JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Communication between theory and practice via deep learning and “deep teaching”

Math + X Symposium on Data Science and Inverse Problems in Geophysics

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Based on joint work with

Maarten de Hoop and Vitaly Katsnelson

Goals

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- Describing how deep learning might help experimentalists and theorists learn from each other.
- Finding a reliable way to reconstruct as much information as possible from the spectrum of free oscillations.
- The audience telling me whether any of this is possible.

Outline

1. The problem
   - The spectrum of free oscillations
   - Geometry
   - Reconstructing geometry from spectrum
2. A mathematical result
3. Deep learning deep structures
4. Appendix A: Anisotropy and geometry
5. Appendix B: Ray transforms in low regularity and drums
6. Appendix C: More detailed spectral rigidity

The spectrum of free oscillations

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- The amplitudes of different modes vary between different events, but the frequencies are always the same.
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- About 10 000 first frequencies are known.

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- In this "elastic geometry" the distance between two points is the shortest travel time. All other geometrical quantities we need can be derived from length.
- There is one geometry for each wave speed. Due to several polarizations there are several geometries. In the simplest case there is a P-geometry and an S-geometry.

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Problem (Meta problem)

How can theoretical and practical endeavors with the problem benefit from each other?

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- We would like to reconstruct the geometry in the Cartesian coordinates. Alas, this is not possible in anisotropic elasticity without severe constraints. The elastic model is not sensitive to the underlying Euclidean geometry and is therefore blind to it.
- Once one has reconstructed the elastic geometry, one would like to interpret it physically and chemically. I believe it is beneficial to separate the two steps: first find the geometry, then interpret it.

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- A spherically symmetric Earth is $M=\bar{B}(0,1)\subset\mathbb{R}^3$ with $g=c^{-2}e$ and $c(x)=c(|x|)$.
- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n=2$) diffeomorphism $\phi\colon M\to M$ so that ϕ^*g is of this form.

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- The modes of free oscillations are modeled as scalar Neumann eigenfunctions of the Laplace–Beltrami operator of (M, g).
- Reconstructing a Riemannian manifold (M, g) from the Neumann spectrum of Δ_g

  $\approx$

  Reconstructing elliptically inhomogeneous elastic object from the spectrum of free oscillations.

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Intuitively, this means that geodesics (straight lines) curve outwards.

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- The radial Preliminary Reference Earth Model (PREM) is not smooth. Both pressure and shear waves have jump discontinuities.
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Apart from these problems (jumps and liquid) both shear and pressure wave speeds do satisfy the Herglotz condition everywhere.

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Let M be the closed unit ball in $\mathbb{R}^3$. Let $c_s(r)$ be a family of radial sound speeds depending C^∞-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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If each c_s gives rise to the same spectrum (of the corresponding Laplace–Beltrami operator), then $c_s = c_0$ for all s.

This simple model of the round Earth is spectrally rigid!

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- There are interfaces where the sound speed jumps. The real elastic geometry is non-smooth.
- The proof only works with strong assumptions, but the same auxiliary quantities might be useful to look at in any setting.

Outline

Data combination

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- Mathematical models, solutions to inverse problems, and explicit reconstruction algorithms tend to be simple-minded: a single type of data is used to find a single quantity.
- All data can be used together in an algorithm based on machine learning or iteration. This method is rarely provably reliable.

Extracting useful quantities

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- This reliable reconstruction from partial data is useful *data* for a machine learning algorithm.
- Also less final data can be useful. We need not know how to use it.
- What are useful (and reliably computable) quantities?

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Isospectral deformations of a spherically symmetric Riemannian manifold are trivial.

In a weak sense in spherical symmetry:
The spectrum determines the geometry.

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1. From eigenvalues $\lambda_0, \lambda_1, \lambda_2, \dots$ compute the function

$$f(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k} \cdot t).$$

This is the trace of the (formal) operator $\cos(\sqrt{-\Delta_g} \cdot t)$.

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3. Linearized length spectral data is periodic broken ray transform data.
4. The periodic broken ray transform can be inverted explicitly.

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- The points where f behaves badly (singularities).
- The length spectrum (also accessible in other ways).

