \subset X$.

Instead covering $E \cap B(w, R)$ with balls of radius $0 < r < R$, this leads to the concepts of Assouad codimension.
Let $X = (X, \mu, d)$ be a metric measure space and let $E \subset X$.

Instead covering $E \cap B(w, R)$ with balls of radius $0 < r < R$, we may consider the measure $\mu(E_r \cap B(x, R))$, where $E_r = \{x \in X : d(x, E) < r\}$ is the $r$-neighborhood of $E$.

This leads to the concepts of Assouad codimension.
Assouad revisited

Let $\mu$ is doubling and $E \subset X$. In [KLV] we introduce the following concepts:

The lower Assouad codimension $\text{codim}_A^\mu(E)$ is the supremum of $t \geq 0$ for which there is $C > 0$ s.t.

$$\frac{\mu(E_r \cap B(x, R))}{\mu(B(x, R))} \leq C \left( \frac{r}{R} \right)^t$$

for every $x \in E$ and all $0 < r < R < \text{diam}(E)$.

The upper Assouad codimension $\text{codim}_A^\mu(E)$ is the infimum of $s \geq 0$ for which there is $C > 0$ s.t.

$$\frac{\mu(E_r \cap B(x, R))}{\mu(B(x, R))} \geq C \left( \frac{r}{R} \right)^s$$

for every $x \in E$ and all $0 < r < R < \text{diam}(E)$. 
Lemma (KLV)

If $\mu$ is a doubling measure on $X$ and $E \subset X$, then

$$\dim_{\text{reg}}(\mu) \leq \co \dim_{\mathbb{A}}^\mu(E) + \dim_{\mathbb{A}}(E) \leq \dim_{\text{reg}}(\mu),$$

$$\dim_{\text{reg}}(\mu) \leq \co \dim_{\mathbb{A}}^\mu(E) + \dim_{\mathbb{A}}(E) \leq \dim_{\text{reg}}(\mu).$$

(4)

In particular, if $\mu$ is $s$-regular, then the above lemma implies

$$\limsup_{\mathbb{A}}(E) = s - \co \dim_{\mathbb{A}}^\mu(E),$$

$$\liminf_{\mathbb{A}}(E) = s - \co \dim_{\mathbb{A}}^\mu(E)$$

for all $E \subset X$. The first equation was also proven in [LT]. On the other hand, it is not hard to give examples where $\mu$ is doubling and any given inequality in (4) is strict for a set $E \subset X$. 
Porosity and Assouad dimensions

Porous sets have upper bounds for their (upper) Assouad dimension in regular spaces:

Proposition (KLV, strongly based on [JJKRRS 2010])

If $X$ is $s$-regular, then there is a constant $c > 0$ such that
\[ \dim_A(E) \leq s - c \rho^s \]
for all $\rho$-porous sets $E \subset X$.

If $\mu$ is (only) doubling, then it is still true that each $\varrho$-porous set $E \subset X$ satisfies
\[ \text{co dim}_{A}^{\mu}(E) \geq t, \]
where $t > 0$ only depends on $\varrho$ and the doubling constant of $\mu$ (again observed in [KLV] but based on [JJKRRS 2010]).
In [LT] it was shown that the lower Assouad codimension $\text{co dim}_A^\mu(E)$ (and thus $s - \dim_A(E)$ in an $s$-regular space) can be characterized as the supremum of $q \geq 0$ for which there is $C \geq 1$ s.t.

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} \text{dist}(y, E)^{-q} \, d\mu(y) \leq Cr^{-q}$$

(5)

for every $x \in E$ and all $0 < r < \text{diam}(E)$. (We interpret the integral to be $+\infty$ if $q > 0$ and $E$ has positive measure.)

A concept of dimension defined via integrals as in (5) was first used in [Aikawa 1991] for subsets of $\mathbb{R}^n$ in connection to the so-called quasiadditivity property of (Riesz) capacity.

(Thus in [LT] the lower Assouad codimension is actually called the Aikawa codimension.)
The Hausdorff content of codimension $q$ for $E \subset X$ can be defined as

$$\mathcal{H}^{\mu,q}_R(E) = \inf \left\{ \sum_k \text{rad}(B_k)^{-q} \mu(B_k) : E \subset \bigcup_k B_k, \text{rad}(B_k) \leq R \right\}.$$

The Hausdorff codimension is $\text{co dim}_H(E) = \sup \{ q \geq 0 : \mathcal{H}^{\mu,q}_R(E) = 0 \}$.

It was recently established in [L] that if $q > \text{co dim}_A(E)$, then there is $C > 0$ s.t.

$$\mathcal{H}^{\mu,q}_R(E \cap B(w, R)) \geq CR^{-q} \mu(B(w, R))$$

(6)

for every $w \in E$ and all $0 < R < \text{diam}(E)$. (Recall that we had a similar condition for $0 < t < \dim_A(E)$ and $\mathcal{H}^{t}_R(E)$.)

In fact, we have that $\text{co dim}_A(E) = \inf\{ q \geq 0 : (6) \text{ holds} \}$.

Let us remark here that the uniform estimate (6) for an exponent $1 < q < p$ (and for all $0 < R < \infty$) is equivalent to the set $E$ being uniformly $p$-fat (a capacity condition).
3. Applications: Hardy inequalities
Hardy inequalities

In an open set $\Omega \subset \mathbb{R}^n$ the $(p, \beta)$-Hardy inequality, for $1 < p < \infty$ and $\beta \in \mathbb{R}$, reads as

$$\int_{\Omega} |u(x)|^p \, d\Omega(x)^{\beta-p} \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \, d\Omega(x)^{\beta} \, dx,$$

where $d\Omega(x) = \text{dist}(x, \partial \Omega)$.

