

# The annular decay property and capacity estimates for thin annuli

Anders Björn

*Department of Mathematics, Linköping University,  
SE-581 83 Linköping, Sweden; anders.bjorn@liu.se*

Jana Björn

*Department of Mathematics, Linköping University,  
SE-581 83 Linköping, Sweden; jana.bjorn@liu.se*

Juha Lehrbäck

*Department of Mathematics and Statistics, University of Jyväskylä,  
P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland; juha.lehrback@jyu.fi*

**Abstract.** We obtain upper and lower bounds for the nonlinear variational capacity of thin annuli in weighted  $\mathbf{R}^n$  and in metric spaces, primarily under the assumptions of an annular decay property and a Poincaré inequality. In particular, if the measure has the 1-annular decay property at  $x_0$  and the metric space supports a pointwise 1-Poincaré inequality at  $x_0$ , then the upper and lower bounds are comparable and we get a two-sided estimate for thin annuli centred at  $x_0$ . This generalizes the known estimate for the usual variational capacity in unweighted  $\mathbf{R}^n$ . We also characterize the 1-annular decay property and provide examples which illustrate the sharpness of our results.

*Key words and phrases:* Annular decay property, capacity, doubling measure, metric space, Newtonian space, Poincaré inequality, Sobolev space, thin annulus, upper gradient, variational capacity, weighted  $\mathbf{R}^n$ .

Mathematics Subject Classification (2010): Primary: 31E05; Secondary: 30L99, 31C15, 31C45.

## 1. Introduction

We assume throughout the paper that  $1 \leq p < \infty$  and that  $X = (X, d, \mu)$  is a metric space equipped with a metric  $d$  and a positive complete Borel measure  $\mu$  such that  $0 < \mu(B) < \infty$  for all balls  $B \subset X$ . We also let  $x_0 \in X$  be a fixed but arbitrary point and  $B_r = B(x_0, r) = \{x : d(x, x_0) < r\}$ .

In this paper, we continue the study of sharp estimates for the variational capacity  $\text{cap}_p(B_r, B_R)$ , which we started in Björn–Björn–Lehrbäck [3]. Therein we concentrated on the case  $0 < 2r \leq R$ , while in the present work we are interested in the case where the annulus  $B_R \setminus B_r$  is thin, that is,  $0 < \frac{1}{2}R \leq r < R$ .

Assume for a moment that the measure  $\mu$  is doubling and that the space  $X$  supports a  $p$ -Poincaré inequality. Then it is well known that  $\text{cap}_p(B_r, B_{2r}) \simeq \mu(B_r)r^{-p}$  holds for all  $0 < r < \frac{1}{8} \text{diam } X$ . If in addition the exponents  $0 < q \leq q' <$

$\infty$  are such that

$$\left(\frac{r}{R}\right)^{q'} \lesssim \frac{\mu(B_r)}{\mu(B_R)} \lesssim \left(\frac{r}{R}\right)^q, \quad \text{if } 0 < r \leq R < \text{diam } X, \quad (1.1)$$

then, by [3, Theorem 1.1],

$$\text{cap}_p(B_r, B_R) \simeq \begin{cases} \mu(B_r)r^{-p}, & \text{if } p < q, \\ \mu(B_R)R^{-p}, & \text{if } p > q', \end{cases} \quad (1.2)$$

when  $0 < 2r \leq R < \frac{1}{4} \text{diam } X$ . However, when  $r$  is close to  $R$  these estimates are no longer valid; in particular, typically  $\text{cap}_p(B_r, B_R) \rightarrow \infty$  when  $r \rightarrow R$  and  $p > 1$ . Moreover, the difference in the growth bounds in (1.1) does not play any role when  $r$  is close to  $R$ , and so it is obvious that other properties of the space determine the capacities of thin annuli.

In (unweighted)  $\mathbf{R}^n$  the following equalities hold for capacities of annuli for all  $0 < r < R < \infty$  (see e.g. [11, p. 35]):

$$\text{cap}_p(B_r, B_R) = \begin{cases} C(n, p) |R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)}|^{1-p}, & \text{if } p \notin \{1, n\}, \\ C(n, p) \left(\log \frac{R}{r}\right)^{1-n}, & \text{if } p = n, \\ C(n, p) r^{n-1}, & \text{if } p = 1. \end{cases}$$

When  $0 < \frac{1}{2}R \leq r < R$ , these yield the estimate

$$\text{cap}_p(B_r, B_R) \simeq \left(1 - \frac{r}{R}\right)^{1-p} \frac{m(B_R)}{R^p}, \quad (1.3)$$

where  $m$  is the  $n$ -dimensional Lebesgue measure.

The main goal in this paper is to find general conditions for the space  $X$  under which estimates similar to (1.3) hold. One such condition is the following measure decay property, which will play a crucial role in our results.

**Definition 1.1.** Let  $0 < \eta \leq 1$ . The measure  $\mu$  has the  $\eta$ -annular decay ( $\eta$ -AD) property at  $x \in X$ , if there is a constant  $C$  such that for all radii  $0 < r < R$  we have

$$\mu(B(x, R) \setminus B(x, r)) \leq C \left(1 - \frac{r}{R}\right)^\eta \mu(B(x, R)). \quad (1.4)$$

If there is a common constant  $C$  such that (1.4) holds for all  $x \in X$  (and all radii  $0 < r < R$ ), then  $\mu$  has the *global  $\eta$ -AD property*.

For most of the results in this paper it will be enough to require a pointwise AD property at  $x_0$ , often together with pointwise versions of (reverse) doubling and Poincaré inequalities. This resembles the situation in [3], where for capacity estimates for nonthin annuli, such as (1.2), it was enough to require doubling (and reverse-doubling) and Poincaré inequalities to hold pointwise.

The global AD property was introduced (under the name volume regularity property) in Colding–Minicozzi [6, p. 125] for manifolds and independently by Buckley [5], who called it the annular decay property, for general metric spaces. A variant of the global AD property was already used in David–Journé–Semmes [7, p. 41]. Later, the global AD-property has been used by many other authors. See e.g. Buckley [5] and Routin [15] for more information and applications of the global AD property. We have not seen any considerations related to the pointwise AD property in the literature.