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If the length of the interval is L, the eigenvalues are

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Suppose $L = \frac{1}{2}$ and we have measured the numbers $0, 4\pi^2, 16\pi^2, \ldots$

We compute and plot the trace function

$$f(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k} \cdot t).$$

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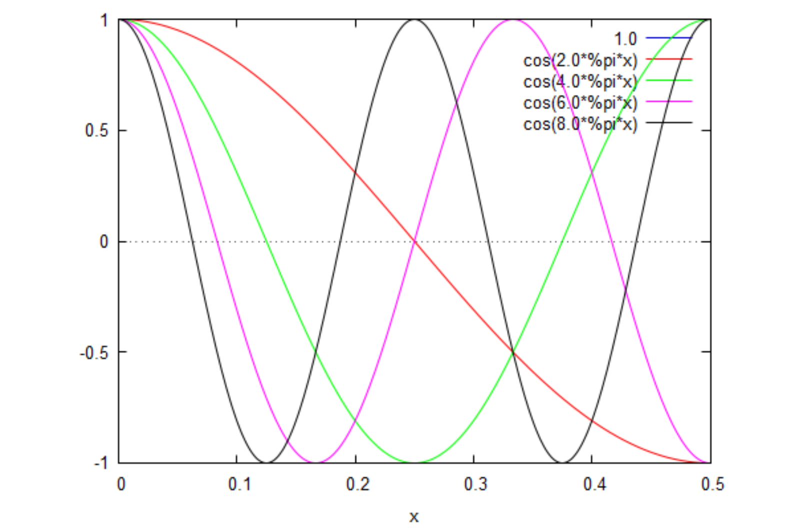

Eigenfunctions for $k = 0, 1, 2, 3, 4$.

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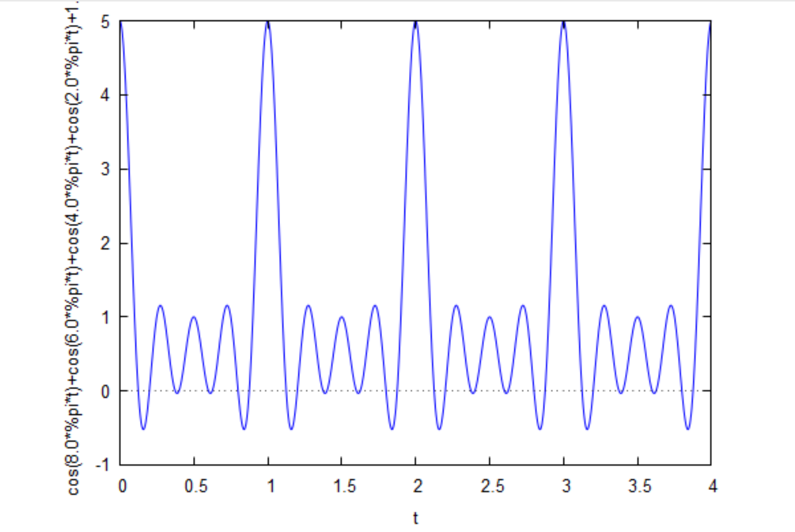

Trace function computed from $k = 0, 1, 2, 3, 4$.

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- The singularities give the length spectrum.
- The length spectrum determines the geometry (length) of the interval.
- Perhaps the trace would be a useful quantity to extract for machine learning with spectral data. The singularities contain geometrical information, and we want to reconstruct the geometry.
- Perhaps other theorists have other potentially useful quantities to throw at a machine and improve performance.

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- The learning process can be unnecessarily (unfeasibly?) heavy.

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For example, we do not know whether the trace function is useful in more complicated spectral inverse problems.

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  Examples:
  1. To extract structural information from spectral data, look at the trace function and its singularities.
  2. Deep teaching: If a machine learning algorithm learns what data is useful, it can tell it to those collecting data and working with theory.

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Elliptic and general elastic anisotropy

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    - General anisotropy corresponds to a Finsler manifold (a manifold with a Finsler metric).
    - Riemannian manifolds are a very special subclass of Finsler manifolds.
- A material is isotropic if sound speed is independent of direction. This can be modeled by a conformally Euclidean metric.

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- To model elastic waves in general anisotropy, one needs a manifold with two Finsler metrics, one for pressure and one for shear waves.
- In fact, the shear wave speed might not even by a Finsler metric in the traditional sense.

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- A fully anisotropic model can *never* be reconstructed from boundary measurements uniquely. The data is always invariant under changes of coordinates.
- The best one can hope for is reconstruction up to changes of coordinates.
- The Earth is spherically symmetric to a good approximation, but the best (elliptically anisotropic) radial model might not be conformally Euclidean. After a radial change of coordinates the metric becomes conformal — and Cartesian coordinates are lost.

Our model

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- Isotropic P-wave speed. — Conformally Euclidean metric.
- Spherical symmetry.
- Reconstruction possible in the natural Cartesian coordinates. — No gauge freedom.

Outline

X-ray transforms

Theorem (de Hoop–I.(2017))

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Let M be a rotation symmetric non-trapping manifold with a piecewise $C^{1,1}$ metric and strictly convex boundary. Then the geodesic X-ray transform is injective on $L^2(M)$.

X-ray transforms

Theorem (de Hoop–I.(2017))

Let M be a rotation symmetric non-trapping manifold with a piecewise $C^{1,1}$ metric and strictly convex boundary. Then the geodesic X-ray transform is injective on $L^2(M)$.

