



JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Towards a mathematical theory of seismology with a single seismometer

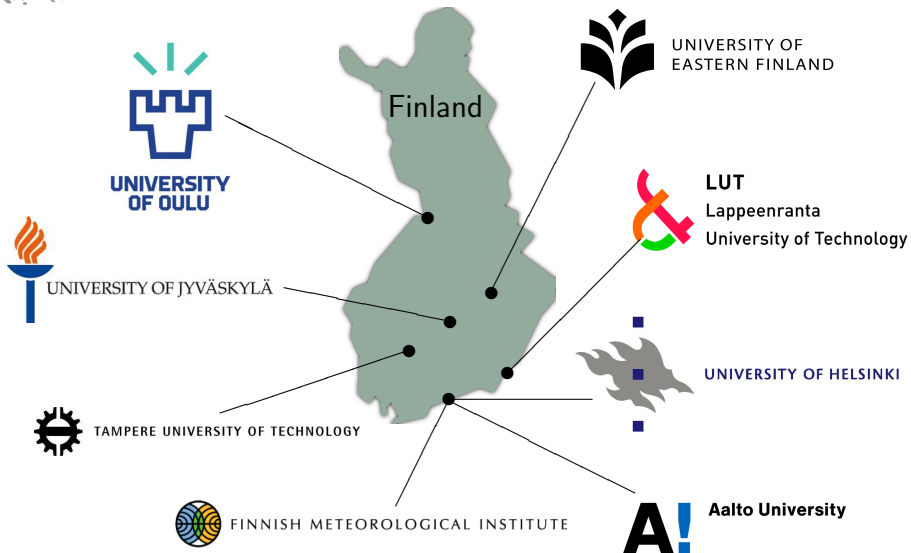
Math + X Symposium on Inverse Problems
and Deep Learning in Space Exploration

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Based on joint work with
Maarten de Hoop and Vitaly Katsnelson

Finnish Centre of Excellence in Inverse Modelling and Imaging 2018-2025



Conference announcement

The annual inverse problems conference “Inverse Days” will be organized in Jyväskylä this year.

Preliminary dates: 16–18 December, 2019.

All kinds of inverse problems in all fields are welcome!

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- Identifying useful data sets can help future mission planning.
- Grand goal: A mathematical theory of seismic planetary exploration.

- 1 Seeing the radial Martian mantle with InSight
 - A small but reliable step
 - Method A: Linearized travel time tomography
 - Method B: Spectral data
 - Method C: Meteorite impacts
 - Summary
- 2 Steps further

A small but reliable step

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- Mars is roughly spherically symmetric. There are reliable ways to reconstruct a radial model of the (upper) mantle from a single station. (The mantle determines the CMB.)

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- Mars is roughly spherically symmetric. There are reliable ways to reconstruct a radial model of the (upper) mantle from a single station. (The mantle determines the CMB.)
- I will ignore noise, model errors, finiteness, stability, and many other practical things.

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- The set of all periodic travel times is the length spectrum.

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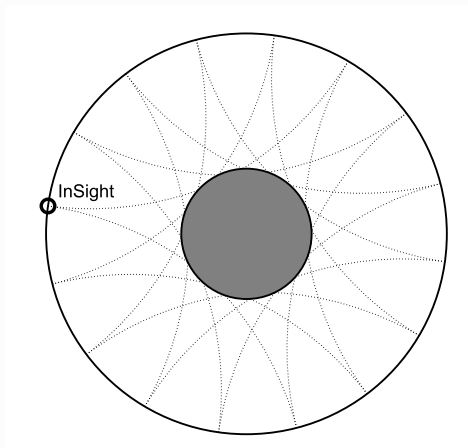
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- Solution: Linearize!
- Linearized data: Pairs of periodic broken rays and integrals over them.
Unknown: Variations of wave speed (a function).

Method A: Linearized travel time tomography



Periodic seismic ray reflecting on the surface and CMB.

Method A: Linearized travel time tomography

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Solving the linearized problem gives an iterative algorithm to solve the nonlinear one. (Uniqueness should be provable for the non-linear one, too.)

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- The different modes are excited differently in different events, but one thing remains: the set of frequencies — the spectrum of free oscillations. (We are at first interested in properties of the planet, not properties of the events.)
- The spectrum of free oscillations can be measured from any single point.

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- Again wave speed = geometry!

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If a family of wave speeds $c_s(r)$ have the same spectrum, are they equal? Is the (Martian) mantle spectrally rigid?

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Consider the annulus (mantle) $M = \bar{B}(0, 1) \setminus B(0, R) \subset \mathbb{R}^3$. Let $c_s(r)$ be a family of radial sound speeds depending C^∞ -smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [R, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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This simple model of the round Martian mantle is spectrally rigid!