If  $X$  is a length space and  $\mu$  is globally doubling, then  $\mu$  has the global  $\eta$ -AD property for some  $\eta > 0$ , see the proof of Lemma 3.3 in [6]. The AD property implies the following upper bound for the variational capacity.

**Proposition 1.2.** *Assume that  $\mu$  has the  $\eta$ -AD property at  $x_0$ . Then*

$$\text{cap}_p(B_r, B_R) \lesssim \left(1 - \frac{r}{R}\right)^{\eta-p} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < r < R. \quad (1.5)$$

*If  $\mu$  has the global  $\eta$ -AD property, then the implicit constant is independent of  $x_0$ .*

The proof of this result is quite simple, and it is perhaps more interesting that there are similar lower bounds and that the estimate is sharp, as we show in Example 3.3. The sharpness is true even if one assumes that  $\mu$  has the global  $\eta$ -AD property.

Lower bounds for capacities are in general considerably more difficult to obtain than upper bounds. Here we use relatively simple means to obtain lower bounds similar to the upper bounds, so that we obtain two-sided estimates as in (1.3). The key assumption is, as usual, some type of Poincaré inequality. When both the 1-AD property and the 1-Poincaré inequality are available, our upper and lower bounds coincide, and we obtain the following generalization of (1.3), which is our main result.

**Theorem 1.3.** *Assume that  $X$  supports a global 1-Poincaré inequality and that  $\mu$  has the global 1-AD property. Then*

$$\text{cap}_p(B_r, B_R) \simeq \left(1 - \frac{r}{R}\right)^{1-p} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{3}. \quad (1.6)$$

As in Proposition 1.2 it is actually enough to require pointwise versions of the assumptions, but then the result is a bit more complicated to formulate; see Theorem 4.3 for the exact statement. Nevertheless, even with global assumptions the parameters are sharp, see Example 6.1.

The 1-AD property, which is the best possible annular decay and which Buckley [5] calls “strong annular decay”, is essential in both the upper and lower bounds of Theorem 1.3. To further illustrate this useful property, we establish several characterizations of the 1-AD property in Section 5.

*Acknowledgement.* A. B. and J. B. were supported by the Swedish Research Council. J. L. was supported by the Academy of Finland (grant no. 252108) and the Väisälä Foundation of the Finnish Academy of Science and Letters. Part of this research was done during several visits of J. L. to Linköping University in 2012–15, and one visit of A. B. to the University of Jyväskylä in 2015.

## 2. Preliminaries

In this section we introduce the necessary background notation on metric spaces and in particular on Sobolev spaces and capacities in metric spaces. See the monographs Björn–Björn [1] and Heinonen–Koskela–Shanmugalingam–Tyson [13] for more extensive treatments of these topics, including proofs of most of the results mentioned in this section.

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable, and thus each curve can be parameterized by its arc length  $ds$ . The metric space  $X$  is a *length space* if whenever  $x, y \in X$  and  $\varepsilon > 0$ , there is a curve between  $x$  and  $y$  with length less than  $(1 + \varepsilon)d(x, y)$ .

A property is said to hold for  *$p$ -almost every curve* if it fails only for a curve family  $\Gamma$  with zero  $p$ -modulus, i.e. there exists  $0 \leq \rho \in L^p(X)$  such that  $\int_\gamma \rho ds = \infty$  for every curve  $\gamma \in \Gamma$ . Following Heinonen–Koskela [12], we introduce upper gradients as follows (they called them very weak gradients).

**Definition 2.1.** A Borel function  $g: X \rightarrow [0, \infty]$  is an *upper gradient* of a function  $f: X \rightarrow [-\infty, \infty]$  if for all curves  $\gamma: [0, l_\gamma] \rightarrow X$ ,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \quad (2.1)$$

where the left-hand side is considered to be  $\infty$  whenever at least one of the terms therein is infinite. If  $g: X \rightarrow [0, \infty]$  is measurable and (2.1) holds for  $p$ -almost every curve, then  $g$  is a  *$p$ -weak upper gradient* of  $f$ .

The  $p$ -weak upper gradients were introduced in Koskela–MacManus [14]. It was also shown there that if  $g \in L^p(X)$  is a  $p$ -weak upper gradient of  $f$ , then one can find a sequence  $\{g_j\}_{j=1}^\infty$  of upper gradients of  $f$  such that  $g_j \rightarrow g$  in  $L^p(X)$ . If  $f$  has an upper gradient in  $L^p(X)$ , then it has an a.e. unique *minimal  $p$ -weak upper gradient*  $g_f \in L^p(X)$  in the sense that for every  $p$ -weak upper gradient  $g \in L^p(X)$  of  $f$  we have  $g_f \leq g$  a.e., see Shanmugalingam [17] and Hajlasz [9]. Following Shanmugalingam [16], we define a version of Sobolev spaces on the metric measure space  $X$ .

**Definition 2.2.** For a measurable function  $f: X \rightarrow [-\infty, \infty]$ , let

$$\|f\|_{N^{1,p}(X)} = \left( \int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients  $g$  of  $f$ . The *Newtonian space* on  $X$  is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The quotient space  $N^{1,p}(X)/\sim$ , where  $f \sim h$  if and only if  $\|f - h\|_{N^{1,p}(X)} = 0$ , is a Banach space and a lattice, see Shanmugalingam [16]. In this paper we assume that functions in  $N^{1,p}(X)$  are defined everywhere, not just up to an equivalence class in the corresponding function space. This is needed for the definition of upper gradients to make sense. If  $f, h \in N_{\text{loc}}^{1,p}(X)$ , then  $g_f = g_h$  a.e. in  $\{x \in X : f(x) = h(x)\}$ , in particular  $g_{\min\{f,c\}} = g_f \chi_{\{f < c\}}$  for  $c \in \mathbf{R}$ .

