Broken ray tomography: Approaches and applications

PDE/Analysis Mini-school
The University of North Carolina at Chapel Hill

Joonas Ilmavirta

University of Jyväskylä

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Material for the lecture

Slides are available at my homepage:
`http://users.jyu.fi/~jojapeil`

Publications > Conferences and talks > the first talk on the list

If you have any questions, interrupt at any time (during or after the lecture). If you want to know more, ask me after the lecture.

Outline

X-ray transform

- X-ray tomography problem: Can we recover a function from its integrals over all lines?

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- Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain.
- Let Γ be the set of all lines in $\mathbb{R}^n$. (Which can be parametrized, for example, by $\mathbb{R}^n \times S^{n-1}$.)
- For a continuous function $f : \bar{\Omega} \to \mathbb{R}$, the X-ray transform of f is a function $\mathcal{I}f : \Gamma \to \mathbb{R}$ defined by

  $$\mathcal{I}f(\gamma) = \int_\gamma f \mathrm{d}s$$

  (where $\mathrm{d}s$ is the usual length measure along the line).
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  (where $\mathrm{d}s$ is the usual length measure along the line).
- We could also use line segments with endpoints on $\partial\Omega$.
- X-ray tomography problem: Is $\mathcal{I}$ injective?
- Answer: Yes in many cases, for example in bounded Euclidean domains.

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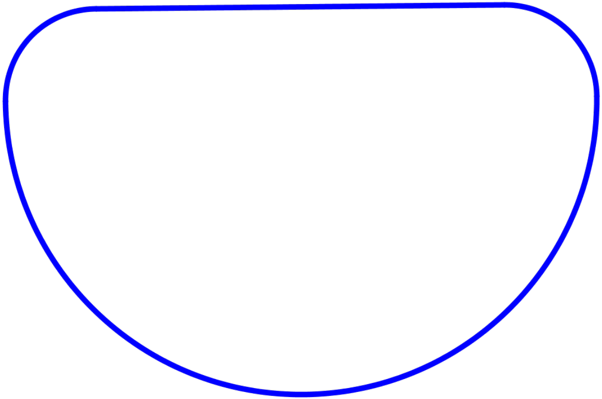
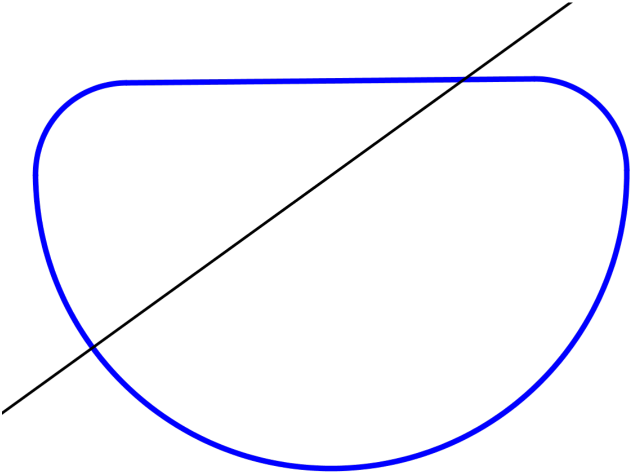
- Let Ω be as before. Split its boundary in two disjoint parts: $\partial\Omega = E \cup R$.
- Broken rays have endpoints on E and reflect finitely many times on R (like light reflecting from a perfect mirror).
- If $E = \partial\Omega$ and $R = \emptyset$, then broken rays are just lines through Ω.

Broken ray transform



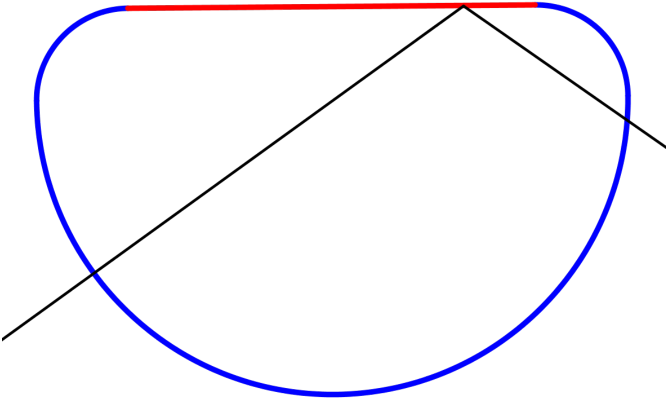
A domain.

