JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

X-ray tomography in periodic slabs

SIAM Conference on Imaging Science

Joonas Ilmavirta

June 6, 2018

Based on joint work with

Gunther Uhlmann

Outline

The slab and rays through it

The slab and rays through it

- Goal: X-ray tomography in *infinite* slabs.

The slab and rays through it

- Goal: X-ray tomography in *infinite* slabs.
- The most familiar kind of infinite slab is $[0,1] \times \mathbb{R}^2 \subset \mathbb{R}^3$.

The slab and rays through it

- Goal: X-ray tomography in *infinite* slabs.
- The most familiar kind of infinite slab is $[0, 1] \times \mathbb{R}^2 \subset \mathbb{R}^3$.
- The same tools work with the infinite strip $[0, 1] \times \mathbb{R} \subset \mathbb{R}^2$.

The slab and rays through it

- Goal: X-ray tomography in *infinite* slabs.
- The most familiar kind of infinite slab is $[0,1] \times \mathbb{R}^2 \subset \mathbb{R}^3$.
- The same tools work with the infinite strip $[0,1] \times \mathbb{R} \subset \mathbb{R}^2$.
- In general, we will consider the $(n+1)$-dimensional slab $[0,1] \times \mathbb{R}^n$ for $n \geq 0$.

The slab and rays through it

- A line through the slab has endpoints at the opposite surfaces (boundary components).

The slab and rays through it

- A line through the slab has endpoints at the opposite surfaces (boundary components).
- The aim is to reconstruct a function from its integrals over all these lines.

The slab and rays through it

- A line through the slab has endpoints at the opposite surfaces (boundary components).
- The aim is to reconstruct a function from its integrals over all these lines.
- This is partial data problem: We have no access to the lines parallel to the boundary.

The slab and rays through it

- A line through the slab has endpoints at the opposite surfaces (boundary components).
- The aim is to reconstruct a function from its integrals over all these lines.
- This is partial data problem: We have no access to the lines parallel to the boundary.
- Even with a finite-sized slab we might only have data through the slab, not parallel to it. A good theoretical understanding of the idealized slab will hopefully help in more realistic settings.

The kernel

The kernel

- Consider a function $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ of the form $f(x,y) = h(x)$.

The kernel

- Consider a function $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ of the form $f(x,y) = h(x)$.
- Consider a ray $\gamma\colon [0,1] \to [0,1] \times \mathbb{R}^n$ of the form $\gamma(t) = (t, a + tv)$, where $a, v \in \mathbb{R}^n$. All the rays through the slab are of this form.

The kernel

- Consider a function $f\colon [0,1]\times\mathbb{R}^n\to\mathbb{R}$ of the form $f(x,y)=h(x)$.
- Consider a ray $\gamma\colon [0,1]\to[0,1]\times\mathbb{R}^n$ of the form $\gamma(t)=(t,a+tv)$, where $a,v\in\mathbb{R}^n$. All the rays through the slab are of this form.
- The integral (X-ray transform) is

$$\int_0^1 f(\gamma(t))\,\mathrm{d}t=\int_0^1 h(t)\,\mathrm{d}t.$$

The kernel

- Consider a function $f\colon [0,1]\times\mathbb{R}^n \to \mathbb{R}$ of the form $f(x,y) = h(x)$.
- Consider a ray $\gamma\colon [0,1] \to [0,1]\times\mathbb{R}^n$ of the form $\gamma(t) = (t, a+tv)$, where $a, v \in \mathbb{R}^n$. All the rays through the slab are of this form.
- The integral (X-ray transform) is

$$\int_0^1 f(\gamma(t))\,\mathrm{d}t = \int_0^1 h(t)\,\mathrm{d}t.$$

- All functions of the form $f(x,y) = h(x)$ with $\int_0^1 h = 0$ are in the kernel.

The question

The question

Problem

If $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ integrates to zero over all lines through the slab, is it of the form $f(x,y) = h(x)$ with $\int_0^1 h = 0$?

The question

Problem

If $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ integrates to zero over all lines through the slab, is it of the form $f(x,y) = h(x)$ with $\int_0^1 h = 0$?

Are there other things in the kernel?

The question

Problem

If $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ *integrates to zero over all lines through the slab, is it of the form* $f(x,y) = h(x)$ *with* $\int_0^1 h = 0$*?*

Are there other things in the kernel?

What regularity do we need to assume?

The question

Problem

If $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ integrates to zero over all lines through the slab, is it of the form $f(x,y) = h(x)$ with $\int_0^1 h = 0$?

Are there other things in the kernel?

What regularity do we need to assume?

Unless $h \equiv 0$, functions of this form have no decay at infinity and cannot be integrable.

