Pestov identities for generalized X-ray transforms

Inverse Problems Seminar
University of Washington

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University of Jyväskylä
("Jyväskylä" = "you vascular" + $\mathcal{O}(1)$)

17 April 2017

Events in 1917

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- Radon publishes *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten* (On the determination of functions from their integral values over certain manifolds), which starts the theory of Radon and X-ray transforms.

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- Finland declares independence.

Outline

1. Ray tomography on simple surfaces
   - The sphere bundle
   - From ray transform to transport equation
   - The Pestov identity
2. Higher dimensions
3. Broken ray tomography
4. Pseudo-Riemannian manifolds
5. Extra material

The sphere bundle

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- If $M \subset \mathbb{R}^n$, then $SM = M \times S^{n-1}$.
- SM is a subbundle of the tangent bundle TM.
- If $\dim(M) = n$, then $\dim(SM) = 2n - 1$.

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These vector fields are Sasaki-orthonormal.

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- If $(x, v) \in SM$, denote by $\gamma_{x,v}$ the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.
- Let $\tau_{x,v} \geq 0$ be the exit time of the geodesic $\gamma_{x,v}$.

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- Since u^f is the integral of f along a geodesic, the fundamental theorem of calculus gives

$$Xu^f(x,v) = -f(x)$$

for all $(x,v) \in SM$. This is the transport equation.

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- Since f integrates to zero over all geodesics, u^f is zero at $\partial(SM)$.

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- The lower order term $\frac{1}{2}X_\perp$ plays a role: if we change its sign, we *always* lose unique solvability.
- There is an energy identity for this PDE that allows us to deduce that solutions are unique on some manifolds.

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Let M be a compact, orientable Riemannian surface with boundary. If $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$, then

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Proof

Calculate $\|VXu\|^2 - \|XVu\|^2$ using integration by parts and commutator relations: $[X,V] = X_\perp$, $[V,X_\perp] = X$, $[X,X_\perp] = -KV$. □

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- Apply the Pestov identity to u^f which satisfies $VXu^f = 0$:

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- If $K \leq 0$, then all terms are non-negative and therefore vanish. In particular, $f = -Xu^f = 0$.
- If the manifold is simple, then

$$\|XVu\|^2 - \langle KVu, Vu \rangle \geq 0$$

for all $u \in C^\infty(SM)$, and the same conclusion holds. [Mukhometov 1977, . . .]

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- In 2D we had $V^* = -V$ and $X_\perp^* = -X_\perp$, but in higher dimensions the vertical and horizontal gradients are different objects than the vertical and horizontal divergences ($\overset{\mathrm{v}}{\mathrm{div}}, \overset{\mathrm{h}}{\mathrm{div}}$).

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- I will be very brief. For more details in higher dimensions, see

  Paternain–Salo–Uhlmann (Math. Ann. 2015): Invariant distributions, Beurling transforms and tensor tomography in higher dimensions

  and references therein.

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where R is an operator given by the Riemann curvature tensor. The norms and inner products are those of $L^2(SM)$.

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If M has non-positive sectional curvature, all terms are non-negative.

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Theorem (Mukhometov 1977)

The X-ray transform is injective on all simple manifolds.

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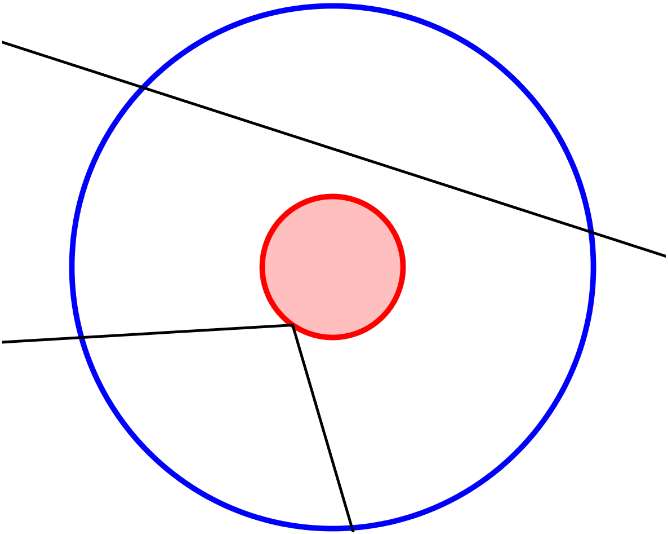
- The Pestov identity gives (solenoidal) injectivity for one-forms as well, but higher order tensors require more care.
- The Pestov identity can also be used on non-compact manifolds. [Lehtonen, 2016]