**Definition 2.3.** The *Sobolev  $p$ -capacity* of an arbitrary set  $E \subset X$  is

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all  $u \in N^{1,p}(X)$  such that  $u \geq 1$  on  $E$ .

The Sobolev capacity is countably subadditive and it is the correct gauge for distinguishing between two Newtonian functions. If  $u \in N^{1,p}(X)$ , then  $u \sim v$  if and only if they differ only in a set of capacity zero. Moreover, if  $u, v \in N^{1,p}(X)$  and  $u = v$  a.e., then  $u \sim v$ . This is the main reason why, unlike in the classical Euclidean setting, we do not need to require the functions admissible in the definition of capacity to be 1 in a neighbourhood of  $E$ . In (weighted or unweighted)  $\mathbf{R}^n$ ,  $C_p$  is the usual Sobolev capacity and  $N^{1,p}(\mathbf{R}^n)$  and  $N^{1,p}(\Omega)$  are the refined Sobolev spaces as in Heinonen–Kilpeläinen–Martio [11, p. 96], see Björn–Björn [1, Theorem 6.7 (ix) and Appendix A.2].

**Definition 2.4.** The measure  $\mu$  is *doubling at  $x$*  if there is a constant  $C > 0$  such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad \text{whenever } r > 0. \quad (2.2)$$

If (2.2) holds with the same constant  $C > 0$  for all  $x \in X$ , we say that  $\mu$  is (*globally*) *doubling*.

We also say that the measure  $\mu$  is *reverse-doubling at  $x$* , if there are constants  $\gamma, \tau > 1$  such that

$$\mu(B(x, \tau r)) \geq \gamma \mu(B(x, r)) \quad \text{for all } 0 < r \leq \text{diam } X/2\tau.$$

The global doubling condition is often assumed in the metric space literature, but for many of our estimates it will be enough to assume that  $\mu$  is doubling at  $x$ . If  $X$  is connected, or more generally uniformly perfect (see Heinonen [10]), and  $\mu$  is globally doubling, then  $\mu$  is also reverse-doubling at every point, with uniform constants. In the connected case, one can choose  $\tau > 1$  arbitrarily and find  $\gamma > 1$  independent of  $x$ , see e.g. Corollary 3.8 in [1]. If  $\mu$  is merely doubling at  $x$ , then the reverse-doubling at  $x$  does not follow automatically and has to be imposed separately whenever needed.

The  $\eta$ -AD property at  $x_0$  easily implies that  $\mu$  is doubling at  $x_0$ . The converse is not true even if  $X$  is a length space, as seen by considering  $m + \delta_1$  on  $\mathbf{R}$ , where  $\delta_1$  is the Dirac measure at 1, which is doubling at 0, but does not have the  $\eta$ -AD property at 0 for any  $\eta > 0$ . (For an absolutely continuous example, consider  $\mathbf{R}$  equipped with the measure  $w dx$ , where  $w(x) = \max\{1, 1/|x-1|(\log|x-1|)^2\}$ .)

**Definition 2.5.** We say that  $X$  supports a  *$p$ -Poincaré inequality at  $x$*  if there exist constants  $C > 0$  and  $\lambda \geq 1$  such that for all balls  $B = B(x, r)$ , all integrable functions  $f$  on  $X$ , and all ( $p$ -weak) upper gradients  $g$  of  $f$ ,

$$\int_B |f - f_B| d\mu \leq Cr \left( \int_{\lambda B} g^p d\mu \right)^{1/p},$$

where  $f_B := \int_B f d\mu / \mu(B)$ . If  $C$  and  $\lambda$  are independent of  $x$ , we say that  $X$  supports a (*global*)  *$p$ -Poincaré inequality*.

It is well known that if  $X$  supports a global  $p$ -Poincaré inequality, then  $X$  is connected, but in fact even a pointwise  $p$ -Poincaré inequality is sufficient for this.

**Proposition 2.6.** *If  $X$  supports a  $p$ -Poincaré inequality at  $x_0$ , then  $X$  is connected and  $C_p(S_R) > 0$  for every sphere  $S_R = \{x : d(x, x_0) = R\}$  with  $R < \text{diam } X/2$ .*

In particular, if  $\mu$  is globally doubling and  $X$  supports a global Poincaré inequality, then  $\mu$  is reverse-doubling and  $\tau > 1$  can be chosen arbitrarily.

*Proof.* The first part is shown in the same way as in Proposition 4.2 in [1]. For the second part, assume that  $C_p(S_R) = 0$ . Then 0 is a  $p$ -weak upper gradient of  $\chi_{S_R}$ , as  $p$ -almost no curve intersects  $S_R$ , see [1, Proposition 1.48]. Thus the  $p$ -Poincaré inequality is violated for  $B_{2R}$ .  $\square$

**Definition 2.7.** Let  $\Omega \subset X$  be open. The *variational  $p$ -capacity* of  $E \subset \Omega$  with respect to  $\Omega$  is

$$\text{cap}_p(E, \Omega) = \inf_u \int_{\Omega} g_u^p d\mu,$$

where the infimum is taken over all  $u \in N^{1,p}(X)$  such that  $\chi_E \leq u \leq 1$  in  $\Omega$  and  $u = 0$  on  $X \setminus \Omega$ ; we call such functions  $u$  *admissible* for  $\text{cap}_p(E, \Omega)$ .

Also the variational capacity is countably subadditive and coincides with the usual variational capacity (see Björn–Björn [2, Theorem 5.1] for a proof valid in weighted  $\mathbf{R}^n$ ).

Throughout the paper, we write  $a \lesssim b$  if there is an implicit constant  $C > 0$  such that  $a \leq Cb$ , where  $C$  is independent of the essential parameters involved. We also write  $a \gtrsim b$  if  $b \lesssim a$ , and  $a \simeq b$  if  $a \lesssim b \lesssim a$ .

Recall that  $x_0 \in X$  is a fixed but arbitrary point and  $B_r = B(x_0, r)$ .

### 3. Upper bounds for capacity

In this section we prove Proposition 1.2 and show its sharpness.

**Lemma 3.1.** *If  $0 < r < R$ , then*

$$\text{cap}_p(B_r, B_R) \leq \frac{\mu(B_R \setminus B_r)}{(R-r)^p}.$$

*Proof.* The function

$$u(x) = \left(1 - \frac{\text{dist}(x, B_r)}{R-r}\right)_+$$

is admissible for  $\text{cap}_p(B_r, B_R)$ , and  $g = (R-r)^{-1}\chi_{B_R \setminus B_r}$  is an upper gradient of  $u$ . We thus obtain that

$$\text{cap}_p(B_r, B_R) \leq \int_{B_R} g^p d\mu = \frac{\mu(B_R \setminus B_r)}{(R-r)^p}. \quad \square$$

*Proof of Proposition 1.2.* Using the  $\eta$ -AD property at  $x_0$  and Lemma 3.1, we obtain that

$$\text{cap}_p(B_r, B_R) \leq \frac{\mu(B_R \setminus B_r)}{(R-r)^p} \lesssim \left(\frac{R-r}{R}\right)^\eta \frac{\mu(B_R)}{(R-r)^p} = \left(\frac{R-r}{R}\right)^{\eta-p} \frac{\mu(B_R)}{R^p}. \quad \square$$

**Remark 3.2.** Note that if  $\mu$  has no AD property (in which case we could say that  $\mu$  has the “0”-AD property), then Lemma 3.1 still gives

$$\text{cap}_p(B_r, B_R) \leq \left(1 - \frac{r}{R}\right)^{-p} \frac{\mu(B_R)}{R^p}.$$

This is sharp by Example 3.3 below.

Moreover, if  $\mu$  has local  $\eta$ -AD at  $x_0$ , in the sense that there is some  $R_0 > 0$  such that (1.4) holds for all  $0 < r < R < R_0$ , then also (1.5) holds whenever  $0 < r < R < R_0$ . Similar local versions hold for our other results, as well.

It follows directly from the proof that the constant  $C$  from Definition 1.1 can be used as the implicit constant in (1.5).

The following example shows that Proposition 1.2 is sharp.

**Example 3.3.** (This example has been inspired by Example 1.3 in Buckley [5].) Let  $x_0 = 0$ ,  $0 < \eta < 1$  and  $d\mu = w dx$  on  $\mathbf{R}^n$ ,  $n \geq 1$ , where  $w(x) = w(|x|)$  and

$$w(\rho) = \max\{1, |\rho - 1|^{\eta-1}\}.$$

This is a Muckenhoupt  $A_1$ -weight, by Theorem II.3.4 in García-Cuerva–Rubio de Francia [8], and thus  $\mu = w dx$  is globally doubling and supports a global 1-Poincaré inequality, by Theorem 4 in J. Björn [4]. It is easily verified that  $\mu(B_R) \simeq R^n$  for all  $R > 0$ . We also see that  $\mu(B_1 \setminus B_r) \simeq (1-r)^\eta$ , if  $\frac{1}{2} \leq r \leq 1$ . One can check that this is the extreme case showing that the measure  $\mu$  has the global  $\eta$ -AD property (and that  $\eta$  is optimal). By Proposition 10.8 in Björn–Björn–Lehrbäck [3], for  $p > 1$  and  $\frac{1}{2} \leq r < 1$ ,

$$\begin{aligned} \text{cap}_{p,w}(B_r, B_1) &\simeq \left( \int_r^1 (w(\rho)\rho^{n-1})^{1/(1-p)} d\rho \right)^{1-p} \\ &\simeq \left( \int_r^1 (1-\rho)^{(\eta-1)/(1-p)} d\rho \right)^{1-p} \simeq (1-r)^{\eta-p}, \end{aligned} \quad (3.1)$$

which shows that the upper bound in Proposition 1.2 is sharp when  $p > 1$ .

For  $p = 1$  we cannot use Proposition 10.8 in [3]. Instead we observe that when  $\frac{1}{2} \leq r < 1$  then

$$\text{cap}_{1,w}(B_r, B_1) \geq w(r) \text{cap}_1(B_r, B_1) \simeq w(r)r^{n-1} \simeq (1-r)^{\eta-1},$$

which shows the sharpness also for  $p = 1$ .

## 4. Lower bounds for capacity

We now turn to lower estimates for capacities of thin annuli. The following is our main estimate for obtaining the lower bound in Theorem 1.3. As usual for lower bounds, a key assumption is some sort of a Poincaré inequality.

**Theorem 4.1.** *Assume that  $1 \leq q < p < \infty$ , that  $X$  supports a  $q$ -Poincaré inequality at  $x_0$ , and that  $\mu$  has the  $\eta$ -AD property at  $x_0$  and is reverse-doubling at  $x_0$  with dilation  $\tau > 1$ . Then*

$$\text{cap}_p(B_r, B_R) \gtrsim \left(1 - \frac{r}{R}\right)^{\eta(q-p)/q} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{2\tau}. \quad (4.1)$$

If  $\mu$  has the global  $\eta$ -AD property and supports a global  $q$ -Poincaré inequality, then the implicit constant is independent of  $x_0$  and we may choose  $\tau > 1$  arbitrarily, see the proof of Theorem 1.3 below.

Example 6.2 shows that the reverse-doubling assumption cannot be dropped, while Example 6.1 shows that it is not enough to assume that  $X$  supports global  $q$ -Poincaré inequalities for all  $q > p$ .

*Proof.* Let  $u$  be admissible for  $\text{cap}_p(B_r, B_R)$ . Then  $u = 1$  in  $B_r$ ,  $u = 0$  outside  $B_R$ , and  $g_u = 0$  a.e. outside  $B_R \setminus B_r$ . The reverse-doubling implies that  $\mu(B_{\tau R} \setminus B_R) \gtrsim \mu(B_R)$  from which it follows that  $|u_{B_{\tau R}}| < c < 1$ , and so  $|u - u_{B_{\tau R}}| > 1 - c > 0$  in  $B_{R/2}$ . Note that the  $\eta$ -AD property implies that  $\mu$  is doubling at  $x_0$ . Thus we obtain from the  $q$ -Poincaré inequality at  $x_0$  and Hölder's inequality that

$$\begin{aligned} 1 &\lesssim \int_{B_{R/2}} |u - u_{B_{\tau R}}| d\mu \lesssim \int_{B_{\tau R}} |u - u_{B_{\tau R}}| d\mu \lesssim R \left( \int_{B_{\tau R}} g_u^q d\mu \right)^{1/q} \\ &\lesssim \frac{R}{\mu(B_R)^{1/q}} \left( \int_{B_R \setminus B_r} g_u^q d\mu \right)^{1/q} \\ &\lesssim \frac{R}{\mu(B_R)^{1/q}} \mu(B_R \setminus B_r)^{1/q-1/p} \left( \int_{B_R \setminus B_r} g_u^p d\mu \right)^{1/p}. \end{aligned}$$

By the  $\eta$ -AD property,  $\mu(B_R \setminus B_r) \lesssim (1-r/R)^\eta \mu(B_R)$ . Inserting this into the above estimate yields

$$\begin{aligned} \left( \int_{B_R \setminus B_r} g_u^p d\mu \right)^{1/p} &\gtrsim \frac{\mu(B_R)^{1/q}}{R} \left(1 - \frac{r}{R}\right)^{\eta(1/p-1/q)} \mu(B_R)^{1/p-1/q} \\ &= \frac{\mu(B_R)^{1/p}}{R} \left(1 - \frac{r}{R}\right)^{\eta(q-p)/pq}, \end{aligned}$$

and (4.1) follows after taking infimum over all admissible  $u$ .  $\square$

Theorem 4.1 establishes the lower bound in Theorem 1.3 when  $p > 1$ . For  $p = 1$  we instead use the following result. In view of Remark 3.2, we can see this as an  $\eta = 0$  version of Theorem 4.1.

**Proposition 4.2.** *Assume that  $X$  supports a  $p$ -Poincaré inequality at  $x_0$ , and that  $\mu$  is doubling at  $x_0$  and reverse-doubling at  $x_0$  with dilation  $\tau > 1$ . Then*

$$\text{cap}_p(B_r, B_R) \gtrsim \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{2\tau}. \quad (4.2)$$

Moreover,  $\text{cap}_p(B_r, B_{2r}) \simeq \mu(B_r)r^{-p}$  when  $0 < r \leq \text{diam } X/4\tau$ .

If  $\mu$  is globally doubling and  $X$  supports a global  $p$ -Poincaré inequality, then the implicit constants are independent of  $x_0$ .

Example 6.1 shows that the  $p$ -Poincaré assumption cannot be weakened, even if it is assumed globally, while Example 6.2 shows that the reverse-doubling assumption cannot be dropped.

*Proof.* Let  $u$  be admissible for  $\text{cap}_p(B_r, B_R)$ . As in the proof of Theorem 4.1 (with  $q$  replaced by  $p$ ), we get that

$$1 \lesssim R \left( \int_{B_{\tau\lambda R}} g_u^p d\mu \right)^{1/p} \lesssim \frac{R}{\mu(B_R)^{1/p}} \left( \int_{B_R \setminus B_r} g_u^p d\mu \right)^{1/p},$$

and (4.2) follows after taking infimum over all admissible  $u$ . That  $\text{cap}_p(B_r, B_{2r}) \simeq \mu(B_r)r^{-p}$  follows from this and Lemma 3.1.  $\square$

**Theorem 4.3.** *Assume that  $X$  supports a 1-Poincaré inequality at  $x_0$  and that  $\mu$  has the 1-AD property at  $x_0$  and is reverse-doubling at  $x_0$  with dilation  $\tau > 1$ . Then*

$$\text{cap}_p(B_r, B_R) \simeq \left(1 - \frac{r}{R}\right)^{1-p} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{2\tau}.$$

Even under global assumptions, as in Theorem 1.3, the 1-Poincaré and 1-AD assumptions cannot be weakened, as shown by Example 6.1. Example 6.2 shows that the reverse-doubling assumption cannot be dropped, and in particular that it is possible that  $\mu$  has the 1-AD property at  $x_0$  and  $X$  supports a 1-Poincaré inequality at  $x_0$ , but that  $\mu$  fails to be reverse-doubling at  $x_0$ .

*Proof.* This follows by combining Proposition 1.2 with Theorem 4.1 (for  $p > 1$ ) and Proposition 4.2 (for  $p = 1$ ).  $\square$

*Proof of Theorem 1.3.* It follows from the global assumptions and Proposition 2.6 that  $X$  is connected. Hence,  $X$  is reverse-doubling at  $x_0$  with  $\tau = \frac{3}{2}$ . As the implicit constants in Theorem 4.3 only depend on the parameters in the assumptions, Theorem 1.3 follows.  $\square$

## 5. Characterizations of the 1-AD property

Our aim in this section is to prove the following characterizations of the 1-AD property.

**Theorem 5.1.** *Let  $f(r) := \mu(B_r)$ . Then the following are equivalent:*

- (a)  $\mu$  has the 1-AD property at  $x_0$ ;
- (b)  $f$  is locally absolutely continuous on  $(0, \infty)$  and  $\rho f'(\rho) \lesssim f(\rho)$  for a.e.  $\rho > 0$ ;
- (c)  $f$  is locally Lipschitz on  $(0, \infty)$  and  $\rho f'(\rho) \lesssim f(\rho)$  for a.e.  $\rho > 0$ .

If moreover  $\mu$  is reverse-doubling at  $x_0$  and  $X$  supports a 1-Poincaré inequality at  $x_0$ , then also the following condition is equivalent to those above:

- (d)  $f$  is locally Lipschitz on  $(0, \infty)$  and  $\rho f'(\rho) \simeq f(\rho)$  for a.e.  $\rho$  with  $0 < \rho < \text{diam } X/2\tau$ .

The assumption of absolute continuity cannot be dropped, as shown by Example 2.6 in Björn–Björn–Lehrbäck [3] where  $X$  is the usual Cantor ternary set and  $f$  is the Cantor staircase function for which  $f'(\rho) = 0 \leq f(\rho)/\rho$  for a.e.  $\rho > 0$ .

Example 6.1 shows that the 1-Poincaré assumption (for the last part) cannot be weakened, even under global assumptions, while Example 6.2 shows that the reverse-doubling assumption cannot be dropped even if  $X$  supports a global 1-Poincaré inequality.

*Proof.* (a)  $\Rightarrow$  (c) By the 1-AD property at  $x_0$  we have for  $0 < \varepsilon < \rho$  that

$$\frac{f(\rho) - f(\rho - \varepsilon)}{\varepsilon} \lesssim \frac{\left(1 - \frac{\rho - \varepsilon}{\rho}\right) f(\rho)}{\varepsilon} = \frac{f(\rho)}{\rho}. \quad (5.1)$$

Since the right-hand side is locally bounded it follows that  $f$  is locally Lipschitz on  $(0, \infty)$ , and thus that  $f'(\rho)$  exists for a.e.  $\rho > 0$ . Moreover, by (5.1) we see that  $f'(\rho) \lesssim f(\rho)/\rho$  whenever  $f'(\rho)$  exists.

(c)  $\Rightarrow$  (b) This is trivial.

(b)  $\Rightarrow$  (a) Assume that  $\rho f'(\rho)/f(\rho) \leq M$  a.e., where  $M \geq 1$ . Then

$$(\rho^{-M} f(\rho))' = \rho^{-M-1} (-M f(\rho) + \rho f'(\rho)) \leq 0 \text{ a.e.},$$

and thus  $f(R)/R^M \leq f(r)/r^M$ , i.e.  $(r/R)^M \leq f(r)/f(R)$  for  $0 < r < R$ . Hence

$$f(R) - f(r) \leq \left(1 - \left(\frac{r}{R}\right)^M\right) f(R) \leq M \left(1 - \frac{r}{R}\right) f(R).$$

Thus we have shown that (a)–(c) are equivalent.

Now assume that  $\mu$  is reverse-doubling at  $x_0$ , and that  $X$  supports a 1-Poincaré inequality at  $x_0$ .

(c)  $\Rightarrow$  (d) Let  $0 < \rho < \text{diam } X/2\tau$  and  $0 < \varepsilon < \frac{1}{2}\rho$ . We have already shown that (c)  $\Rightarrow$  (a), so  $\mu$  has the 1-AD property at  $x_0$ , and in particular  $\mu$  is doubling at  $x_0$ . Thus Lemma 5.2 below (with  $q = 1$ ) yields

$$\frac{f(\rho) - f(\rho - \varepsilon)}{\varepsilon} \gtrsim \frac{\left(1 - \frac{\rho - \varepsilon}{\rho}\right) f(\rho)}{\varepsilon} = \frac{f(\rho)}{\rho},$$

showing that  $f'(\rho) \gtrsim f(\rho)/\rho$  whenever  $f'(\rho)$  exists.

(d)  $\Rightarrow$  (c) If  $X$  is unbounded this is trivial. So assume that  $X$  is bounded. If  $\rho > \text{diam } X$ , then  $f(\rho) = \mu(X)$  and  $f'(\rho) = 0$ . As  $f$  is locally Lipschitz there is a constant  $M$  such that  $f'(\rho) \leq M$  for a.e.  $\rho$  satisfying  $\text{diam } X/2\tau < \rho < \text{diam } X$ . For such  $\rho$  we have that  $f(\rho)/\rho \geq f(B_{\text{diam } X/2\tau})/\text{diam } X$  and thus  $\rho f'(\rho) \lesssim f(\rho)$  for a.e.  $\rho > \text{diam } X/2\tau$ . Together with (d) this yields (c).  $\square$

The following estimate completes the proof of Theorem 5.1. It also complements the 1-AD property. In particular, if  $q = 1$  then this lower bound, together with the 1-AD property, leads to a sharp two-sided estimate for the measures of thin annuli.

**Lemma 5.2.** *Assume that  $X$  supports a  $q$ -Poincaré inequality at  $x_0$  for some  $1 \leq q < \infty$  and that  $\mu$  is doubling at  $x_0$  and reverse-doubling at  $x_0$ . Then*

$$\mu(B_R \setminus B_r) \gtrsim \left(1 - \frac{r}{R}\right)^q \mu(B_R) \quad \text{when } 0 < \frac{R}{2} \leq r < R < \frac{\text{diam } X}{2\tau}.$$

Example 6.1 shows that the Poincaré assumption cannot be weakened, while Example 6.2 shows that the reverse-doubling condition cannot be omitted. We do not know if the doubling condition can be omitted.

*Proof.* Let

$$u(x) = \left(1 - \frac{\text{dist}(x, B_r)}{R-r}\right)_+.$$

As in the proof of Lemma 3.1, we obtain that

$$\int_X g_u^q d\mu \leq \frac{\mu(B_R \setminus B_r)}{(R-r)^q}.$$

Hence, as in the proof of Theorem 4.1, we get that

$$1 \lesssim R \left( \int_{B_{r\lambda R}} g_u^q d\mu \right)^{1/q} \lesssim \frac{R}{\mu(B_R)^{1/q}} \frac{\mu(B_R \setminus B_r)^{1/q}}{(R-r)},$$

from which the claim follows.  $\square$

## 6. Counterexamples

In this section we provide counterexamples showing that most of our results are sharp. The following example shows the sharpness of the Poincaré and the AD assumptions in Theorem 1.3 and in several other results.

**Example 6.1.** (Weighted bow-tie) Let

$$X = \{(x_1, \dots, x_n) : x_2^2 + \dots + x_n^2 \leq \frac{1}{4}x_1^2 \text{ and } -1 \leq x_1 \leq 2\} \quad (6.1)$$

as a subset of  $\mathbf{R}^n$ ,  $n \geq 2$ , and equip  $X$  with the measure  $d\mu = |x|^\alpha dx$ , where  $\alpha > -n$ . (Additionally, we can make this example into a length space if we equip  $X$  with the inner metric (see [1, Definition 4.41]), which only makes a difference when calculating distances between the two sides of the origin.) Note that the constant 2 in the range of  $x_1$  in (6.1) above was chosen so that we can have  $R = 1 \leq \frac{1}{3} \text{diam } X$  below as required in Theorem 1.3.

If  $q \geq 1$ , then  $X$  supports a global  $q$ -Poincaré inequality if and only if  $q > n + \alpha$  or  $q = 1 \geq n + \alpha$ , see Example 5.7 in [1]. Moreover,  $\mu$  is globally doubling.

Let  $x_0 = (-1, 0, \dots, 0)$  and  $\eta = \min\{1, n + \alpha\}$ . Then for  $0 < r < R < \text{diam } X$  we have

$$\mu(B_R) \simeq R^n \quad \text{and} \quad \mu(B_R \setminus B_r) \lesssim \left(1 - \frac{r}{R}\right)^\eta \mu(B_R),$$

which shows that  $\mu$  has the  $\eta$ -AD property at  $x_0$ . One can check that this is the extreme case showing that  $\mu$  has the global  $\eta$ -AD property (and that  $\eta$  is optimal).

If  $0 < \delta < \frac{1}{2}$ , then

$$\mu(B_1 \setminus B_{1-\delta}) \simeq \int_0^\delta \rho^\alpha \rho^{n-1} d\rho \simeq \delta^{n+\alpha},$$

which shows that the Poincaré assumption in Lemma 5.2 cannot be weakened. Moreover, by Lemma 3.1,

$$\text{cap}_p(B_{1-\delta}, B_1) \lesssim \frac{1}{\delta^p} \mu(B_1 \setminus B_{1-\delta}) \simeq \delta^{n+\alpha-p},$$

which shows that (1.6) fails if  $n + \alpha > 1$ , and thus we cannot replace the assumption of a global 1-Poincaré inequality in Theorem 1.3 by a global  $q$ -Poincaré inequality for any fixed  $q > 1$ .

Conversely, if  $0 < \delta < \frac{1}{2}$  and  $p > n + \alpha$ , i.e.  $X$  supports a global  $p$ -Poincaré inequality, then a simple reflection argument and [3, Proposition 10.8] imply that

$$\text{cap}_p(B_{1-\delta}, B_1) \gtrsim \text{cap}_p(\{0\}, B(0, 2\delta)) \simeq \delta^{n+\alpha-p}$$

and hence

$$\text{cap}_p(B_{1-\delta}, B_1) \simeq \delta^{n+\alpha-p}.$$

If  $\eta = n + \alpha < 1$ , then  $X$  supports a global 1-Poincaré inequality and  $\mu$  has the  $\eta$ -AD property at  $x_0$ , but (1.6) fails. Hence we cannot replace the 1-AD assumption in Theorem 1.3 by the  $\eta$ -AD property for any fixed  $\eta < 1$ .

Next, if  $p = n + \alpha > 1$ , then  $X$  supports a global  $q$ -Poincaré inequality for each  $q > p$ , but not a global  $p$ -Poincaré inequality. Moreover, by Example 5.7 in [1],  $C_p(\{0\}) = 0$  and thus we can test  $\text{cap}_p(B_{1-\delta}, B_1)$  with  $u = \chi_{B_1}$  yielding  $\text{cap}_p(B_{1-\delta}, B_1) = 0$ . It also follows from Proposition 2.6 that  $X$  does not support a  $p$ -Poincaré inequality at  $x_0$ . Hence the  $p$ -Poincaré assumption in Proposition 4.2 cannot be weakened if  $p > 1$ . Moreover, it also follows that it is not enough to assume that  $X$  supports global  $q$ -Poincaré inequalities for all  $q > p$  in Theorem 4.1.

When  $p = 1 < q$  we instead choose  $n$  and  $\alpha$  so that  $1 < n + \alpha < q$ . In particular,  $X$  supports a global  $q$ -Poincaré inequality in this case. As above,  $\text{cap}_1(B_{1-\delta}, B_1) = 0$  and  $X$  does not support a 1-Poincaré inequality at  $x_0$ , showing that the Poincaré assumption in Proposition 4.2 is sharp also for  $p = 1$ .

Finally, let  $f(r) = \mu(B_r)$  as in Theorem 5.1. Let  $q > 1$  and choose  $n$  and  $\alpha$  so that  $1 < n + \alpha < q$ . Then  $\mu$  has the global 1-AD property and  $X$  supports a global  $q$ -Poincaré inequality. For  $\frac{1}{2} \leq r < R \leq 1$ , with  $R$  close to  $r$ , we see that

$$\mu(B_R \setminus B_r) \simeq (1-r)^\alpha m(B_R \setminus B_r) \simeq (1-r)^\alpha (R-r)(1-r)^{n-1},$$

where  $m$  is the  $n$ -dimensional Lebesgue measure. Hence

$$r f'(r) = r \lim_{R \rightarrow r^+} \frac{\mu(B_R \setminus B_r)}{R-r} \simeq r(1-r)^{n+\alpha-1} \not\asymp r \simeq r^n \simeq \mu(B_r) \quad \text{when } \frac{1}{2} < r < 1.$$

Thus condition (d) in Theorem 5.1 fails, which shows that it is not enough to assume that  $X$  supports a global  $q$ -Poincaré inequality for some fixed  $q > 1$  in (the last part of) Theorem 5.1.

We do not have a counterexample to the conclusion of Theorem 4.3 which supports pointwise  $q$ -Poincaré inequalities at  $x_0$  for all  $q > 1$ .

**Example 6.2.** Let  $w$  be a positive nonincreasing weight function on  $X = [0, \infty)$ ,  $d\mu = w dx$  and  $x_0 = 0$ . Assume that  $\mu(B_1) < \infty$ . As  $w$  is nonincreasing it is easy to see that  $\mu$  is doubling at  $x_0$ . Let  $f$  be an integrable function on  $X$  with an upper gradient  $g$ , and let  $B = B(x, r) \subset X$  be a ball. Then either  $B = (a, b)$  with  $0 \leq a < b$  or  $B = [a, b)$  with  $a = 0 < b$ . In either case we have

$$\begin{aligned} \int_B |f - f(a)| d\mu &\leq \frac{1}{\mu(B)} \int_a^b \int_a^t g(x) dx d\mu(t) = \frac{1}{\mu(B)} \int_a^b \int_x^b d\mu(t) g(x) dx \\ &\leq \frac{1}{\mu(B)} \int_a^b 2rw(x)g(x) dx = 2r \int_B g d\mu. \end{aligned}$$

It thus follows from Lemma 4.17 in [1] that  $X$  supports a global 1-Poincaré inequality.

Moreover, if  $0 < \frac{1}{2}R \leq r < R$ , then

$$\mu(B_R \setminus B_r) \leq w(r)(R-r) \leq \frac{\mu(B_r)}{r}(R-r) \leq 2\left(1 - \frac{r}{R}\right)\mu(B_R).$$

On the other hand, if  $0 < 2r < R$ , then

$$\mu(B_R \setminus B_r) \leq \mu(B_R) \leq 2\left(1 - \frac{r}{R}\right)\mu(B_R).$$

Hence  $\mu$  has the 1-AD property at  $x_0$ .

So far we have just assumed that  $w$  is nonincreasing, but now assume that  $w(x) = \min\{1, 1/x\}$ . If  $R > 2$ , then by Lemma 3.1,

$$\text{cap}_p(B_{R/2}, B_R) \lesssim \frac{\mu(B_R \setminus B_{R/2})}{R^p} = \frac{\log 2}{R^p},$$

while the right-hand sides (with  $r = \frac{1}{2}R$ ) in Theorem 4.1, Proposition 4.2 and Theorem 4.3 are larger than this when  $R$  is large enough, since  $\mu(B_R) \rightarrow \infty$  as  $R \rightarrow \infty$ . In particular it follows that  $\mu$  cannot be reverse-doubling at  $x_0$  (which also follows directly from  $\mu(B_R \setminus B_{R/2}) = \log 2$ ) and that the reverse-doubling assumption in Theorem 4.1, Proposition 4.2 and Theorem 4.3 cannot be dropped.

Moreover, as  $\mu(B_R \setminus B_{R/2}) = \log 2$  and  $\mu(B_R) \rightarrow \infty$  as  $R \rightarrow \infty$ , the inequality in Lemma 5.2 fails in this case (for every  $q \geq 1$ ), showing that the reverse-doubling assumption in Lemma 5.2 cannot be dropped either.

Finally, write  $f(r) = \mu(B_r)$ , as in Theorem 5.1. Then  $f(r) = 1 + \log r$  for  $r \geq 1$ . For  $\rho > 1$  we have  $\rho f'(\rho) = 1 \not\geq f(\rho)$ , so condition (d) in Theorem 5.1 fails. Thus the reverse-doubling assumption in (the last part of) Theorem 5.1 cannot be dropped.

## References

1. A. BJÖRN and J. BJÖRN, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics **17**, European Math. Soc., Zürich, 2011.
2. A. BJÖRN and J. BJÖRN, The variational capacity with respect to nonopen sets in metric spaces, *Potential Anal.* **40** (2014), 57–80.
3. A. BJÖRN, J. BJÖRN and J. LEHRBÄCK, Sharp capacity estimates for annuli in weighted  $\mathbf{R}^n$  and in metric spaces, *Preprint*, 2013, [arXiv:1312.1668](https://arxiv.org/abs/1312.1668).
4. J. BJÖRN, Poincaré inequalities for powers and products of admissible weights, *Ann. Acad. Sci. Fenn. Math.* **26** (2001), 175–188.
5. S. BUCKLEY, Is the maximal function of a Lipschitz function continuous?, *Ann. Acad. Sci. Fenn. Math.* **24** (1999), 519–528.
6. T. H. COLDING and W. P. MINICOZZI II, Liouville theorems for harmonic sections and applications, *Comm. Pure Appl. Math.* **51** (1998), 113–138.
7. G. DAVID, J.-L. JOURNÉ and S. SEMMES, Opérateurs de Calderón–Zygmund, fonctions para-accrétives et interpolation, *Rev. Mat. Iberoam.* **1:4** (1985), 1–56.
8. J. GARCÍA-CUERVA and J. L. RUBIO DE FRANCIA, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.
9. P. HAJLASZ, Sobolev spaces on metric-measure spaces, in *Heat Kernels and Analysis on Manifolds, Graphs and Metric Spaces (Paris, 2002)*, Contemp. Math. **338**, pp. 173–218, Amer. Math. Soc., Providence, RI, 2003.
10. J. HEINONEN, *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.
11. J. HEINONEN, T. KILPELÄINEN and O. MARTIO, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, 2nd ed., Dover, Mineola, NY, 2006.
12. J. HEINONEN and P. KOSKELA, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.
13. J. HEINONEN, P. KOSKELA, N. SHANMUGALINGAM and J. T. TYSON, *Sobolev Spaces on Metric Measure Spaces*, New Math. Monographs **27**, Cambridge Univ. Press, Cambridge, 2015.
14. P. KOSKELA and P. MACMANUS, Quasiconformal mappings and Sobolev spaces, *Studia Math.* **131** (1998), 1–17.
15. E. ROUTIN, Distribution of points and Hardy type inequalities in spaces of homogeneous type, *J. Fourier Anal. Appl.* **19** (2013), 877–909.

16. N. SHANMUGALINGAM, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoam.* **16** (2000), 243–279.
17. N. SHANMUGALINGAM, Harmonic functions on metric spaces, *Illinois J. Math.* **45** (2001), 1021–1050.