Outline

1. X-ray tomography in a slab
2. Periodicity
   - Periodic functions or slabs
   - Kernel characterization
   - Tensor tomography
3. Proof

Periodic functions or slabs

Periodic functions or slabs

- We make a simplifying assumption: The function $f \colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is periodic with respect to all the n variables on $\mathbb{R}^n$.

Periodic functions or slabs

- We make a simplifying assumption: The function $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is periodic with respect to all the n variables on $\mathbb{R}^n$.
- After rescaling, all the periods can be assumed to be 1.

Periodic functions or slabs

- We make a simplifying assumption: The function $f\colon [0,1]\times\mathbb{R}^n\to\mathbb{R}$ is periodic with respect to all the n variables on $\mathbb{R}^n$.
- After rescaling, all the periods can be assumed to be 1.
- A 1-periodic function $\mathbb{R}^n\to\mathbb{R}$ can be regarded as a function $\mathbb{R}^n/\mathbb{Z}^n\to\mathbb{R}$.

Periodic functions or slabs

- We make a simplifying assumption: The function $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is periodic with respect to all the n variables on $\mathbb{R}^n$.
- After rescaling, all the periods can be assumed to be 1.
- A 1-periodic function $\mathbb{R}^n \to \mathbb{R}$ can be regarded as a function $\mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}$.
- The quotient $\mathbb{R}^n/\mathbb{Z}^n$ is the flat torus $\mathbb{T}^n$.

Periodic functions or slabs

- It is the same thing to consider periodic functions on $\mathbb{R}^n$ as it is to consider all functions on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Periodic functions or slabs

- It is the same thing to consider periodic functions on $\mathbb{R}^n$ as it is to consider all functions on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
- The most familiar example is with $n = 1$, when $\mathbb{T}^1 = S^1$.

Periodic functions or slabs

- It is the same thing to consider periodic functions on $\mathbb{R}^n$ as it is to consider all functions on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
- The most familiar example is with $n = 1$, when $\mathbb{T}^1 = S^1$.
- When we assume periodicity in the functions, we are effectively doing X-ray tomography in the periodic slab $[0, 1] \times \mathbb{T}^n$.

Kernel characterization

Theorem (I.–Uhlmann, 2018)

Kernel characterization

Theorem (I.–Uhlmann, 2018)

A scalar L^2 function $f\colon [0,1] \times \mathbb{T}^n \to \mathbb{R}$ integrates to zero over all lines through the slab if and only if it is of the form $f(x,y) = h(x)$ with $\int_0^1 h(x)\,\mathrm{d}x = 0$.

Kernel characterization

Theorem (I.–Uhlmann, 2018)

A scalar L^2 function $f\colon [0,1]\times\mathbb{T}^n\to\mathbb{R}$ integrates to zero over all lines through the slab if and only if it is of the form $f(x,y)=h(x)$ with $\int_0^1 h(x)\,\mathrm{d}x=0$.

Proof

Write the function as a Fourier series in all variables. Take a suitable Fourier transform of the data and show that $\hat{f}$ vanishes in most places. □

Kernel characterization

Theorem (I.–Uhlmann, 2018)

A scalar L^2 function $f\colon [0,1]\times\mathbb{T}^n\to\mathbb{R}$ integrates to zero over all lines through the slab if and only if it is of the form $f(x,y)=h(x)$ with $\int_0^1 h(x)\,\mathrm{d}x=0$.

Proof

Write the function as a Fourier series in all variables. Take a suitable Fourier transform of the data and show that $\hat{f}$ vanishes in most places. □

Fun fact: Also true for the Möbius strip.

Tensor tomography

Tensor tomography

- The integral of a scalar function f over a line $\gamma\colon [0,T] \to [0,1] \times \mathbb{T}^n$ is

$$\int_0^T f(\gamma(t))\,\mathrm{d}t \in \mathbb{R}.$$

Tensor tomography

- The integral of a scalar function f over a line $\gamma\colon [0,T] \to [0,1] \times \mathbb{T}^n$ is

$$\int_0^T f(\gamma(t))\,\mathrm{d}t \in \mathbb{R}.$$

- The integral of a vector field f over a line $\gamma\colon [0,T] \to [0,1] \times \mathbb{T}^n$ is

$$\int_0^T f(\gamma(t)) \cdot \dot{\gamma}(t)\,\mathrm{d}t \in \mathbb{R}.$$

Tensor tomography

- The integral of a scalar function f over a line $\gamma\colon [0,T] \to [0,1] \times \mathbb{T}^n$ is

$$\int_0^T f(\gamma(t))\,\mathrm{d}t \in \mathbb{R}.$$

- The integral of a vector field f over a line $\gamma\colon [0,T] \to [0,1] \times \mathbb{T}^n$ is

$$\int_0^T f(\gamma(t)) \cdot \dot{\gamma}(t)\,\mathrm{d}t \in \mathbb{R}.$$

- If $f = \nabla u$, then the integral is $u(\gamma(T)) - u(\gamma(0))$.

Tensor tomography

Theorem (I.–Uhlmann, 2018)

Tensor tomography

Theorem (I.–Uhlmann, 2018)

Let f be an L^2 vector field on $[0,1] \times \mathbb{T}^n$. It integrates to zero over all lines through the slab if and only if it is of the form

$$f = \tilde{h} + \nabla u,$$

Tensor tomography

Theorem (I.–Uhlmann, 2018)

Let f be an L^2 vector field on $[0,1] \times \mathbb{T}^n$. It integrates to zero over all lines through the slab if and only if it is of the form

$$f = \tilde{h} + \nabla u,$$

where

- $\tilde{h}(x,y) = h(x) \in \mathbb{R}^{n+1}$,
- $h \in L^2([0,1])$,
- $\int_0^1 h(x)\,\mathrm{d}x = 0$,
- $u \in H^1([0,1] \times \mathbb{T}^n)$, *and*
- u *vanishes at the boundary.*

Tensor tomography

- There is a similar result for higher rank tensor fields.

Tensor tomography

- There is a similar result for higher rank tensor fields.
- There are two kinds of tensor fields in the kernel:

Tensor tomography

- There is a similar result for higher rank tensor fields.
- There are two kinds of tensor fields in the kernel:
  - Symmetrized covariant derivatives of lower order tensor fields.

Tensor tomography

- There is a similar result for higher rank tensor fields.
- There are two kinds of tensor fields in the kernel:
  - Symmetrized covariant derivatives of lower order tensor fields.
  - Tensor fields that are invariant under translations along the slab.

Tensor tomography

- There is a similar result for higher rank tensor fields.
- There are two kinds of tensor fields in the kernel:
  - Symmetrized covariant derivatives of lower order tensor fields.
  - Tensor fields that are invariant under translations along the slab.
- The X-ray transform is not solenoidally injective in the slab for tensor fields, but the kernel can be characterized for any rank and dimension.

Outline

The setting

The setting

- Let $f \in C^\infty([0,1] \times S^1)$.

The setting

- Let $f \in C^\infty([0,1] \times S^1)$.
- We will show that if f integrates to zero over all lines through the strip, then f is of the required form: $f(x,y) = h(x)$ and $\int_0^1 h = 0$.

The setting

- Let $f \in C^\infty([0,1] \times S^1)$.
- We will show that if f integrates to zero over all lines through the strip, then f is of the required form: $f(x,y) = h(x)$ and $\int_0^1 h = 0$.
- Here $S^1 = \mathbb{R}/\mathbb{Z} = \mathbb{T}^1$. Angles go from zero to one, not from zero to 2π.

The setting

- Let $f \in C^\infty([0,1] \times S^1)$.
- We will show that if f integrates to zero over all lines through the strip, then f is of the required form: $f(x,y) = h(x)$ and $\int_0^1 h = 0$.
- Here $S^1 = \mathbb{R}/\mathbb{Z} = \mathbb{T}^1$. Angles go from zero to one, not from zero to 2π.
- The same idea works for $L^2([0,1] \times \mathbb{T}^n)$.

Fourier series

- Write f as Fourier series:

$$f(x,y)=\sum_{j,k\in\mathbb{Z}}\hat{f}(j,k)e^{2\pi i(jx+ky)}.$$

Fourier series

- Write f as Fourier series:

$$f(x,y) = \sum_{j,k\in\mathbb{Z}} \hat{f}(j,k)e^{2\pi i(jx+ky)}.$$

- Here $x \in [0,1]$ and $y \in \mathbb{R}/\mathbb{Z}$.

Fourier series

- Write f as Fourier series:

$$f(x,y) = \sum_{j,k\in\mathbb{Z}} \hat{f}(j,k)e^{2\pi i(jx+ky)}.$$

- Here $x \in [0,1]$ and $y \in \mathbb{R}/\mathbb{Z}$.
- The function is a sum of functions of the form

$$e_{jk}(x,y) = e^{2\pi i(jx+ky)}$$

and the X-ray transform of e_{jk} can be computed explicitly.

X-ray transform

- The X-ray transform of $f \in C^\infty([0,1] \times S^1)$ is $If \in C^\infty(S^1 \times \mathbb{R})$ defined by

$$If(a,v) = \int_0^1 f(t, a+tv)\,\mathrm{d}t.$$

X-ray transform

- The X-ray transform of $f \in C^\infty([0,1] \times S^1)$ is $If \in C^\infty(S^1 \times \mathbb{R})$ defined by
$$If(a,v) = \int_0^1 f(t, a+tv)\,\mathrm{d}t.$$
- The parameters $(a,v) \in S^1 \times \mathbb{R}$ parametrize all the geodesics through the periodic slab (strip) $[0,1] \times \mathbb{R}$. The initial position is $(0,a)$ and the velocity $(1,v)$.