Outline

The broken ray transform



One reflecting obstacle in a domain. Two broken rays.

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Is a function determined by its integrals over all broken rays?

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- This u^f satisfies $u^f = u^f \circ \rho$ on SR.

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Technical problem: The function u^f is not a priori smooth at all!

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- There is a similar boundary term in higher dimensions.

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- A pseudo-Riemannian metric of signature (n_1, n_2) is like the matrix

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix},$$

with n_1 positive signs and n_2 negative ones.

Pseudo-Riemannian products

For two Riemannian manifolds (M_1, g_1) and (M_2, g_2) we can equip the product manifold $M_1 \times M_2$ with the Riemannian product metric $g_1 \oplus g_2$ or the pseudo-Riemannian product metric $g_1 \ominus g_2$.

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Theorem (I., 2016)

Let M_1 and M_2 be two Riemannian manifolds of non-negative sectional curvature, strictly convex boundary and dimension ≥ 2. Then the null geodesic X-ray transform (light ray transform) is injective on the pseudo-Riemannian product $(M_1 \times M_2, g_1 \ominus g_2)$.

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The proof is based on a Pestov identity. We assume that the signature (n_1, n_2) satisfies $n_1 \geq 2$ and $n_2 \geq 2$. No Pestov identity method is known in Lorentzian geometry ($n_2 = 1$).

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- We can use the natural operators on both sphere bundles: X_1, $\overset{\mathrm{v}}{\mathrm{div}}_1$, $\overset{\mathrm{h}}{\nabla}_2$, R_2 ...
- The null geodesic flow is generated by $X = X_1 + X_2$.

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If $u\colon LM \to \mathbb{R}$ *is smooth and vanishes at the boundary, then*

$$\begin{aligned}
&(n_2-1)\left\|\overset{\mathrm{v}}{\nabla}_1 Xu\right\|^2 + (n_1-1)\left\|\overset{\mathrm{v}}{\nabla}_2 Xu\right\|^2 \\
&= (n_2-1)\left\|X\overset{\mathrm{v}}{\nabla}_1 u\right\|^2 + (n_1-1)\left\|X\overset{\mathrm{v}}{\nabla}_2 u\right\|^2 \\
&\quad - (n_2-1)\left\langle R_1\overset{\mathrm{v}}{\nabla}_1 u, \overset{\mathrm{v}}{\nabla}_1 u\right\rangle - (n_1-1)\left\langle R_2\overset{\mathrm{v}}{\nabla}_2 u, \overset{\mathrm{v}}{\nabla}_2 u\right\rangle \\
&\quad + (n_1-1)(n_2-1)\left\|Xu\right\|^2 .
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  - Products $M \times \mathbb{R}$ where M has an injective Riemannian ray transform. [Oksanen–Kian, unpublished]
- For other Lorentzian and pseudo-Riemannian manifolds the problem is open.

Outline

Vertical and horizontal things

- How to split TTM into vertical and horizontal subbundles?
- Identifying $H(x,v)$ and $V(x,v)$ with T_xM.
- The Sasaki metric.
- What are the different derivatives on SM?

A proof of HD Pestov identity

- The curvature operator.
- Commutator formulas.
- Adjoint operators.

Vector (and tensor) fields

- Integrating a (covariant) tensor field over a function.
- Gauge freedom.
- Proof for one-forms.
- Why does the simple proof not work for tensor fields?

End

Thank you.

Slides are available at `http://users.jyu.fi/~jojapeil`.