Earlier similar results:

- The X-ray transform (Radon et al.): Euclidean metric (c is constant).
- Mukhometov, 1977: Smooth simple metrics (simplicity is stronger than Herglotz).
- Sharafutdinov, 1997: C^∞ metrics and C^∞ functions.

Periodic broken ray transforms

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Let M be a rotation symmetric non-trapping manifold with a $C^{1,1}$ metric and strictly convex boundary and dimension at least three. Assume that there are not too many conjugate points at the boundary. The integrals of a function $f \in L^p(M)$, $p > 3$, over all periodic broken rays determines the even part of the function.

Very little can be recovered of the odd part.

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Tools used:

- Planar average ray transform.
- Abel transform.
- Funk transform.
- Fourier series.

Hearing the shape of a drum

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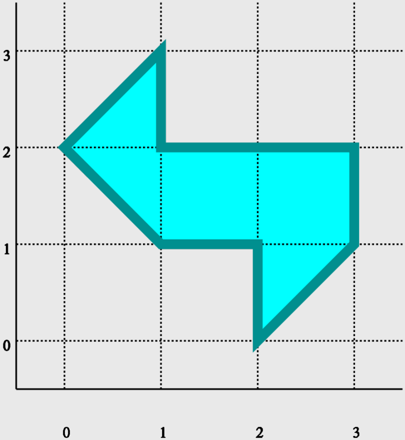
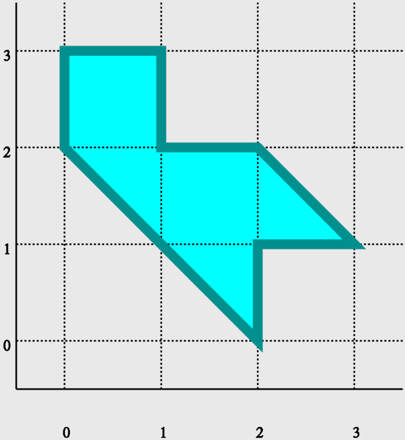
- This is why the answer to Kac's famous question "Can you hear the shape of a drum?" is not trivially "No!".
- ... but it is non-trivially "No!" if there are no geometrical restrictions.

Hearing the shape of a drum





These two drums sound exactly alike. (Wikimedia Commons)

Outline

A simplified model

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- In practice, the Earth is the closed unit ball $M = \bar{B}(0,1) \subset \mathbb{R}^3$. The anisotropic sound speed is modeled with a Riemannian metric g on M.
- Physically, this corresponds to omitting S-waves and including only elliptic anisotropy.

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- The Riemannian metric on M is $g = c^{-2}(x)e$. This makes (M, g) into a radially conformally Euclidean manifold.
- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n = 2$) diffeomorphism $\phi\colon M \to M$ so that $\phi^* g$ is radially conformally Euclidean.

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A radial sound speed $c(r)$ satisfies the Herglotz condition if

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- In addition, the shear wave speed vanishes in the liquid outer core.
- Apart from these problems (jumps and liquid) both shear and pressure wave speeds do satisfy the Herglotz condition everywhere.

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This simple model of the round Earth is spectrally rigid!

Spectral rigidity of the round Earth

Corollary (de Hoop–I.–Katsnelson (2017))

Let M be the closed unit ball in $\mathbb{R}^3$. Let g_s be a family of rotation invariant metrics depending C^∞-smoothly on $s \in (-\varepsilon, \varepsilon)$. Suppose each g_s is non-trapping with strictly convex boundary and assume a generic geometrical condition.

If the spectra of the Laplace–Beltrami operators Δ_{g_s} are all equal, then there is a family of radial diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

Outline of the proof

Lemma (Trace formula)

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Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the positive eigenvalues of the Laplace–Beltrami operator. Define a function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \sum_{k=0}^{\infty} \cos\left(\sqrt{\lambda_k} \cdot t\right).$$

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In particular, the spectrum determines the length spectrum.

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Similar “trace formulas” and related results are known on closed manifolds (eg. Duistermaat–Guillemin 1975) and a weaker version on some manifolds with boundary (eg. Guillemin–Melrose 1979).

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Corollary

Spectral rigidity follows if we can prove length spectral rigidity.

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Let M be the closed unit ball in $\mathbb{R}^n$, $n \geq 2$. Let $c_s(r)$ be a family of radial sound speeds depending $C^{1,1}$-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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This simple model of the round Earth is length spectrally rigid!

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In particular, if the length spectrum does not depend on s, then $\frac{\mathrm{d}}{\mathrm{d}s} c_s^{-2}$ integrates to zero over (almost) all periodic broken rays.

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This concludes the proof.

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- No planet is precisely spherically symmetric.
- The proof only works with strong assumptions, but the same auxiliary quantities might be useful to look at in any setting.

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