Broken ray transform



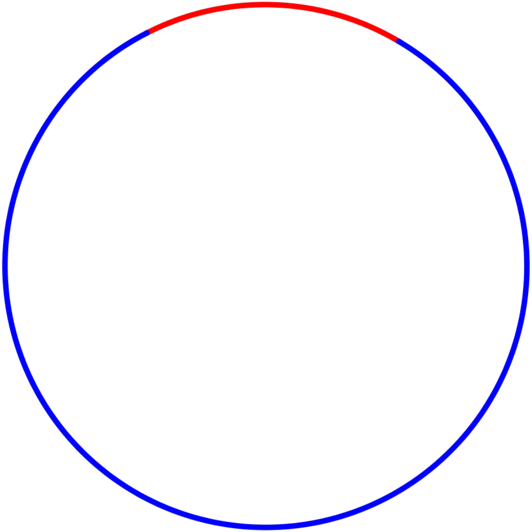
A line through the domain.

Broken ray transform



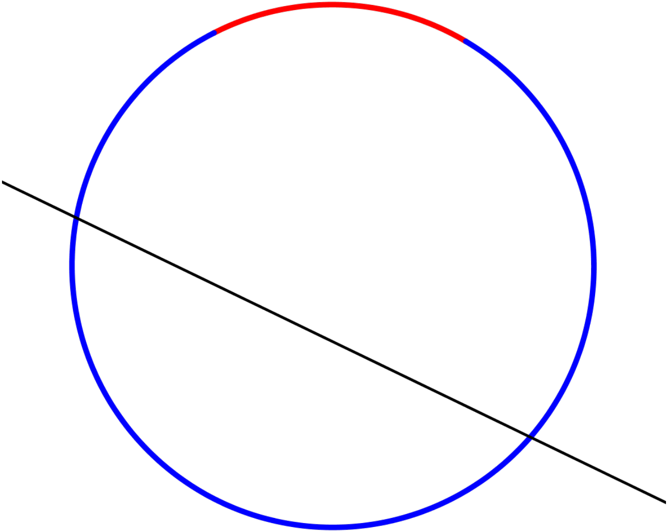
A broken ray. E is blue and R is red.

Broken ray transform



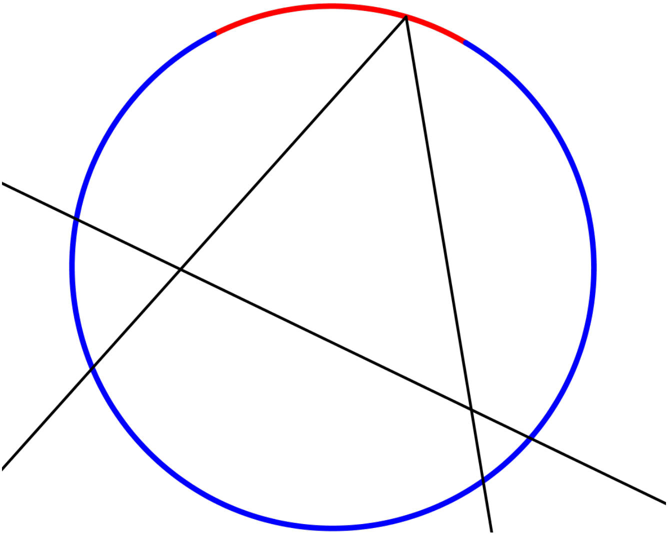
Large E.

Broken ray transform



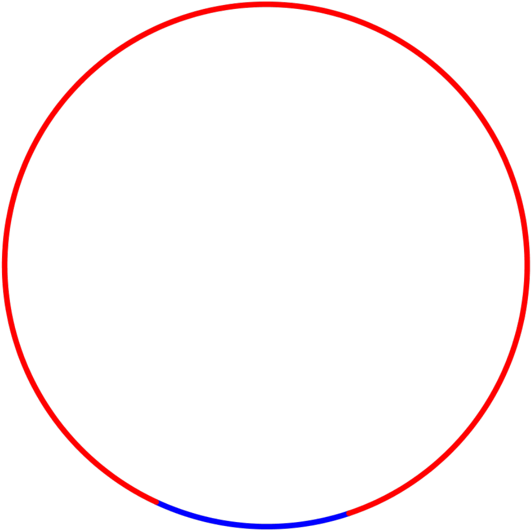
A broken ray.

Broken ray transform



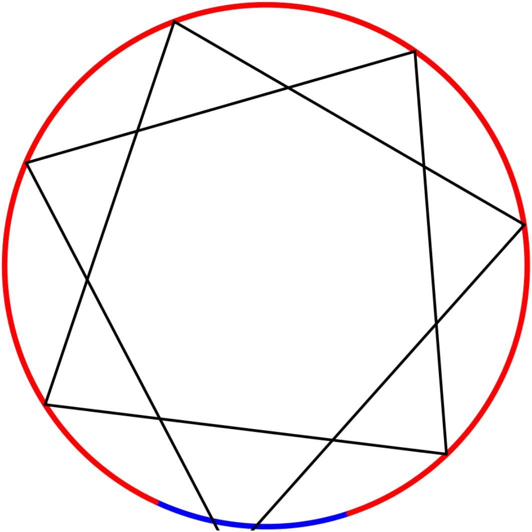
Another broken ray.

Broken ray transform



Small E.

Broken ray transform



A broken ray.

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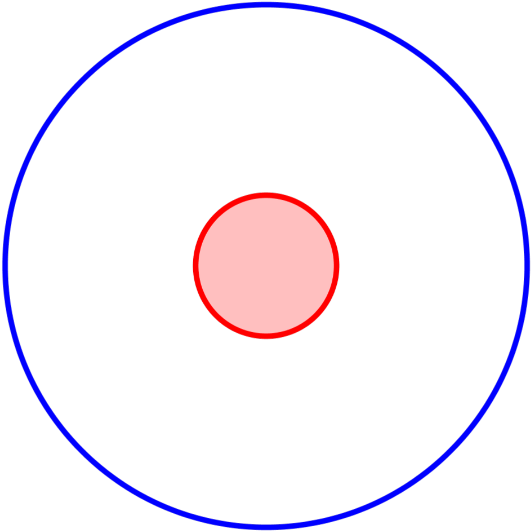
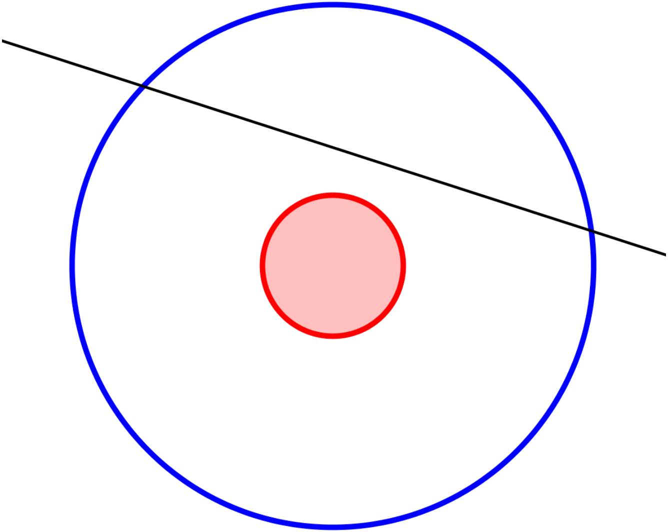


An annulus.

Broken ray transform



A broken ray.

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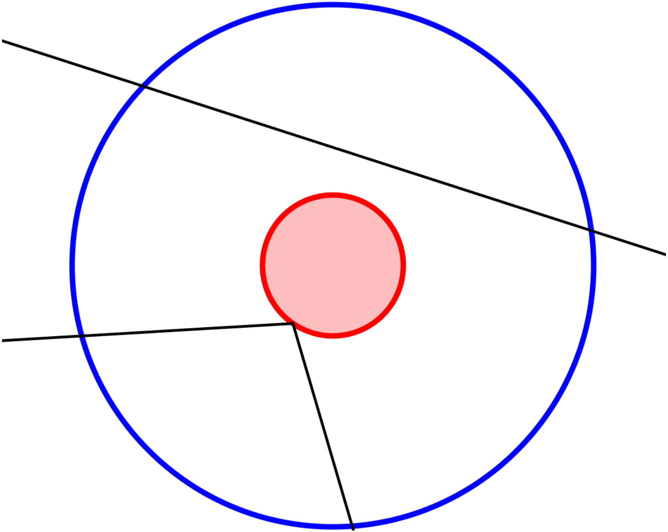

Another broken ray.

Broken ray transform

- Let Γ be the set broken rays in Ω. (This set depends on E and R!)
- The broken ray transform of a continuous function $f : \bar{\Omega} \to \mathbb{R}$ is $\mathcal{G}f : \Gamma \to \mathbb{R}$,

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- Note that the broken ray transform (like the X-ray transform) is linear.
- Broken ray tomography problem: Is $\mathcal{G}$ injective? How does this depend on Ω, E, and regularity assumptions on f?

Technical remarks

- We assumed that all broken rays reflect only finitely many times.
- Sometimes there are trapped broken rays which never find their way back to E.
- We only consider non-trapped broken rays here.

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- We only consider non-trapped broken rays here.
- The Euclidean domain can be replaced with a manifold with boundary.
- X-ray transform and the broken ray transform can also be studied on manifolds, but I will focus on the Euclidean case. Some results are only reasonable for manifolds, but an intuitive idea of a surface is enough.

Outline

1. Definitions
2. Applications
   - Calderón's problem
   - IBVP for electromagnetic Schrödinger's equation
   - Partial data problems in general
3. Approaches
4. Suggested reading

Calderón's problem

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- Calderón's problem: Can we find the conductivity at every point inside an object by making electrical measurements on the boundary? (This is an inverse boundary value problem, IBVP.)
- What if we can only do measurements on a part of the boundary? (Partial data problem.)
- Kenig and Salo (2013) showed that this partial data problem on a tube-shaped domain can be reduced to the injectivity of the broken ray transform on the cross section of the tube.
- For details, see the paper.

IBVP for electromagnetic Schrödinger's equation

- Can we find the electric and magnetic potential (up to gauge) inside an object by sending electromagnetic waves through it? (This is another inverse boundary value problem, but the underlying PDE is different.)

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- Can we find the electric and magnetic potential (up to gauge) inside an object by sending electromagnetic waves through it? (This is another inverse boundary value problem, but the underlying PDE is different.)
- What if there are known obstacles in the domain? (Partial data problem.)
- Eskin (2004) showed that under some geometrical assumptions this problem can be reduced to the injectivity of the broken ray transform.
- For details, see the paper.

- Many problems with full data have been previously reduced to the X-ray transform.
- It seems that in the case of partial data, the X-ray transform is often replaced by the broken ray transform.
- There not many examples yet. There is work to be done...

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   - Four angles of attack
   - Reflection
   - Direct calculation
   - Boundary reconstruction
   - Energy estimate for a PDE
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The main goal is to reduce the problem to a more tractable one, like the X-ray transform, manipulation of Fourier series, or unique solvability of a PDE.

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- Can we make the domain reflect but keep the rays straight?
- Yes, at least if R is flat.

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Proposition

The broken ray transform is injective for the domain $\Omega = \{x \in B(0,1) \subset \mathbb{R}^2; x_1 > 0\}$ *with* $R = \{x \in \partial\Omega; x_1 = 0\}$*.*

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Proof

Suppose $f \in C(\bar{\Omega})$ integrates to zero over all broken rays. Let $\tilde{\Omega} = B(0,1)$ and let $\pi : \bar{\tilde{\Omega}} \to \bar{\Omega}$ be the folding map $\pi(x_1, x_2) = (|x_1|, x_2)$. Define $\tilde{f} : \bar{\tilde{\Omega}} \to \mathbb{R}$ by $\tilde{f}(x) = f(\pi(x))$. Now $\tilde{f}$ integrates to zero over all lines in $\tilde{\Omega}$. Because the X-ray transform is injective, $\tilde{f} = 0$. Thus also $f = 0$. □

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The idea of reducing the broken ray transform on Ω to the X-ray transform of a reflected domain $\tilde{\Omega}$ works in great generality.

Reflection

Lemma (Helgason's support theorem)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported continuous function and $K \subset \mathbb{R}^n$ a compact, convex set. If the integral of f is zero over every line that doesn't meet K, then f is zero outside K.

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Theorem

Let $C \subset \mathbb{R}^2$ be a cone. (In polar coordinates, $r > 0$ and $0 < \theta < \theta_0$ for some θ_0.) Let $\Omega \subset C$ be a bounded domain. Then the broken ray transform in Ω with $R = \partial C \cap \partial \Omega$ is injective.

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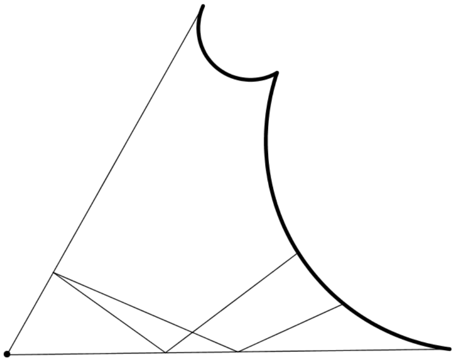
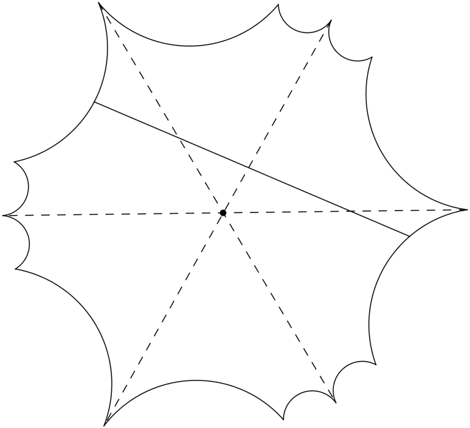
Proof will be given as pictures. For details, see the paper.

Reflection



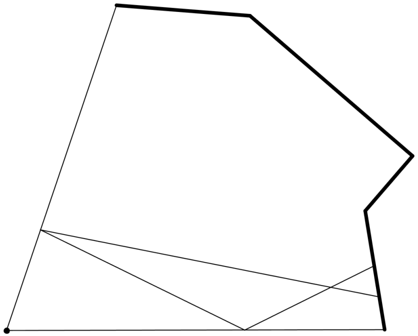
A domain with opening angle $\theta_0 = \pi/m$ for an integer m.

Reflection



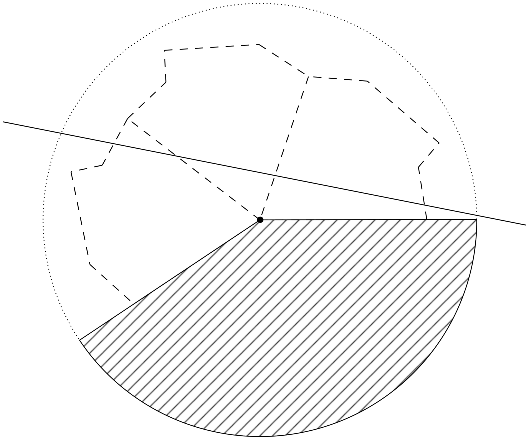
Glue together m copies of Ω (every other one reflected). The broken ray folds out to a straight line. Use injectivity of X-ray transform.

Reflection



A domain with general opening angle θ_0.

Reflection



Glue together enough copies of Ω so that the shaded sector becomes convex.

Apply Helgason's support theorem.

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- A line (a chord) in the disc can be parametrized by the polar coordinates of the closest point to the center.

Direct calculation

- The integral of f over a line given by (r, θ) is
$$\mathcal{I}f(r,\theta) = \sum_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{A}_{|k|}a_k(r)$$
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where $\mathcal{A}_k$ is a generalized Abel transform. ($\mathcal{A}_0$ is the usual Abel transform.)
- These integral transforms $\mathcal{A}_k$ are injective, and this procedure can be used to invert the X-ray transform: $\mathcal{I}f = 0 \iff$ all Fourier components of $\mathcal{I}f$ are zero $\iff \mathcal{A}_{|k|}a_k = 0$ for all $k \iff a_k = 0$ for all $k \iff f = 0$.

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- A broken ray consists of finitely many line segments which are rotations of each other. The integral of f over a broken ray can be expressed explicitly, but it is more complicated than the above sum for the X-ray transform.
- Careful analysis of this expression shows that the broken ray transform is injective.

Boundary reconstruction

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- We can control how well it stays near the boundary. This is similar how an ideally bouncing ball stays near the surface in slowly varying gravitation. (Now gravitation is replaced by the second fundamental form of the boundary.)
- If we take a sequence of broken rays that stay closer and closer to the boundary, the limit curve is a curve on the boundary $\partial\Omega$.
- This curve is actually a geodesic on $\partial\Omega$. Boundary geodesics as limits of broken rays (= billiard trajectories) are known by many names: 'glancing billiards', 'creeping rays', 'whispering gallery trajectories', 'gliding rays'.
- These boundary geodesics can be assumed to have endpoints on ∂E.

Boundary reconstruction

- If we know the broken ray transform of f, we can find its integrals over boundary geodesics.
- If the X-ray transform on $\partial\Omega$ (or more properly R) is injective, we can recover f at $\partial\Omega$ from its broken ray transform.

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Boundary reconstruction

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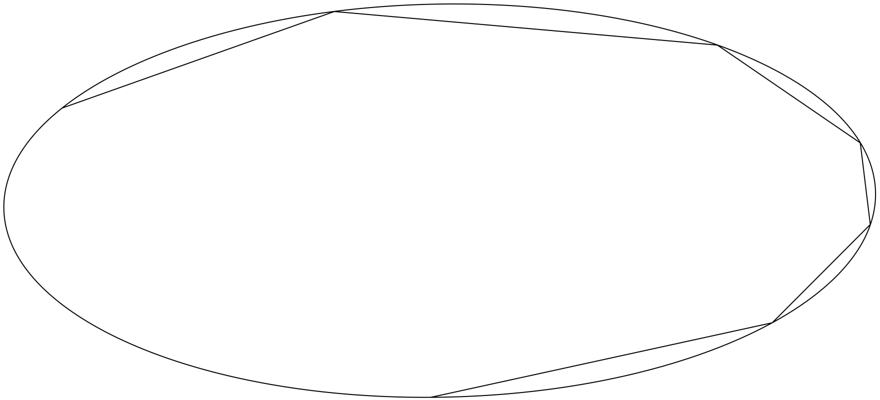
- Boundary reconstruction results are common first steps towards interior reconstruction results in inverse problems.
- If certain weighted X-ray transforms on R are injective, then we can also recover the normal derivatives (of all orders) of f at the boundary.
- The weight depends on curvature (the second fundamental form). Intuitively, this is because a broken ray is on average further away from the boundary where the curvature is largest.

Boundary reconstruction



A broken ray in an ellipse.

Energy estimate for a PDE

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- For $(x, v) \in SM$, let $\gamma_{x,v}$ be the unique broken ray starting at x in direction v. We assume that every broken ray reaches E in finite time.

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- For $x \in \partial M$, let $\rho : S^1 \to S^1$ be the reflection map $v \mapsto v - 2(v \cdot \nu_x)\nu_x$ where ν_x is the (inner or outer) unit normal vector at x.

Energy estimate for a PDE

- For a C^2 function $f : M \to \mathbb{R}$, let $u^f : SM \to \mathbb{R}$ be the integral of f along this broken ray:

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- Define the differential operators $X = v \cdot \nabla_x$ and $V = \partial_v$. (These are known as the geodesic vector field and the vertical vector field.)
- For $x \in \operatorname{int} M$ this function satisfies $Xu^f(x, v) = -f(x)$.
- If we think of f as a function on SM, then $Vf = 0$. Thus $VXu^f = 0$ in $\operatorname{int} SM$.

Energy estimate for a PDE

- For $x \in R$ this function satisfies $u^f(x,v) = u^f(x,\rho(v))$ because of the way broken ways reflect.
- If we assume f to have vanishing broken ray transform, then $u_f(x,v) = 0$ whenever $x \in E$.

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- We have ended up with the PDE $VXu^f = 0$ with boundary conditions on $\partial(SM) = (\partial M) \times S^1$.
- If we can show that this PDE has only the zero solution, then we know that $f = -Xu^f = -X0 = 0$ and the broken ray transform is injective!

Energy estimate for a PDE

Lemma

If $u : SM \to \mathbb{R}$ has sufficient regularity and satisfies the boundary conditions on $\partial(SM)$ mentioned above for u^f, then

$$\begin{aligned}\|VXu\|^2_{L^2(SM)} = \|XVu\|^2_{L^2(SM)} + \|Xu\|^2_{L^2(SM)} \\ - \int_{SM} K(x)\,|Vu(x,v)|^2 - \int_{\partial(SM)} \kappa(x)\,|Vu(x,v)|^2,\end{aligned}$$

where K is the curvature of M and κ is the curvature of ∂M.

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Lemma

If $u : SM \to \mathbb{R}$ has sufficient regularity and satisfies the boundary conditions on $\partial(SM)$ mentioned above for u^f, then

$$\|VXu\|^2_{L^2(SM)} = \|XVu\|^2_{L^2(SM)} + \|Xu\|^2_{L^2(SM)} - \int_{SM} K(x)\,|Vu(x,v)|^2 - \int_{\partial(SM)} \kappa(x)\,|Vu(x,v)|^2\,,$$

where K is the curvature of M and κ is the curvature of ∂M.

Suppose $K \leq 0$ and $\kappa \leq 0$. If f has vanishing broken ray transform, applying the lemma to u^f gives

$$0 = \left\|Xu^f\right\|^2_{L^2(SM)} + \text{something positive}$$

so $f = -Xu^f = 0$.

Outline

Broken ray transform in general

- J. Ilmavirta: **On the broken ray transform**, PhD thesis at the University of Jyväskylä 2014, arXiv:1409.7500. (Introductory part, essentially a survey on the topic.)

Applications

- C. E. Kenig, M. Salo: **The Calderón problem with partial data on manifolds and applications**, *Analysis & PDE* 2013, arXiv:1211.1054.
- G. Eskin: **Inverse boundary value problems for domains with several obstacles**, *Inverse Problems* 2004.

The four approaches

- Reflection:
  - J. Ilmavirta: **A reflection approach to the broken ray transform**, to appear in *Mathematica Scandinavica*, arXiv:1306.0341.
  - M. Hubenthal: **The Broken Ray Transform in n Dimensions with Flat Reflecting Boundary**, *Inverse Problems and Imaging* 2015, arXiv:1310.7156.
  - M. Hubenthal: **The Broken Ray Transform on the Square**, *Journal of Fourier Analysis and Applications* 2014, arXiv:1302.6193.

The four approaches

- Calculation: J. Ilmavirta: **Broken ray tomography in the disc**, *Inverse Problems* 2013, arXiv:1210.4354.
- Boundary reconstruction: J. Ilmavirta: **Boundary reconstruction for the broken ray transform**, *Annales Academiae Scientiarum Fennicae Mathematica* 2014, arXiv:1310.2025.
- Energy method: J. Ilmavirta, M. Salo: **Broken ray transform on a Riemann surface with a convex obstacle**, to appear in *Communications in Analysis and Geometry*, arXiv:1403.5131.

If you got interested...

I'm happy to discuss this topic with anyone who is interested. Come and ask me or send email any time.

End

Thank you.

Questions?

Slides and papers available at `http://users.jyu.fi/~jojapeil`.