If there exists a constant $C > 0$ such that this holds for all $u \in C^\infty_0(\Omega)$, we say that $\Omega$ admits a $(p, \beta)$-Hardy inequality.

In a metric space $X$, with a doubling measure $\mu$, smooth functions are replaced with Lipschitz functions with compact support in $\Omega$, and $|\nabla u(x)|$ is replaced with an upper gradient $g$ of $u$:

$$\int_{\Omega} |u(x)|^p \, d\Omega(x)^{\beta-p} \, d\mu \leq C \int_{\Omega} g(x)^p \, d\Omega(x)^{\beta} \, d\mu.$$
Sufficient conditions I

We have the following recent result from [L]:

**Theorem**

Let $1 \leq p < \infty$, $\beta < p - 1$, and assume that $X$ is an unbounded doubling metric space. If $\beta \leq 0$, we further assume that $X$ supports a $p$-Poincaré inequality, and if $\beta > 0$ we assume that $X$ supports a $(p - \beta)$-Poincaré inequality. If $\Omega \subset X$ is an open set satisfying

$$\text{co dim}^\mu_A(X \setminus \Omega) > p - \beta,$$

then $\Omega$ admits a $(p, \beta)$-Hardy inequality.

This has been previously known in $\mathbb{R}^n$ (with different terminology) in the case $\beta = 0$ by [Aikawa 1991] and [Koskela–Zhong 2003], and for general $\beta$ under some additional geometric assumptions [L. 2008].
Conversely, a combination of some previously known results (e.g. [L. PAMS (to apper)]) based on Hausdorff content density / uniform fatness and the link between these and the upper Assouad codimension gives the following formulation:

**Theorem**

Let $1 \leq p < \infty$, $\beta < p - 1$, and assume that $X$ is a doubling metric space which supports a $p$-Poincaré inequality if $\beta \leq 0$, and a $(p - \beta)$-Poincaré inequality if $\beta > 0$. Let $\Omega \subset X$ be an open set satisfying

$$\text{co dim}_A^\mu(X \setminus \Omega) < p - \beta;$$

in case $\Omega$ is unbounded, we require in addition that $X \setminus \Omega$ is unbounded as well. Then $\Omega$ admits a $(p, \beta)$-Hardy inequality.
Sufficient conditions in $\mathbb{R}^n$

In the Euclidean case, we can reformulate the previous results as follows:

**Corollary**

Let $1 \leq p < \infty$ and $\beta < p - 1$, and let $\Omega \subset \mathbb{R}^n$ be an open set. If

$$\overline{\dim}_A(\Omega^c) < n - p + \beta \quad \text{or} \quad \underline{\dim}_A(\Omega^c) > n - p + \beta,$$

then $\Omega$ admits a $(p, \beta)$-Hardy inequality; in the latter case, if $\Omega$ is unbounded, then we require that also $\Omega^c$ is unbounded.

In [LS] we established an equivalence between $p$-Hardy inequalities ($\beta = 0$) and the quasiadditivity of the variational $p$-capacity (in metric spaces). This provides a link between the work of Aikawa (where essentially the condition $\overline{\dim}(\Omega^c) < n - p$ was used) and our recent considerations.
Necessary conditions

The above sufficient conditions (i.e. $\text{co dim}_A^\mu(\Omega^c) > p - \beta$ or $\text{co dim}_A^\mu(\Omega^c) < p - \beta$) are rather natural for $(p, \beta)$-Hardy inequalities. In fact, the following necessary conditions hold as well:

**Theorem (LT ($\beta = 0$), L)**

Let $1 < p < \infty$ and $\beta \neq p$, and suppose that a domain $\Omega \subset X$ admits a $(p, \beta)$-Hardy inequality. Then

$$\text{co dim}_H(\Omega^c) < p - \beta \quad \text{or} \quad \text{co dim}_A^\mu(\Omega^c) > p - \beta.$$  

Moreover, such a dichotomy also holds locally, i.e. for each ball $B_0 \subset X$

$$\text{co dim}_H(4B_0 \cap \Omega^c) < p - \beta \quad \text{or} \quad \text{co dim}_A^\mu(B_0 \cap \Omega^c) > p - \beta.$$
A blast from the past

In my talk in the Finnish Mathematical Days 2010 I asked:
... samaa ideaa käyttäen saadaan \( \mathbb{R}^n \)-ssä esimerkkejä, joissa [reunan osan dimensio] \( \mu \geq n - 1 \). Tällöin paksu osa reunasta saadaan ’piiloon’ pienien osan taakse, eikä \((p, \beta)\)-Hardy pärde millekään
\[ \beta \geq p - n + \mu \] [vaikka siis olisi \( \text{dim}_A(\Omega^c) = \mu < n - p + \beta \)].

Toisaalta, jos pieni osa reunaa on \( \mu \)-ulotteinen \((0 \leq \mu < n)\) ja tämän osan läheisistä päästään \( \lambda \)-paksun reunan osan läheelle \((\mu < \lambda)\), pätee \((p, \beta)\)-Hardy, kun
\[ p - n + \mu < \beta < p - n + \lambda. \]

Kysymys: Päteekö edellä \((p, \beta)\)-Hardy kaikille
\[ p - n + \mu < \beta < p - n + \lambda \] ilman lisäähtoa, jos \( \mu < n - 1 \)?

Edellisten tulosten perusteella osaan nyt vastata:
KYLLÄ, kunhan \( \beta < p - 1 \)
(ja jos \( \lambda = \text{dim}_A(\Omega^c) > n - 1 \) niin ei välttämättä kun \( \beta \geq p - 1 \).)
Some references:


