One loop neutrino self energies in coherent quasiparticle approximation

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Abstract

In this research training report we calculate leading order corrections to neutrino self energies in coherent quasiparticle approximation (cQPA). These corrections are needed for a treatment of neutrino oscillation in matter in finite temperature; such a treatment in cQPA will take coherence into account more carefully than the standard approach.

We first briefly review cQPA, and to this end will briefly discuss the role of coherence in quantum mechanics and the different formulations and phenomena of thermal field theory. Using the Feynman rules for evaluating self energy corrections in cQPA, we identify the relevant diagrams and calculate the corrections.

Finally we discuss the corrections and compare them to the standard approach to neutrino oscillations in matter, which does not similarly take into account nonlocal coherence and the fermionic nature of neutrinos. Application of the obtained results to neutrino oscillations are unfortunately beyond the scope of this work, and will hopefully be discussed in a further study.

Tiivistelmä

Tässä erikoistyössä laskemme johtavan kertaluvun korjaukset neutriinojen itseisenergiaan koherentissa kvasihiukkasapproksimaatiossa (coherent quasiparticle approximation, cQPA). Näitä korjauksia tarvitaan neutriino-oskillaatioiden tutkimiseen väliaineessa ja äärellisessä lämpötilassa; tällainen tarkastelu cQPA:ssa mahdollistaa koherenssin huomioimisen huolellisemmin kuin tavanomaisessa lähestymistavassa.

Ensin luomme lyhyen katsauksen cQPA:han, ja sitä varten tarkastelemme lyhyesti koherenssin merkitystä kvanttimekaniikassa sekä erilaisia termisen kenttäteorian muotoiluja ja ilmiöitä. Käyttäen cQPA:n Feynmanin sääntöjä itseisenergiakorjausten laskemiseen etsimme tarpeelliset diagrammit ja laskemme korjaukset.

Lopuksi tarkastelemme korjauksia ja vertaamme niitä tavalliseen tapaan käsitellä neutriino-oskillaatioita väliaineessa, joka ei samalla tavoin ota huomioon epälokaalia koherenssia ja neutriinojen fermionista luonnetta. Saatujen tulosten soveltaminen neutriino-oskillatioiden tutkimiseen jää ikävä kyllä tämän työn ulkopuolelle, ja sitä tutkitaan toivottavasti lähemmin myöhemmissä tutkimuksissa.

Contents

1 Introduction

Quantum mechanical coherence between distinct states is what allows non-classical behavior in quantum mechanical systems; the probabilities emerging from a state written as a coherent superposition in some basis is in the very heart of the phenomenology of quantum mechanics (QM). Quantum field theory (QFT), as a relativistic formulation of many-body QM, is therefore also expected to take coherence phenomena into account. In the presence of nonlinearities due to interactions, however, the equations of motion in QFT defy analytic solutions, necessitating the use of various approximation schemes.

Neutrino oscillations present an excellent example of coherence[1]. In a weak charged current interaction process a neutrino is produced in pure flavor state. The mass of such a state is ill-defined and therefore the time evolution is rather complicated; it is best expressed as a superposition of neutrino mass eigenstates. The slightly different time evolution of different mass states is what leads to the observed oscillation in flavor basis. Having different kinematical properties, these different mass states tend to drift apart as the neutrino propagates. As the overlap between the wave packets of different mass states is gradually lost, the oscillations cease and the probability distribution in flavor basis no longer evolves in time. The classically unexpected yet significant phenomenon of neutrino oscillation thus vitally depends on coherence.

In this paper we study coherent quasiparticle approximation (cQPA), an approximation scheme in QFT in non-zero temperature. In particular, we calculate the first order corrections to neutrino self energies in this scheme and find out that taking nonlocal coherence properly into account may lead to phenomenology substantially different from what would be expected when coherence is neglected.

The structure of this report is as follows: In Section 2 we briefly describe cQPA and list the momentum space Feynman rules needed here. Section 3 is devoted to the calculation of leading order corrections to neutrino self energy in the framework of cQPA, and the result is briefly discussed in Section 4. Finally, a summary and outlook are given in Section 5. A concise summary of the results is presented in Appendix A.

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[1] A more thorough discussion of coherence in neutrino oscillations can be found e.g. in Ref. [1].

2 Coherent quasiparticle approximation

The description of many physical situations require simultaneously taking into account finite temperature, special relativity, nonlocal quantum mechanical coherence and even thermodynamics out of equilibrium. Such situations include, for example, particle creation in the early universe and neutrino propagation in spatially or temporally varying background.

Coherent quasiparticle approximation (cQPA) is an approximation scheme capable of treating such physical systems. It was introduced by Herranen, Kainulainen, and Rahkila in Ref. [2] and reformulated in a more easily calculable form in Ref. [3].

The diagrammatic methods developed in Ref. [3] are used here to calculate leading order corrections to neutrino self energies due to weak interactions with the medium. The result differs from the traditional result, where nonlocal coherence has not been taken into account; this will be discussed in greater detail in Section 4 after the calculations are done.

2.1 Thermal field theory

When doing QFT in vacuum in zero temperature, one is typically interested in scattering processes where long-lived particles interact by interchanging virtual particles. In non-zero temperature there is, however, a thermal distribution of various particles, and a propagating particle does not only interact with itself via spontaneous virtual excitations, but also with its surroundings. Moreover, particles in a thermal system are often short-lived, whence the asymptotic in- and out-states familiar from scattering theory are no longer meaningful. Similar phenomena take place also in zero temperature, when particles propagate and interact in a medium. Such phenomena can be investigated using thermal field theory (TFT).

There are two main formalisms for TFT: imaginary and real time. The imaginary time formalism is the one adopted in most introductory treatments of TFT (such as Refs. [4] and [5]). In this formulation one writes the time coordinate as $t = x^0 = -i\tau = -ix_4$ for some real $\tau = x_4$ (which is periodic with period β). Similarly one replaces p^0 with $-ip_4$ in the momentum space, and the Minkowskian structure of spacetime becomes an Euclidean one: $t^2 - \vec{x}^2 = -(\tau^2 + \vec{x}^2)$ and similarly for p. In this formulation the integration over energy appearing in the path integral representation of the propagators is replaced by a sum over discrete energies in a Euclidean space; this gives rise to the Matsubara (or imaginary time) propagator.

In the time integral appearing in the partition function it may be more convenient to choose a more complicated path in the complex plane than a (possibly slightly tilted) horizontal or vertical line. The Keldysh path $\mathcal{C}$, composed of three line segments joining $-T+i\varepsilon$, $+T$, $-T-i\varepsilon$, and $-T-i\beta$, where $-T$ is some large negative initial time (T is let tend to infinity), $\varepsilon > 0$ is small parameter which is let tend to zero, and β is the inverse temperature. Due to the boundary condition $\varphi(t, \vec{x}) = \varphi(t - i\beta, \vec{x})$ for bosonic fields φ this time path is periodic. A generic propagator $\Delta_{\mathcal{C}}(t, \vec{x}; t', \vec{x}')$ splits to four parts: it is Δ^{++} (Δ^{--}) when both t and t' lie on upper (lower) horizontal line segment and $\Delta^{<} = \Delta^{+-}$ when t is on the upper and t' on the lower line segment (and vice versa for $\Delta^{>} = \Delta^{-+}$). This is one formulation of the real time formalism.

Simple calculations tend to be easier to do in the imaginary time formalism, but more involved ones are often easier to handle in the real time formalism. The real time formalism also preserves the Minkowskian structure of the spacetime more explicitly.

All phenomena present in vacuum and zero temperature are also present when temperature is increased or a medium introduced. In the real time formalism vacuum and thermal phenomena can be separated (for example, the propagator can be written as a sum of a vacuum propagator and a thermal propagator) thus making it more straightforward to study changes to vacuum behavior due to finite temperature or medium effects. In this report we will follow this method.

For details on TFT beyond this relatively naive introduction, see for example the books by Kapusta [4] and Le Bellac [5].

2.2 A brief introduction to cQPA[2]

This is only a short introduction to cQPA. The practical Feynman rules needed here are given in Section 2.3 below. For more details on cQPA, see Refs. [2, 6, 7, 8, 9, 10, 3] and references therein. Here we follow the notational conventions of Ref. [3].

2.2.1 Propagators and self energies

In the study of non-equilibrium TFT, the fermionic Wightman functions $iS^{<}(u, v) = \left\langle \bar{\psi}(v)\psi(u) \right\rangle$ and $iS^{>}(u, v) = \left\langle \psi(u)\bar{\psi}(v) \right\rangle$ are of central interest[3]. These functions in a way describe the self-correlation of the fermionic field ψ between points u and v in a Minkowskian spacetime. The expectation values $\langle \cdot \rangle$ are calculated with respect to an unknown density operator.

[2]This introduction follows mainly Ref. [3].

[3]It is a common convention to define $iS^{<}$ with an additional minus sign. See e.g. Ref [11].

We can also express the Wightman functions in terms of the relative and average coordinates $r = u - v$ and $x = (u+v)/2$; this is particularly convenient after a Fourier transformation in r (a Wigner transformation):

$$S^{<,>}(k,x) = \int \mathrm{d}^4k\, e^{ik\cdot r} S^{<,>}(x + \frac{r}{2}, x - \frac{r}{2}). \tag{1}$$

In analogue to $iS^{<,>}$ we define the time ordered Green's function (Feynman propagator) iS^t and in turn the hermitian Green's function $S^h = S^t - (S^> - S^<)/2$. The self energies corresponding to iS (with any of the indices $<$, $>$, t, and h) are denoted by $i\Sigma$ (with the same indices).

Similarly we may define the retarded and advanced propagators as $S^{r,a} = S^t \pm S^{<,>}$ (so that $S^h = (S^r + S^a)/2$) and the anti-Feynman propagator $S^{\bar{t}}$ (with inverse time ordering). The antihermitian Green's function

$$\mathcal{A} = \frac{i}{2}(S^> + S^<) \tag{2}$$

is known as the the spectral function[4].

In multiflavor formalism we include flavor indices so that in $iS_{ij}(u,v)$ the flavor index i corresponds to the coordinate u and similarly j to v. The flavor indices are suppressed where they can easily be inferred from the context.

For a more elaborate description of the various Green's functions, see Ref. [11].

2.2.2 Equations of motion and mass shell structure

We define the diamond operator (cf. Poisson brackets) as

$$\Diamond = \frac{1}{2}(\partial_x^{(1)} \cdot \partial_k^{(2)} - \partial_k^{(1)} \cdot \partial_x^{(2)}). \tag{3}$$

It acts on a pair of functions (the bracketed indices refer to these functions) which depend on x and k. For two functions $f(k,x)$ and $g(k,x)$, for example,

$$\Diamond\{f\}\{g\} = \frac{1}{2}(\partial_x f \cdot \partial_k g - \partial_k f \cdot \partial_x g). \tag{4}$$

Using Eq. (3) we may similarly define $\Diamond^n\{f\}\{g\}$ for any $n \in \mathbb{N}$, and so also $e^{-i\Diamond}$.

We denote by $m = m(x)$ the possibly space- and time-dependent and complex mass matrix, and write its hermitian and antihermitian parts as

[4]In the following we will only consider spectral functions for fermionic fields, whence it is written shortly $\mathcal{A} = \mathcal{A}^\psi$.

$m_h = (m + m^\dagger)/2$ and $m_a = (m - m^\dagger)/(2i)$. Using these, we define the mass operators

$$\hat{m}_{0,5} f(k,x) = e^{-i\diamond}\{m_{h,a}(x)\}\{f(k,x)\}, \tag{5}$$

where we take $\partial_k m_{h,a} = 0$.

With these notations, the Wightman functions obey the equations

$$\begin{aligned}(\not{k} + \frac{i}{2}\not{\partial}_x - \hat{m}_0 - i\hat{m}_5\gamma^5)S^{<,>} - e^{-i\diamond}\{\Sigma^h\}\{S^{<,>}\} \\ - e^{-i\diamond}\{\Sigma^{<,>}\}\{S^h\} = \pm\mathcal{C}_{\text{coll}},\end{aligned} \tag{6}$$

where the collision term is

$$\mathcal{C}_{\text{coll}} = \frac{1}{2}e^{-i\diamond}(\{\Sigma^>\}\{S^<\} - \{\Sigma^<\}\{S^>\}). \tag{7}$$

Eq. (6) is the most fundamental equation of motion, but in practice impossible to solve in full generality.

It turns out [3] that in the mass eigenbasis and with suitable approximations the phase space structure of the homogeneous and isotropic Wightman functions is more complicated than naively expected. The phase space constraint equation for $iS^<_{ij}(k,x)$ in Eq. (6) is

$$\left(k^2 - \frac{m_i^2 + m_j^2}{2}\right)k_0^2 + \frac{1}{4}\left(\frac{m_i^2 - m_j^2}{2}\right)^2 = 0. \tag{8}$$

Defining $\omega_i = \omega_i(\vec{k}) = \sqrt{m_i^2 + \vec{k}^2}$, this gives rise to dispersion relations

$$k_0 = \pm\frac{1}{2}(\omega_i + \omega_j) \tag{9}$$

and

$$k_0 = \pm\frac{1}{2}(\omega_i - \omega_j). \tag{10}$$

In the case $m_i = m_j$ the dispersion relation of Eq. (9) gives the standard relation $k^2 = m_i^2$.

Corresponding to the four dispersion relations in Eqs. (9) and (10) there are four distribution functions describing the different shell occupations. These functions for Eq. (9) are $f^{m<}_{ijh\pm}$, which describe coherence between the mass eigenstates with on-shell energies $\pm\omega_i$ and $\pm\omega_j$ and helicity h. For Eq. (10) the corresponding functions are $f^{c<}_{ijh\pm}$, and they describe the coherence between the mass eigenstates with on-shell energies $\pm\omega_i$ and $\mp\omega_j$ and helicity h. No coherence between helicities h and $-h$ appears in this approximation.

Using the Feynman-Stückelberg interpretation we identify negative energy particles as antiparticles, and relate the elements of the distribution functions f on the flavor diagonal to the particle phase space densities by

$$n_{ih\vec{k}} = \frac{m_i}{\omega_i} f^{m<}_{iih+}, \qquad \bar{n}_{ih\vec{k}} = 1 + \frac{m_i}{\omega_i} f^{m<}_{iih-}. \tag{11}$$

The distribution functions for $iS^{>}$ are $f^{m>}_{ijh\pm} = \delta_{ij} - f^{m<}_{ijh\pm}$ and $f^{c>}_{ijh\pm} = -f^{c<}_{ijh\pm}$. One may also find the hermiticity relations $f^{m<}_{jih\pm} = (f^{m<}_{ijh\pm})^*$ and $f^{c<}_{jih\pm} = (f^{c<}_{ijh\mp})^*$.

The Feynman rules, especially Eq. (12), given below in Section 2.3 show how the shell structure appears in the Wightman functions in more detail.

2.3 Feynman rules

The Feynman rules of cQPA for calculating corrections to the fermion self energies $i\Sigma^{<,>}$ given in [3] are as follows (the Feynman rules relevant for the calculations done here are presented in Figs. 1 and 2):

1. Draw all perturbative two-particle irreducible diagrams and associate the usual symmetry factor and sign with them.

2. Associate with each vertex the normal vertex factor (not including a four-momentum conservation delta function). The vertex rules relevant here are listed in Fig. 2.

3. Associate a delta function $(2\pi)^4\delta^4(p_{\rm in} - p_{\rm out})$ with all vertices except the one next to the outgoing external fermion line.[5]

4. For all pole type propagators such as the gauge boson propagator in Fig. 1 substitute the propagator in the diagram and integrate over its momentum as usual: $\int \frac{\mathrm{d}^4k}{(2\pi)^4}$.

5. For other propagators such as the generic fermion propagator in Fig. 1 substitute the propagator $iS_f(q,q')$ and integrate over both momenta: $\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4q'}{(2\pi)^4}$.

The vertex rules in Fig. 2 and the gauge boson propagator in Fig. 1 are exactly as they appear in the Standard Model. We will make no specific assumptions about the generic fermion propagator $iS_f(q,q')$ in Fig. 2; this propagator will appear only in the tadpole corrections of Fig. 5.

[5]In the calculations done below it makes no difference to leading order in M_W^{-2} whether we drop the delta function from the end of the incoming or outgoing external fermion line.

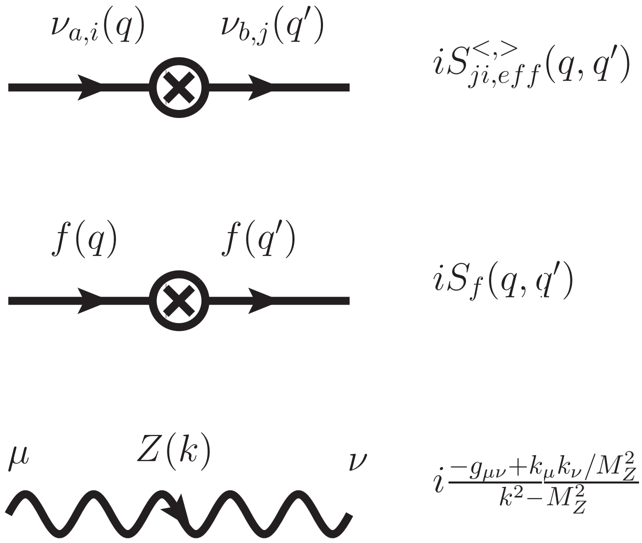


Figure 1: Feynman rules for the effective non-equilibrium neutrino propagator, the generic fermion propagator, and the Z-boson vacuum Feynman propagator in unitary gauge. The W propagator is obtained from the Z propagator by replacing M_Z with M_W. See text for details.

For neutrinos the coefficients $g^f_{V,A}$ both equal $\frac{1}{2}$. The coefficients c_W and $U_{\alpha i}$ in the vertex rules given in Fig. 2 are the cosine of the Weinberg angle and the elements of the leptonic mixing matric (the Pontecorvo–Maki–Nakagawa–Sakata-matrix). No assumptions need to be made of the form of the PMNS-matrix or the number of lepton generations.

The effective neutrino propagator (Wightman function) $iS^{<,>}_{ji,eff}$ in Fig. 1 is [3]

$$iS^{<,>}_{ji,eff} = \mathcal{A}_{jj}(q)F^{<,>}_{ji}(q,q')\mathcal{A}_{ii}(q'), \tag{12}$$

where the spectral function $\mathcal{A}$ is

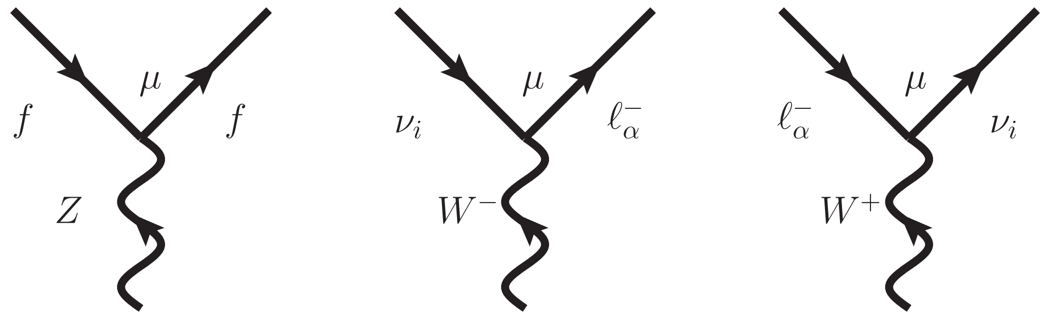
$$\mathcal{A}_{ij}(k) = \pi\,\mathrm{sgn}(k_0)(\not{k} + m_i)\delta(k^2 - m_i^2)\delta_{ij} \tag{13}$$

and the the effective two-point vertex F is defined as

$$F^{<,>}_{ij}(q,q') = 4(2\pi)^3\delta^3(\vec{q} - \vec{q}')\sum_{h,\pm} P_h(\hat{q})\theta^q_\pm(\theta^{q'}_\pm f^{m<}_{ijh\pm}(\vec{q}) + \theta^{q'}_\mp f^{c<}_{ijh\pm}(\vec{q})). \tag{14}$$

Here

$$P_h(\hat{q}) = \frac{1}{2}(1 + h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5) \tag{15}$$



$\frac{-ig}{2c_W}\gamma^\mu(g_V^f - g_A^f\gamma^5)$ $\qquad$ $\frac{-ig}{2\sqrt{2}}U_{\alpha i}\gamma^\mu(1-\gamma^5)$ $\qquad$ $\frac{-ig}{2\sqrt{2}}U^*_{\alpha i}\gamma^\mu(1-\gamma^5)$

Figure 2: Feynman rules for weak interaction vertices.

with $\hat{q} = \vec{q}/\,|\vec{q}|$ is the usual helicity projector and $\theta^q_\pm = \theta(\pm q^0)$.

When calculating corrections to the hermitian (dispersive) self energy Σ^h, we include an additional factor $-i$ to every graph and use the two-point function $F^<_{ji}(q,q')$, from which the vacuum contribution has been removed.

The distribution functions $f^{m,c}$ may depend on time, but this dependence is suppressed here for the sake of simplicity.

3 Neutrino self energies

The dispersive self energy Σ^h of a particle describes the effect of interactions between the particle and its environment to its mass and potential energy. Thus, to describe particle propagation in matter in higher detail, the self energy of the particle must be evaluated with sufficient accuracy. As seen in Eq. (6), the self energy enters the equation of motion for the propagator, which by Eq. (11) describes, among others, neutrino density.

To study neutrino propagation in matter, we must start with evaluating neutrino self energy. In the mass basis the neutrino Hamiltonian is diagonal in flavor indices[6] in the absence of interactions. Such a Hamiltonian gives rise to neutrino oscillations in vacuum. The self energy Σ^h gives rise to a correction to the Hamiltonian, describing the effects of finite temperature and medium.

[6] Flavor states of charged leptons and corresponding neutrinos will be denoted by Greek indices (α, β, ...) and neutrino mass states by Latin indices (i, j, ...). For brevity, both of these indices will be referred to as flavor indices, although the term ‘mass state index’ would be more accurate for the Latin indices.

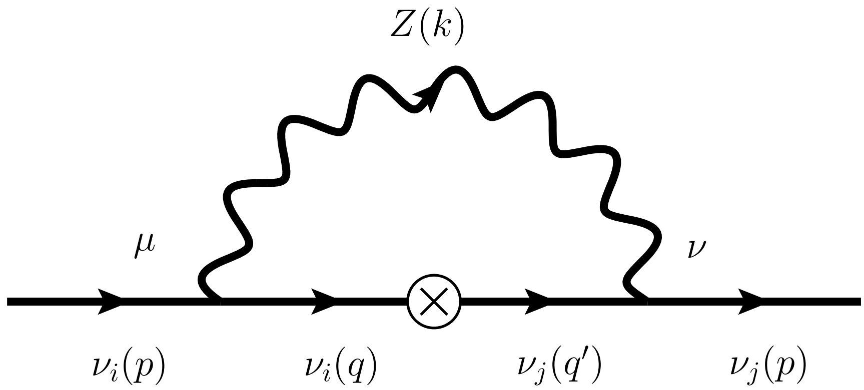


Figure 3: The Z-loop diagram contributing to neutrino self energies.

When neutrinos traverse dense matter (e.g. the Sun or the Earth), interactions with the surroundings must be taken into account. Since neutrinos only experience weak interactions, it suffices to consider one-loop corrections to the self energy to obtain reasonably accurate results. In what follows we do this in the context of cQPA as described in Section 2.

3.1 One loop diagrams

There are three one loop diagrams contributing to neutrino self energies: a Z-loop diagram (see Fig. 3), a W-loop diagram (see Fig. 4), and a tadpole diagram (see Fig. 5). All of these have an incoming neutrino of momentum p and flavor i, and outgoing flavor j. The flavors and momenta of internal lines are marked on the Feynman diagrams.

We will only calculate the corrections to the thermal part of the self energy. This is sufficient when calculationg medium- and temperature-induced corrections to neutrino propagation in zero-temperature vacuum. Moreover, we assume the temperature to be so low that the thermal distribution of gauge bosons $W^{\pm}$ and Z essentially vanish, whence we only use the vacuum propagator of Fig. 1 for these particles.

The corresponding corrections to hermitian neutrino self energies $\Sigma^h = \Sigma^h_{ji}(p)$ are denoted by $\Delta_Z\Sigma^h$, $\Delta_W\Sigma^h$, and $\Delta_{\rm tad}\Sigma^h$. The flavor indices i and j and momentum p always refer to the external lines of the self energy diagrams.

According to the Feynman rules given in Section 2.3, the Z-loop correction

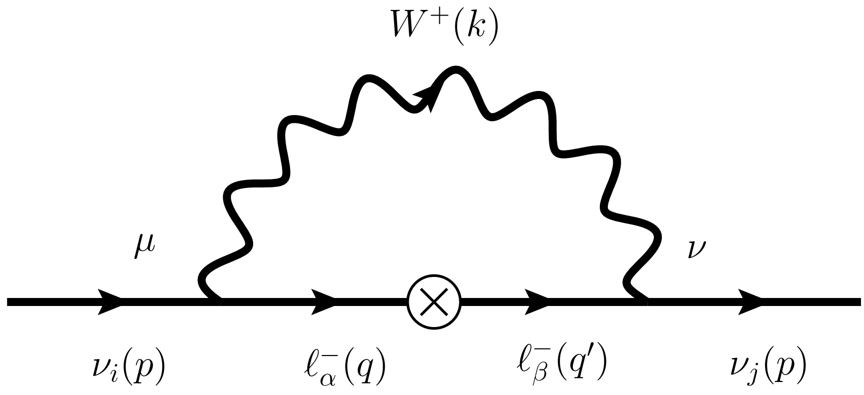


Figure 4: The W-loop diagram contributing to neutrino self energies.

of Fig. 3 can be written as

$$\begin{aligned}\Delta_Z \Sigma^h = -i \int & \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 q'}{(2\pi)^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-ig}{4c_W} \gamma^\nu (1-\gamma^5) \\ & \times \mathcal{A}_{jj}(q') F^<_{ji}(q',q) \mathcal{A}_{ii}(q) \frac{-ig}{4c_W} \gamma^\mu (1-\gamma^5) \\ & \times i \frac{-g_{\mu\nu} + k_\mu k_\nu / M_Z^2}{k^2 - M_Z^2} (2\pi)^4 \delta^4(k+q-p). \end{aligned} \tag{16}$$

The W-loop correction of Fig. 3 is similar, but we have to sum over the charged lepton flavor indices α and β. We obtain

$$\begin{aligned}\Delta_W \Sigma^h = -i \sum_{\alpha,\beta} \int & \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 q'}{(2\pi)^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-ig}{2\sqrt{2}} U^*_{\beta j} \gamma^\nu (1-\gamma^5) \\ & \times \mathcal{A}_{\beta\beta}(q') F^<_{\beta\alpha}(q',q) \mathcal{A}_{\alpha\alpha}(q) \frac{-ig}{2\sqrt{2}} U_{\alpha i} \gamma^\mu (1-\gamma^5) \\ & \times i \frac{-g_{\mu\nu} + k_\mu k_\nu / M_W^2}{k^2 - M_W^2} (2\pi)^4 \delta^4(k+q-p). \end{aligned} \tag{17}$$

The tadpole correction of Fig. 5 is different. We have to sum over all possible fermions f in the loop[7]. We denote the fermion propagator in the

[7] Here the fermion label f contains all quantum numbers of the fermion, including possible flavor indices. We do not sum over antifermions separately to quarantee a symmetry factor 1.

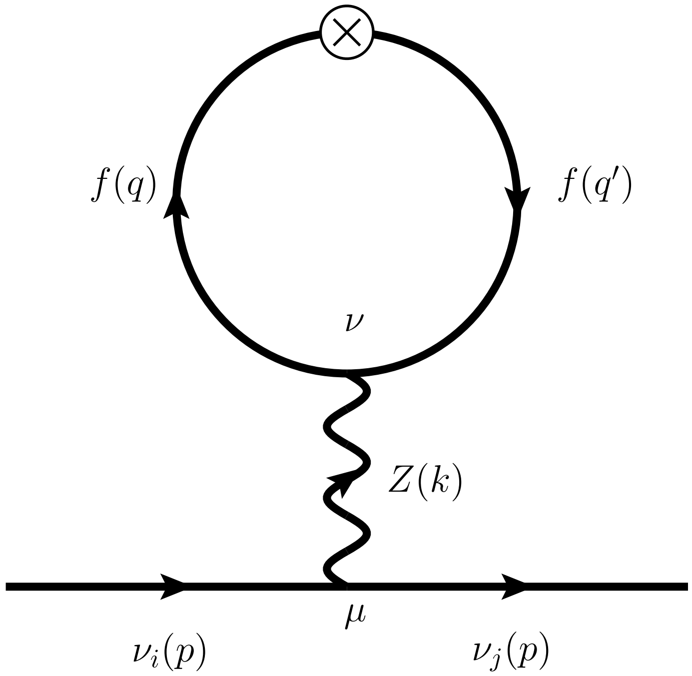


Figure 5: The tadpole diagram contributing to neutrino self energies.

loop by $iS_f(q,q')$. We thus write the tadpole correction as

$$\begin{aligned}\Delta_{\rm tad}\Sigma^h = &-i\sum_f\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4q'}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-ig}{4c_W}\delta_{ij}\gamma^\mu(1-\gamma^5)\\ &\times i\frac{-g_{\mu\nu}+k_\mu k_\nu/M_Z^2}{k^2-M_Z^2}(-1)\,\mathrm{Tr}(iS_f(q,q')\frac{-ig}{2c_W}\gamma^\nu(g_V^f-g_A^f\gamma^5))\\ &\times(2\pi)^4\delta^4(q-q'-k).\end{aligned} \tag{18}$$

We note that since we do not associate a delta function with the lower vertex, the Z-boson may carry a non-zero four-momentum k if the propagator S_f does not conserve four-momentum.

After carrying out the k-integrals using the delta functions the corrections are

$$\begin{aligned}\Delta_Z\Sigma^h = &\frac{-g^2}{16c_W^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4q'}{(2\pi)^4}\gamma^\nu(1-\gamma^5)\\ &\times\mathcal{A}_{jj}(q')F^<_{ji}(q',q)\mathcal{A}_{ii}(q)\gamma^\mu(1-\gamma^5)\\ &\times\frac{-g_{\mu\nu}+(p-q)_\mu(p-q)_\nu/M_Z^2}{(p-q)^2-M_Z^2},\end{aligned} \tag{19}$$

$$
\begin{aligned}
\Delta_W \Sigma^h = \frac{-g^2}{8} \sum_{\alpha,\beta} U_{\alpha i} U^*_{\beta j} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 q'}{(2\pi)^4} \gamma^\nu (1-\gamma^5) \\
\times \mathcal{A}_{\beta\beta}(q') F^<_{\beta\alpha}(q', q) \mathcal{A}_{\alpha\alpha}(q) \gamma^\mu (1-\gamma^5) \\
\times \frac{-g_{\mu\nu} + (p-q)_\mu (p-q)_\nu / M_W^2}{(p-q)^2 - M_W^2},
\end{aligned} \tag{20}
$$

and, after carrying out the q'-integration in the tadpole correction,

$$
\begin{aligned}
\Delta_{\mathrm{tad}} \Sigma^h = \frac{g^2}{8c_W^2} \delta_{ij} \gamma^\mu (1-\gamma^5) \sum_f \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \\
\times \frac{-g_{\mu\nu} + k_\mu k_\nu / M_Z^2}{k^2 - M_Z^2} \\
\times \mathrm{Tr}(i S_f(q, q-k) \gamma^\nu (g_V^f - g_A^f \gamma^5)).
\end{aligned} \tag{21}
$$

From the last expression we see that $\Delta_{\mathrm{tad}} \Sigma^h$ is diagonal in flavor indices and does not depend on i, j and p in any other way.

We do not assume any particular form for for the fermion propagator S_f and cannot hence carry out the summation over all fermions in Eq. (21).

3.2 Inserting fermion propagators

We are now ready to use the definitions of the spectral function $\mathcal{A}$ and the two-point vertex F in Eqs. (13) and (14). We observe that

$$
\int \mathrm{d}q_0 \ \mathrm{sgn}(q_0) \theta^q_\pm \delta(q^2 - m_i^2) G(k_0) = \frac{G(\pm\omega_i)}{2\omega_i} \tag{22}
$$

for any function $G(k_0)$, where $\omega_i = \omega_i(\vec{q}) = \sqrt{\vec{q}^2 + m_i^2}$, and so for any functions $G_1(q_0, q_0', \vec{q}, \vec{q}')$ and $G_2(q_0, q_0', \vec{q}, \vec{q}')$ we have

$$
\begin{aligned}
&\int \mathrm{d}^3 q' \, \mathrm{d}q_0' \, \mathrm{d}q_0 \ \mathrm{sgn}(q_0) \delta(q^2 - m_i^2) \, \mathrm{sgn}(q_0') \delta(q'^2 - m_j^2) \delta^3(\vec{q} - \vec{q}') \\
&\quad \times \theta^{q'}_\pm (\theta^q_\pm G_1(q_0, q_0', \vec{q}, \vec{q}') + \theta^q_\mp G_2(q_0, q_0', \vec{q}, \vec{q}')) \\
&= \frac{1}{2\omega_i 2\omega_j} (G_1(\pm\omega_i, \pm\omega_j, \vec{q}, \vec{q}) + G_2(\mp\omega_i, \pm\omega_j, \vec{q}, \vec{q})).
\end{aligned} \tag{23}
$$

Using these results we can carry out all but the $\vec{q}$-integration in Eq. (19). To simplify the expressions, we define the on-shell four-vector $q^\mu_{i\pm} = (\pm\omega_i, \vec{q})$, for which clearly $q_{i\pm}^2 = m_i^2$. We obtain

$$\begin{aligned}
\Delta_Z \Sigma^h = \frac{-g^2}{32c_W^2} \sum_{h,\pm} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2\omega_i 2\omega_j} \\
\times \gamma^\nu (1-\gamma^5)(\not{q}_{j\pm} + m_j)(1 + h\gamma^0 \hat{q}\cdot\vec{\gamma}\gamma^5) \\
\times \left[f^{m<}_{jih\pm}(\vec{q})(\not{q}_{i\pm} + m_i) + f^{c<}_{jih\pm}(\vec{q})(\not{q}_{i\mp} + m_i) \right] \\
\times \gamma^\mu (1-\gamma^5) \frac{-g_{\mu\nu} + (p-q_{i\pm})_\mu (p-q_{i\pm})_\nu / M_Z^2}{(p-q_{i\pm})^2 - M_Z^2}.
\end{aligned} \tag{24}$$

Similarly, Eq. (20) can be simplified to

$$\begin{aligned}
\Delta_W \Sigma^h = \frac{-g^2}{16} \sum_{\alpha,\beta,h,\pm} U_{\alpha i} U^*_{\beta j} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2\omega_\alpha 2\omega_\beta} \\
\times \gamma^\nu (1-\gamma^5)(\not{q}_{\beta\pm} + m_\beta)(1 + h\gamma^0 \hat{q}\cdot\vec{\gamma}\gamma^5) \\
\times \left[f^{m<}_{h\beta\alpha\pm}(\vec{q})(\not{q}_{\alpha\pm} + m_\alpha) + f^{c<}_{h\beta\alpha\pm}(\vec{q})(\not{q}_{\alpha\mp} + m_\alpha) \right] \\
\times \gamma^\mu (1-\gamma^5) \frac{-g_{\mu\nu} + (p-q_{\alpha\pm})_\mu (p-q_{\alpha\pm})_\nu / M_W^2}{(p-q_{\alpha\pm})^2 - M_W^2}.
\end{aligned} \tag{25}$$

3.3 Dirac algebra

We then turn to simplify the Dirac structure of the corrections. Gamma matrices in Eqs. (24) and (25) appear only in the pattern

$$\gamma^\nu (1-\gamma^5)(\not{q}_{j\pm} + m_j)(1 + h\gamma^0 \hat{q}\cdot\vec{\gamma}\gamma^5)(\not{q}_{i\pm'} + m_i)(1+\gamma^5)\gamma^\mu, \tag{26}$$

where the signs $\pm$ and $\pm'$ are independent.

The gamma matrices γ^1, γ^2, and γ^3 appear here only in the combination $\gamma_q = \vec{q}\cdot\vec{\gamma}$, whence we end up with an 8-dimensional subalgebra spanned by 1, γ^0, γ_q, γ^5, and their products. We denote the norm of the vector $\vec{q}$ by Q.

In this subalgebra it is easy to simplify Eq. (26):

$$\begin{aligned}
&(1-\gamma^5)(\not{q}_{j\pm} + m_j)(1 + h\gamma^0 \hat{q}\cdot\vec{\gamma}\gamma^5)(\not{q}_{i\pm'} + m_i)(1+\gamma^5) \\
&= (1-\gamma^5)(\pm\omega_j\gamma^0 - \gamma_q + m_j)(1 + \frac{h}{Q}\gamma^0\gamma_q\gamma^5) \\
&\quad \times (\pm'\omega_i\gamma^0 - \gamma_q + m_i)(1+\gamma^5) \\
&= 2(-hQ(m_i+m_j) \pm m_i\omega_j \pm' m_j\omega_i)(\gamma^0 + \frac{h}{Q}\gamma_q)(1+\gamma^5).
\end{aligned} \tag{27}$$

The essential Dirac structure has thus simplified to $\gamma^\nu(\gamma^0 + \frac{h}{Q}\gamma_q)(1+\gamma^5)\gamma^\mu$. Also note how the dependece on the sign $\pm'$ factors to the scalar coeffecient. This will make the Dirac structure of $\Delta_Z\Sigma^h$ and $\Delta_W\Sigma^h$ easier to handle.

The Lorentz indices of the gamma matrices γ^ν and γ^μ are contracted in two ways: by $g_{\mu\nu}$ and $k_\mu k_\nu$, where $k = p - q_{i\pm}$. In analogue to γ_q and Q, we define $\gamma_p = \vec{p}\cdot\vec{\gamma}$ and $P = |\vec{p}|$. Using

$$\gamma^\mu\gamma^\lambda(1+\gamma^5)\gamma_\mu = -2\gamma^\lambda(1-\gamma^5) \tag{28}$$

and

$$\not{k}\gamma^\lambda(1+\gamma^5)\not{k} = (2a^\lambda\not{k} - k^2\gamma^\lambda)(1-\gamma^5) \tag{29}$$

we obtain

$$\begin{aligned}
&\gamma^\nu(\gamma^0 + \frac{h}{Q}\gamma_q)(1+\gamma^5)\gamma^\mu(-g_{\mu\nu} + k_\mu k_\nu/M_Z^2) \\
&= \left((2 - \frac{k^2}{M_Z^2})(\gamma^0 + \frac{h}{Q}\gamma_q) + \frac{2}{M_Z^2}(k^0 - \frac{h}{Q}\vec{k}\cdot\vec{q})\not{k}\right)(1-\gamma^5) \\
&= (B^1_{p,\vec{q},i,Z,\pm}\gamma^0 + B^2_{p,\vec{q},i,Z,\pm}\gamma_q + B^3_{p,\vec{q},i,Z,\pm}\gamma_p) \\
&\quad\times(1-\gamma^5),
\end{aligned} \tag{30}$$

where the coefficients $B^{1,2,3}$ are

$$\begin{aligned}
&B^1_{p,\vec{q},i,Z,\pm} = \\
&2 + M_Z^{-2}[(p^0 \mp \omega_i)^2 + P^2 + Q^2 - 2\vec{p}\cdot\vec{q} - 2\frac{h}{Q}(p^0 \mp \omega_i)(\vec{p}\cdot\vec{q} - Q^2)],
\end{aligned} \tag{31}$$

$$\begin{aligned}
&hQ\cdot B^2_{p,\vec{q},i,Z,\pm} = \\
&2 + M_Z^{-2}[-(p^0 \mp \omega_i)^2 - P^2 - Q^2 + 2\vec{p}\cdot\vec{q} + hQ(\pm 2p^0 - 2\omega_i) \mp (\vec{p}\cdot\vec{q} - Q^2)],
\end{aligned} \tag{32}$$

and

$$B^3_{p,\vec{q},i,Z,\pm} = -2M_Z^{-2}\left((p^0 \mp \omega_i) - \frac{h}{Q}(\vec{p}\cdot\vec{q} - Q^2)\right). \tag{33}$$

We can now use the above results to simplify Eqs. (24) and (25). The above calculations were done for the Z-loop correction, but when we change $i, j, Z \to \alpha, \beta, W$, we get the corresponding W-loop term.

Inserting these results to Eqs. (24) and (25) yields

$$\begin{aligned}
\Delta_Z\Sigma^h = \frac{-g^2}{16c_W^2}\sum_{h,\pm}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\left((p - q_{i\pm})^2 - M_Z^2\right)^{-1} g^<_{jih\pm}(\vec{q}) \\
\times(B^1_{p,\vec{q},i,Z,\pm}\gamma^0 + B^2_{p,\vec{q},i,Z,\pm}\gamma_q + B^3_{p,\vec{q},i,Z,\pm}\gamma_p)(1-\gamma^5)
\end{aligned} \tag{34}$$

and

$$\Delta_W \Sigma^h = \frac{-g^2}{8} \sum_{\alpha,\beta,h,\pm} U_{\alpha i} U^*_{\beta j} \times \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \left((p - q_{\alpha\pm})^2 - M_W^2\right)^{-1} g^<_{\beta\alpha h\pm}(\vec{q}) \times (B^1_{p,\vec{q},\alpha,W,\pm}\gamma^0 + B^2_{p,\vec{q},\alpha,W,\pm}\gamma_q + B^3_{p,\vec{q},\alpha,W,\pm}\gamma_p)(1 - \gamma^5), \tag{35}$$

where

$$g^<_{jih\pm}(\vec{q}) = \frac{1}{2\omega_i 2\omega_j}[(-hQ(m_i + m_j) \pm m_i\omega_j \pm m_j\omega_i) f^{m<}_{jih\pm}(\vec{q}) + (-hQ(m_i + m_j) \pm m_i\omega_j \mp m_j\omega_i) f^{c<}_{jih\pm}(\vec{q})]. \tag{36}$$

With no assumptions about the fermion propagator S_f, the dirac structure of Eq. (21) cannot be simplified beyond the trivial observation that it must be of the form $\not{a}(1-\gamma^5)$ for some four-vector a. The tadpole corrections will be dealt with in Sections 3.6 and 3.8.

3.4 Symmetries in the three-momentum integral

Let us assume that the distribution functions $f^{m<}_{jih\pm}(\vec{q})$ and $f^{c<}_{jih\pm}(\vec{q})$ are spherically symmetric and depend only on Q. Then the gauge boson propagator in Eqs. (34) and (35) induces an asymmetry in the directions of $\vec{p}$, but there is a useful symmetry in the $\vec{q}$-integrals in directions perpendicular to $\vec{p}$.

Let us write $\vec{q} = \vec{q}_\parallel + \vec{q}_\perp$, so that $\vec{q}_\parallel$ and $\vec{q}_\perp$ are parallel and perpendicular to $\vec{p}$, respectively.[8] The only dependence on $\vec{q}_\perp$ in the integrand of Eqs. (34) and (35) is in the gamma matrix γ_q. The $\vec{q}_\perp$-integral of the term $\vec{q}_\perp \cdot \vec{\gamma}$ is antisymmetric and thus vanishes, whence we may replace γ_q by $\vec{q}_\perp \cdot \vec{\gamma} = \vec{p} \cdot \vec{q} P^{-2} \gamma_p$.

We can easily manifest this by defining

$$\tilde{B}^3_{p,\vec{q},i,Z,\pm} = B^3_{p,\vec{q},i,Z,\pm} + \frac{\vec{p}\cdot\vec{q}}{P^2} B^2_{p,\vec{q},i,Z,\pm} \tag{37}$$

and replacing

$$B^1_{p,\vec{q},i,Z,\pm}\gamma^0 + B^2_{p,\vec{q},i,Z,\pm}\gamma_q + B^3_{p,\vec{q},i,Z,\pm}\gamma_p \tag{38}$$

in Eq. (34) by

$$B^1_{p,\vec{q},i,Z,\pm}\gamma^0 + \tilde{B}^3_{p,\vec{q},i,Z,\pm}\gamma_p, \tag{39}$$

and similarly in Eq. (35) by replacing i and Z by α and W.

[8] We can explicitly define $\vec{q}_\parallel = \vec{p} \cdot \vec{q} P^{-2} \vec{p}$ and $\vec{q}_\perp = \vec{q} - \vec{q}_\parallel$.

Above we implicitly assumed that $\vec{p} \neq 0$. If $\vec{p} = 0$, we may drop both γ_q and γ_p from Eqs. (34) and (35).

These results were expected on physical grounds: if the distributions f are isotropic, the only direction that can appear in $\Delta_W \Sigma^h$ and $\Delta_Z \Sigma^h$ is that of the external three-momentum $\vec{p}$.

3.5 Leading order approximation in gauge boson mass

If the gauge boson masses M_Z and M_W are much larger than temperature or any momentum scale in the system, we can expand the corrections $\Delta_Z \Sigma^h$ and $\Delta_W \Sigma^h$ in powers of $M_{W,Z}^{-2}$. For $k^2 < M^2$, we can expand the denominator of the gauge boson propagator as

$$\frac{1}{k^2 - M^2} = -\frac{1}{M^2} \sum_{n=0}^{\infty} \left(\frac{k^2}{M^2} \right)^n . \tag{40}$$

To leading order in M_Z^{-2} we can replace the denominator $k^2 - M_Z^2$ by $-M_Z^2$ and drop all terms containing M_Z^{-2} in the coefficients $B^{1,2,3}$ defined in Eqs. (31), (32), and (33) to get

$$B^1_{p,\vec{q},i,Z,\pm} \gamma^0 + \tilde{B}^3_{p,\vec{q},i,Z,\pm} \gamma_p = 2(\gamma^0 + \frac{h \vec{p} \cdot \vec{q}}{P^2 Q} \gamma_p). \tag{41}$$

Assuming isotropicity in the distribution functions $f^{m<}_{jih\pm}(\vec{q})$ and $f^{c<}_{jih\pm}(\vec{q})$ as in Section 3.4, the integral of the γ_p-term vanishes due to antisymmetry caused by $\vec{p} \cdot \vec{q}$, whence we arrive at

$$\Delta_Z^{LO} \Sigma^h_{ji} = \frac{g^2}{8 c_W^2 M_Z^2} \sum_{h,\pm} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} g^<_{jih\pm}(Q) \gamma^0 (1 - \gamma^5) \tag{42}$$

and

$$\Delta_W^{LO} \Sigma^h_{ji} = \frac{g^2}{4 M_W^2} \sum_{\alpha,\beta,h,\pm} U_{\alpha i} U^*_{\beta j} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} g^<_{\beta\alpha h\pm}(Q) \gamma^0 (1 - \gamma^5), \tag{43}$$

where the flavor indices have been reintroduced in the propagator.

Using the familiar result $M_W = c_W M_Z$, we observe a relation between $\Delta_Z \Sigma^h$ and $\Delta_W \Sigma^h$ to leading order in M_W^{-2}:

$$\Delta_W^{LO} \Sigma^h_{ji} = 2 \sum_{\alpha,\beta} U_{\alpha i} U^*_{\beta j} \Delta_Z^{LO} \Sigma^h_{\beta\alpha}. \tag{44}$$

This equation is to be understood in the way that the functions are of the same form, not that they are exaclty the same functions.

3.6 The tadpole correction

We can use the leading order approximation considered in Section 3.5 to simplify the tadpole correction in Eq. (21) to

$$\begin{aligned} \Delta^{LO}_{\rm tad}\Sigma^h = \frac{g^2}{8M_W^2}&\delta_{ij}\gamma^\mu(1-\gamma^5) \\ &\times \sum_f \mathrm{Tr}(I_f\gamma_\mu(g_V^f - g_A^f\gamma^5)), \end{aligned} \tag{45}$$

where the momentum integrals have been collected to the coefficient

$$I_f = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{d}^4 q'}{(2\pi)^4} iS_f(q,q'). \tag{46}$$

Assuming isotropicity and in the propagator we conclude that

$$I_f = \frac{1}{4}(Q_f^S + Q_f^V\gamma^0 + Q_f^P\gamma^5 + Q_f^A\gamma^0\gamma^5) \tag{47}$$

for some coefficients $Q_f^{S,V,P,A}$. But then we have

$$\mathrm{Tr}(I_f\gamma_\mu(g_V^f - g_A^f\gamma^5)) = (Q_f^V g_V^f + Q_f^A g_A^f)\delta^0{}_\mu, \tag{48}$$

and the tadpole correction reduces to

$$\Delta^{LO}_{\rm tad}\Sigma^h = \frac{g^2}{8M_W^2}\left(\sum_f Q_f^V g_V^f + Q_f^A g_A^f\right)\delta_{ij}\gamma^0(1-\gamma^5). \tag{49}$$

3.7 Summary of results to leading order in M_W^{-2}

Collecting Eqs. (42), (43), (44), and (49), we can express the one-loop correction to neutrino self energy to leading order in M_W^{-2} as

$$\begin{aligned} &\Delta^{LO}_{tot}\Sigma^h_{ji} = \Delta^{LO}_Z\Sigma^h_{ji} + \Delta^{LO}_W\Sigma^h_{ji} + \Delta^{LO}_{\rm tad}\Sigma^h_{ji} \\ &= \frac{g^2}{8M_W^2}\left(I^{LO}_{ji} + 2\sum_{\alpha,\beta} U_{\alpha i}U^*_{\beta j} I^{LO}_{\beta\alpha} + \delta_{ij}\sum_f (Q_f^V g_V^f + Q_f^A g_A^f)\right)\gamma^0(1-\gamma^5), \end{aligned} \tag{50}$$

where I^{LO}_{ji} abbreviates the integral

$$I^{LO}_{ji} = \sum_{h,\pm}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} g^<_{jih\pm}(Q). \tag{51}$$

This integral is isotropic, so we may also write it as

$$I_{ji}^{LO} = \sum_{h,\pm} \int_0^\infty \frac{Q^2 \mathrm{d}Q}{2\pi^2} g_{jih\pm}^{<}(Q). \tag{52}$$

The generalization to the case when $g_{jih\pm}^{h}(\vec{q})$ is not isotropic is given in Appendix A.

We note that the Dirac structure of the correction in Eq. (50) is simply $\gamma^0(1-\gamma^5)$. On grounds of the discussion in Section 3.4 and Eqs. (34) and (35) we also expect a $\gamma_p(1-\gamma^5)$-term to appear. It will do so only in subleading orders in M_W^{-2}. If we do not assume isotropicity in the functions f, terms of the form $\vec{a}\cdot\vec{\gamma}(1-\gamma^5)$ for some vector $\vec{a}$ may appear already in leading order.

3.8 Subleading terms in M_W^{-2}

We will now go to the second order in the expansion described in Section 3.5. Keeping only terms proportional to M_Z^{-4} and assuming the isotropicity of the functions f discussed in Section 3.4, Eq. (34) gives

$$\begin{aligned}
\Delta_Z^{NLO}\Sigma^h = {} & \frac{g^2}{16 c_W^2 M_Z^4} \sum_{h,\pm} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} g_{jih\pm}^{<}(Q) \\
& \times \Bigg\{ 2((p^0 \mp \omega_i)^2 - P^2 - Q^2 + 2\vec{p}\cdot\vec{q})(\gamma^0 + \frac{h\vec{p}\cdot\vec{q}}{P^2 Q}\gamma_p) \\
& \quad + [(p^0 \mp \omega_i)^2 + P^2 + Q^2 - 2\vec{p}\cdot\vec{q} \\
& \qquad - 2\frac{h}{Q}(p^0 \mp \omega_i)(\vec{p}\cdot\vec{q} - Q^2)]\gamma^0 \\
& \quad + \Bigg[\frac{h\vec{p}\cdot\vec{q}}{P^2 Q}\big(- (p^0 \mp \omega_i)^2 - P^2 - Q^2 + 2\vec{p}\cdot\vec{q} \\
& \qquad + hQ(\pm 2p^0 - 2\omega_i) \mp (\vec{p}\cdot\vec{q} - Q^2)\big) \\
& \qquad - 2\left((p^0 \mp \omega_i) - \frac{h}{Q}(\vec{p}\cdot\vec{q} - Q^2) \right) \Bigg] \gamma_p \Bigg\} \\
& \times (1 - \gamma^5).
\end{aligned} \tag{53}$$

Dropping the parts antisymmetric in $\vec{q}$ we are left with

$$\begin{aligned}\Delta_Z^{NLO}\Sigma^h = & \frac{g^2}{16c_W^2 M_Z^4}\sum_{h,\pm}\int\frac{\mathrm{d}^3q}{(2\pi)^3}g^<_{jih\pm}(Q) \\ & \times\Bigg\{[3(p^0\mp\omega_i)^2 - P^2 - Q^2 + 2hQ(p^0\mp\omega_i)]\gamma^0 \\ & \quad + \left[(6\mp 1)\frac{h(\vec{p}\cdot\vec{q})^2}{P^2Q} - 2(p^0\mp\omega_i + hQ)\right]\gamma_p\Bigg\} \\ & \times(1-\gamma^5).\end{aligned} \tag{54}$$

We see that a nonzero $\gamma_p(1-\gamma^5)$-term arises in $\Delta_Z^{NLO}\Sigma^h$ in next-to-leading order in M_Z^{-2}. The same happens for the W-loop correction, for which

$$\begin{aligned}\Delta_W^{NLO}\Sigma^h = & \frac{g^2}{8M_W^4}\sum_{\alpha,\beta,h,\pm}U_{\alpha i}U^*_{\beta j}\sum_{h,\pm}\int\frac{\mathrm{d}^3q}{(2\pi)^3}g^<_{\beta\alpha h\pm}(Q) \\ & \times\Bigg\{[3(p^0\mp\omega_\alpha)^2 - P^2 - Q^2 + 2hQ(p^0\mp\omega_\alpha)]\gamma^0 \\ & \quad + \left[(6\mp 1)\frac{h(\vec{p}\cdot\vec{q})^2}{P^2Q} - 2(p^0\mp\omega_\alpha + hQ)\right]\gamma_p\Bigg\} \\ & \times(1-\gamma^5).\end{aligned} \tag{55}$$

If we wish to express these corrections in terms of γ^0 and $\not{p}$ rather than γ^0 and γ_p, we may simply write $\gamma_p = p^0\gamma^0 - \not{p}$.

In analogue to Eq. (44) we find (with the same caveat)

$$\Delta_W^{NLO}\Sigma^h_{ji} = 2c_W^{-2}\sum_{\alpha,\beta}U_{\alpha i}U^*_{\beta j}\Delta_Z^{NLO}\Sigma^h_{\beta\alpha}. \tag{56}$$

The next-to-leading order term of the tadpole correction in Eq. (21) is

$$\begin{aligned}\Delta_{\mathrm{tad}}^{NLO}\Sigma^h = & \frac{g^2}{8c_W^2M_Z^4}\delta_{ij}\gamma^\mu(1-\gamma^5) \\ & \times\sum_f \mathrm{Tr}(I^{NLO}_{f,\mu\nu}\gamma^\nu(g_V^f - g_A^f\gamma^5)),\end{aligned} \tag{57}$$

where the momentum integrals have been collected to the coefficient

$$I^{NLO}_{f,\mu\nu} = \int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}(k^2g_{\mu\nu} - k_\mu k_\nu)iS_f(q,q-k). \tag{58}$$

By the same symmetry argument as used in Section 3.6, we have

$$I_{f,\mu\nu}^{NLO} = C_f^{11} g_{\mu\nu} + C_f^{12} g_{\mu\nu}\gamma^0 + C_f^{21}\delta^0{}_\mu\delta^0{}_\nu + C_f^{22}\delta^0{}_\mu\delta^0{}_\nu\gamma^0 \tag{59}$$

for some coefficients $C_f^{11,12,21,22}$, whence

$$\mathrm{Tr}(I_{f,\mu\nu}^{NLO}\gamma^\nu(g_V^f - g_A^f\gamma^5)) = 4(C_f^{12} + C_f^{22})g_V^f\delta^0{}_\mu \tag{60}$$

and

$$\Delta_{\mathrm{tad}}^{NLO}\Sigma^h = \frac{g^2}{2c_W^2 M_Z^4}\left(\sum_f (C_f^{12} + C_f^{22})g_V^f\right)\delta_{ij}\gamma^0(1-\gamma^5). \tag{61}$$

We immediately note that the leading and next-to-leading order corrections in Eqs. (49) and (61) differ only by the factors C_f. This is not unexpected: the tadpole correction in Eq. (21) does not depend on the external four-momentum p, whence isotropy in the fermion propagator $S_f(q,q')$ forbids any terms of the form $\vec{\gamma}(1-\gamma^5)$, which were found above to appear in $\Delta_Z^{NLO}\Sigma^h$ and $\Delta_W^{NLO}\Sigma^h$.

We can easily use this procedure of simplifying tadpole corrections to Eq. (21) directly. We only have to express the integral

$$I_{h,\mu\nu}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-g_{\mu\nu} + k_\mu k_\nu/M_Z^2}{k^2 - M_Z^2} iS_f(q, q-k) \tag{62}$$

as a sum of some gamma matrices. Assuming isotropicity in $S_f(q,q')$, the only possible terms are 1, γ^0, γ^5, and $\gamma^0\gamma^5$. The result can then be simplified using

$$\mathrm{Tr}((C_f^S + C_f^V\gamma^0 + C_f^P\gamma^5 + C_f^A\gamma^0\gamma^5)\gamma_\nu(g_V^f - g_A^f\gamma^5)) = 4(C_f^V g_V^f + C_f^A g_A^f)\delta^0{}_\nu, \tag{63}$$

so that the only difficulty that remains is evaluating the coefficients of the gamma matrices. However, isotropicity also guarantees that $S_f(q,q')$ is proprtional to the delta function $\delta^3(\vec{q} - \vec{q}')$. This implies $\vec{k} = 0$, whence

$$\frac{-g_{\mu\nu} + k_\mu k_\nu/M_Z^2}{k^2 - M_Z^2} = M_Z^{-2}\,\mathrm{diag}\,(1, -a, -a, -a)\,, \tag{64}$$

where $a = (1 - (k^0/M_Z)^2)^{-1}$. If, moreover, $k^0 = 0$, which happens if we use the cQPA effective propagator as iS_f, we find

$$\frac{-g_{\mu\nu} + k_\mu k_\nu/M_Z^2}{k^2 - M_Z^2} = g_{\mu\nu}/M_Z^2. \tag{65}$$

This implies $\Delta_{tad}^{NLO}\Sigma^h = 0$.

3.9 Summary of the subleading corrections

The leading order correction to Σ^h was given in Eq. (50). This correction only contains a term of the form $\gamma^0(1-\gamma^5)$. Subleading terms of the same form are found in $\Delta_Z^{NLO}\Sigma^h$ and $\Delta_W^{NLO}\Sigma^h$ in Eqs. (54) and (55).

Without fixing the propagator $S_f(q,q')$ little can be said about the magnitude of the tadpole corrections. They are, however, diagonal in flavor and contain only terms of the form $\gamma^0(1-\gamma^5)$ if $S_f(q,q')$ is isotropic.

We may assume the distribution functions $f^{m<}_{ijh\pm}(\vec{q})$ and $f^{c<}_{ijh\pm}(\vec{q})$ to decay exponentially at high momenta. Thus, if the temperature is well below M_W, so is Q in the integrand. If also the external momentum scale defined by P and p^0 is small compared to M_W, we may estimate the relative magnitude of leading and next-to-leading order terms in $\Delta_Z\Sigma^h$ and $\Delta_W\Sigma^h$.

We have actually already made these assumptions more or less implicitly. The expansion in gauge boson mass presented in Section 3.5 will only give a reasonable approximation—or even converge—if all relevant momentum (and hence temperature) scales are much smaller than M_W. Moreover, we can no longer neglect the thermal part of the gauge boson propagator when $T \gtrsim M_W$.

If temperature, P, and p^0 are near some scale Λ, a naive comparison between Eqs. (42) and (54) suggests an order of magnitude estimation

$$\left|\Delta_Z^{NLO}\Sigma^h\right| \sim (\Lambda/M_W)^2 \left|\Delta_Z^{LO}\Sigma^h\right|, \tag{66}$$

and similarly for $\Delta_W\Sigma^h$. The next-to-leading order terms are thus expected to be small in comparison with the leading order terms. But they also behave differently, so they are not omitted.

We may now use Eqs. (54), (56), and (61) to write the next-to-leading order self energy correction in the isotropic case as

$$\Delta_{tot}^{NLO}\Sigma^h_{ji} = \frac{g^2}{8c_W^2 M_Z^4}\left(I^{NLO}_{ji} + 2c_W^{-2}\sum_{\alpha,\beta} U_{\alpha i}U^*_{\beta j} I^{NLO}_{\beta\alpha}\right)(1-\gamma^5), \tag{67}$$

where

$$\begin{aligned} I^{NLO}_{ji} = \frac{1}{2}\sum_{h,\pm}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} g^<_{jih\pm}(Q) \\ \times \Bigg\{ [3(p^0 \mp \omega_i)^2 - P^2 - Q^2 + 2hQ(p^0 \mp \omega_i)]\gamma^0 \\ + \left[(6\mp 1)\frac{h(\vec{p}\cdot\vec{q})^2}{P^2 Q} - 2(p^0 \mp \omega_i + hQ)\right]\gamma_p\Bigg\}. \end{aligned} \tag{68}$$

If we denote by ϑ the angle between $\vec{p}$ and $\vec{q}$, we may also write

$$
\begin{aligned}
I_{ji}^{NLO} &= \frac{1}{2}\sum_{h,\pm}\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\int_{0}^{\infty}\frac{Q^2\mathrm{d}Q}{2\pi^2}g^{<}_{jih\pm}(Q) \\
&\quad\times\Bigg\{[3(p^0\mp\omega_i)^2-P^2-Q^2+2hQ(p^0\mp\omega_i)]\gamma^0 \\
&\qquad+\left[(6\mp 1)\frac{h(\vec{p}\cdot\vec{q})^2}{P^2Q}-2(p^0\mp\omega_i+hQ)\right]\gamma_p\Bigg\} \\
&= \frac{1}{2}\sum_{h,\pm}\int_{0}^{\infty}\frac{Q^2\mathrm{d}Q}{(2\pi)^2}g^{<}_{jih\pm}(Q) \\
&\quad\times\Bigg\{[3(p^0\mp\omega_i)^2-P^2-Q^2+2hQ(p^0\mp\omega_i)]\gamma^0 \\
&\qquad+\left[-2p^0\pm 2\omega_i\mp\frac{1}{3}hQ\right]\gamma_p\Bigg\}.
\end{aligned}
\tag{69}
$$

With Eqs. (50) and (67) we can present the full self energy correction containing each term to their leading order:

$$\Delta\Sigma^h = \Delta_{tot}^{LO}\Sigma^h + \Delta_{tot}^{NLO}\Sigma^h. \tag{70}$$

4 Overview of the self energy corrections

A main application of corrections to neutrino self energies is the calculation of effective potential of neutrinos in matter. This leads to a dependence of neutrino mixing on the medium neutrinos are propagating through, and may significantly differ from neutrino oscillations in vacuum.

In the usual approach, the non-diagonality of Σ_{ji}^h arises as follows: the charged leptons in the medium are mainly electrons, whence only electron neutrinos having a charged current interaction. This can be manifested in Eq. (50) by dropping the tadpole part and assuming the functions I_{ji}^{LO} and $I_{\beta\alpha}^{LO}$ of Eq. (52) to be diagonal in the flavor indices. Then the nondiagonality of Σ_{ji}^h is only due to non-diagonal elements $U_{\alpha i}$ of the leptonic mixing matrix. The next-to-leading order term in gauge boson mass given in Eq. (67) is omitted.

If, however, the non-diagonal elements of I_{ji}^{LO} are significant, the terms I_{ji}^{LO} and $2\sum_{\alpha,\beta}U_{\alpha i}U_{\beta j}^{*}I_{\beta\alpha}^{LO}$ in Eq. (52) may be of the same order of magnitude when $i\neq j$, rendering the standard approach rather coarse.

If the elements of I_{ji}^{LO} are significantly large, it will not only alter the description of neutrino oscillations quantitatively, but also qualitatively: I_{ji}^{LO}

describes self coupling of the neutrinos, while $I^{LO}_{\beta\alpha}$ describes coupling to background fields not containing neutrinos. Before any conclusions can be drawn about the significance of such a phenomenon, a numerical analysis of neutrino propagation is required.

The tadpole corrections are diagonal in flavor and equal for all flavors (i.e. $\Delta_{\rm tad}\Sigma^h_{ji} \sim \delta_{ij}$), and hence of little intrest in exploring the flavor dependence of neutrino self energies. Self-coupling of the neutrinos can also be found in the tadpole correction (as also neutrino loops are summed over), but these do not give rise to a complicated flavor structure like the Z-loop correction.

If one considers a neutrino beam propagating in matter, the assumption that the distributions $f^{c,m}$ be symmetric is unphysical. The leading order result for asymmetric distributions is easily obtained from the calculations done in Section 3.

All self enrgy corrections obtained above are presented concisely in Appendix A.

5 Conclusions

We have evaluated neutrino self energies to the leading non-trivial order in weak interaction. This has been done mainly for use in studying neutrino oscillations, and what remains to be done is to actually use the obtained results to see if including nonlocal coherence phenomena the way it is done in cQPA has a significant impact on neutrino propagation.

To make sure that the results provide a good starting point for further studies and that CQPA itself can be considered reliable, it must be checked that the results are physically meaningful. Comparing Eqs. (11) and (36) suggests that the diagonal elements $g^<_{\alpha\alpha h\pm}(Q)$ are proportional to the density of charged leptons in the background, which is expected since these terms in (50) describe neutrino coupling to the leptonic background. Furthermore, after suitable approximations the results reduce to the standard matter effect description in neutrino oscillations.

The corrections obtained for neutrino self energies seem promising. Their application and a more thorough discussion, however, have to be left outside this work.

Besides obtaining the self energy corrections, we have seen that calculating such corrections in cQPA is relatively straightforward and leads to reasonable results. This opens a new possibility to properly include coherence in problems regarding mixed fermions in varying background, and may lead to a better understanding of, for example [9], the electroweak phase transition and neutrino oscillations in the early Universe.

A Collected results

To facilitate an easier overview of the self energy corrections obtained in Section 3, the results of the calculations are collected here.

The correction to neutrino self energies containing each term to their leading order is (see Eq. (70))

$$\Delta\Sigma^h = \Delta_{tot}^{LO}\Sigma^h + \Delta_{tot}^{NLO}\Sigma^h, \tag{71}$$

where (see Eq. (50))

$$\begin{aligned}\Delta_{tot}^{LO}\Sigma_{ji}^h = &\sqrt{2}G_f\gamma^0 P_L \\ &\times \left(I_{ji}^{LO} + 2\sum_{\alpha,\beta} U_{\alpha i}U_{\beta j}^* I_{\beta\alpha}^{LO} + \delta_{ij}\sum_f (Q_f^V g_V^f + Q_f^A g_A^f) \right).\end{aligned} \tag{72}$$

Here we have used the Fermi coupling constant $G_F = \sqrt{2}g^2/(8M_W^2)$. The integral I_{ji}^{LO} is defined as (see Eq. (52))

$$I_{ji}^{LO} = \sum_{h,\pm}\int_0^\infty \frac{Q^2 \mathrm{d}Q}{2\pi^2} g_{jih\pm}^<(Q) \tag{73}$$

with (see Eq. (36))

$$\begin{aligned}g_{jih\pm}^<(Q) = &\frac{1}{2\omega_i 2\omega_j}[(-hQ(m_i+m_j) \pm m_i\omega_j \pm m_j\omega_i) f_{jih\pm}^{m<}(Q) \\ &+ (-hQ(m_i+m_j) \pm m_i\omega_j \mp m_j\omega_i) f_{jih\pm}^{c<}(Q)].\end{aligned} \tag{74}$$

If we assume no symmetry in the distribution functions $f^{m,c}$, we have to change $I_{ji}^{LO}\gamma^0$ in Eq. (50) to (compare with Eq. (41))

$$\sum_{h,\pm}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} g_{jih\pm}^<(\vec{q})(\gamma^0 + \frac{h\vec{p}\cdot\vec{q}}{P^2 Q}\gamma_p), \tag{75}$$

and similarly for $I_{\beta\alpha}^{LO}\gamma^0$. The tadpole correction remains proportional to $\gamma^0(1-\gamma^5)$.

The next-to-leading order term is (see Eq. (67))

$$\Delta_{tot}^{NLO}\Sigma_{ji}^h = \frac{g^2}{8c_W^2 M_Z^4}\left(I_{ji}^{NLO} + 2c_W^{-2}\sum_{\alpha,\beta} U_{\alpha i}U_{\beta j}^* I_{\beta\alpha}^{NLO} \right)(1-\gamma^5), \tag{76}$$

where (see Eq. (68))

$$
\begin{aligned}
I_{ji}^{NLO} = \frac{1}{2}\sum_{h,\pm} \int & \frac{\mathrm{d}^3 q}{(2\pi)^3} g^{<}_{jih\pm}(Q) \\
& \times \Bigg\{ [3(p^0 \mp \omega_i)^2 - P^2 - Q^2 + 2hQ(p^0 \mp \omega_i)]\gamma^0 \\
& \quad + \left[(6 \mp 1) \frac{h(\vec{p}\cdot\vec{q})^2}{P^2 Q} - 2(p^0 \mp \omega_i + hQ) \right] \gamma_p \Bigg\}.
\end{aligned} \tag{77}
$$

The factors $Q_f^{V,A}$ appearing above have been defined so that (see Eq. (46))

$$
\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 q'}{(2\pi)^4} iS_f(q, q') = \frac{1}{4}(Q_f^S + Q_f^V \gamma^0 + Q_f^P \gamma^5 + Q_f^A \gamma^0 \gamma^5). \tag{78}
$$

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