X-ray transform

- The X-ray transform of $f \in C^\infty([0,1] \times S^1)$ is $If \in C^\infty(S^1 \times \mathbb{R})$ defined by
$$If(a,v) = \int_0^1 f(t, a + tv)\,\mathrm{d}t.$$
- The parameters $(a,v) \in S^1 \times \mathbb{R}$ parametrize all the geodesics through the periodic slab (strip) $[0,1] \times \mathbb{R}$. The initial position is $(0,a)$ and the velocity $(1,v)$.
- The X-ray transform of $e_{jk}(x,y) = e^{2\pi i(jx+ky)}$ is
$$Ie_{jk}(a,v) = e^{2\pi ika}\phi(j+kv),$$
where $\phi(z) = (2\pi iz)^{-1}(e^{2\pi iz} - 1)$.

X-ray transform

- Using the Fourier series gives

$$0 = If(a, v) = \sum_{j,k \in \mathbb{Z}} \hat{f}(j, k)\phi(j + kv)e^{2\pi ika}$$

for all $a \in S^1$ and $v \in \mathbb{R}$.

X-ray transform

- Using the Fourier series gives

$$0 = If(a, v) = \sum_{j,k \in \mathbb{Z}} \hat{f}(j, k)\phi(j + kv)e^{2\pi ika}$$

  for all $a \in S^1$ and $v \in \mathbb{R}$.
- Taking the Fourier series with respect to a gives

$$\sum_{j \in \mathbb{Z}} \hat{f}(j, k)\phi(j + kv) = 0$$

  for all $k \in \mathbb{Z}$ and $v \in \mathbb{R}$.

X-ray transform

- Using the Fourier series gives

$$0 = If(a, v) = \sum_{j,k \in \mathbb{Z}} \hat{f}(j, k)\phi(j + kv)e^{2\pi ika}$$

  for all $a \in S^1$ and $v \in \mathbb{R}$.
- Taking the Fourier series with respect to a gives

$$\sum_{j \in \mathbb{Z}} \hat{f}(j, k)\phi(j + kv) = 0$$

  for all $k \in \mathbb{Z}$ and $v \in \mathbb{R}$.
- We choose v so that $kv \in \mathbb{Z}$. For $z \in \mathbb{Z}$ we have $\phi(z) = \delta_{z0}$.

X-ray transform

- Using the Fourier series gives

$$0 = If(a, v) = \sum_{j,k\in\mathbb{Z}} \hat{f}(j,k)\phi(j+kv)e^{2\pi i k a}$$

  for all $a \in S^1$ and $v \in \mathbb{R}$.
- Taking the Fourier series with respect to a gives

$$\sum_{j\in\mathbb{Z}} \hat{f}(j,k)\phi(j+kv) = 0$$

  for all $k \in \mathbb{Z}$ and $v \in \mathbb{R}$.
- We choose v so that $kv \in \mathbb{Z}$. For $z \in \mathbb{Z}$ we have $\phi(z) = \delta_{z0}$.
- We get $\hat{f}(-kv, k) = 0$ when $k \in \mathbb{Z}$ and $kv \in \mathbb{Z}$. Thus $\hat{f}(j,k)$ can only be non-zero when $k = 0$ and $j \neq 0$.

X-ray transform

- Using the Fourier series gives

$$0 = If(a, v) = \sum_{j,k\in\mathbb{Z}} \hat{f}(j,k)\phi(j+kv)e^{2\pi ika}$$

for all $a \in S^1$ and $v \in \mathbb{R}$.

- Taking the Fourier series with respect to a gives

$$\sum_{j\in\mathbb{Z}} \hat{f}(j,k)\phi(j+kv) = 0$$

for all $k \in \mathbb{Z}$ and $v \in \mathbb{R}$.

- We choose v so that $kv \in \mathbb{Z}$. For $z \in \mathbb{Z}$ we have $\phi(z) = \delta_{z0}$.
- We get $\hat{f}(-kv, k) = 0$ when $k \in \mathbb{Z}$ and $kv \in \mathbb{Z}$. Thus $\hat{f}(j,k)$ can only be non-zero when $k = 0$ and $j \neq 0$.
- If $\hat{h}(j) = \hat{f}(j, 0)$, then $\int_0^1 h = 0$ and $f(x,y) = h(x)$.

DISCOVERING MATH at JYU. Since 1863.

Slides and papers available at

`http://users.jyu.fi/~jojapeil`