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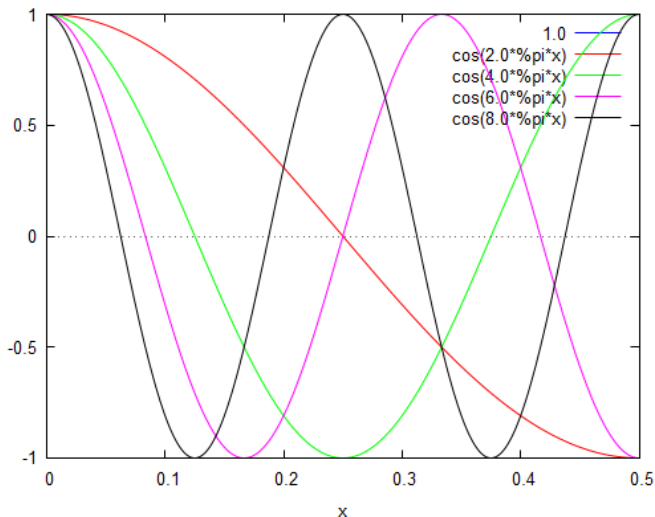
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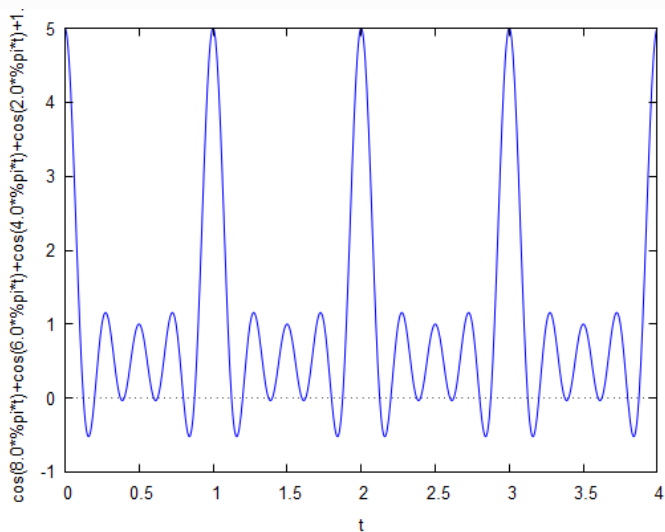
In particular, the spectrum determines the length spectrum. It suffices to prove length spectral rigidity.

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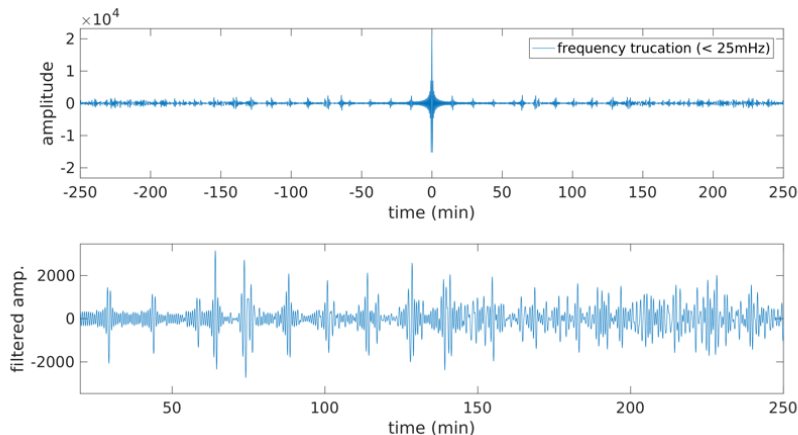
Neumann eigenfunctions for the interval $[0, \frac{1}{2}]$ with $k = 0, 1, 2, 3, 4$.
The length spectrum is \mathbb{Z} .

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Trace function $f(t) = \sum_k \cos(\sqrt{\lambda_k} \cdot t)$ computed from $k = 0, 1, 2, 3, 4$.

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The trace computed from the spectrum of free oscillations in PREM.
Singularities are visible.

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The data set is independent although the method is related.

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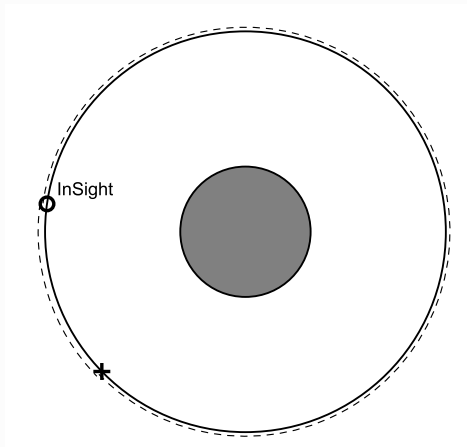
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- Multiple arrivals or a priori information tells the time T around the great circle.

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Two surface wave arrivals from the same event.

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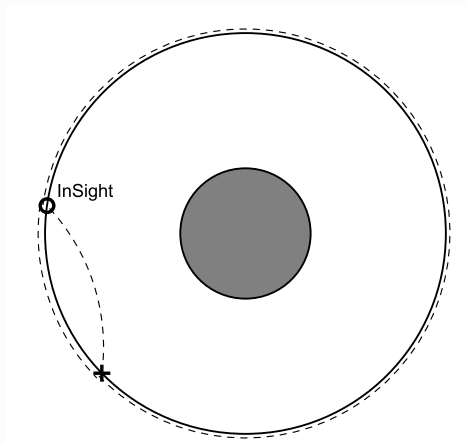
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- To get here, we needed to assume spherical symmetry only on the surface, but the arising problem is easiest to solve if the symmetry extends inside.

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The body wave whose initial point and time were located with surface waves.

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- The linearized problem is X-ray tomography, and can also be solved explicitly. (e.g. de Hoop–I., 2017)

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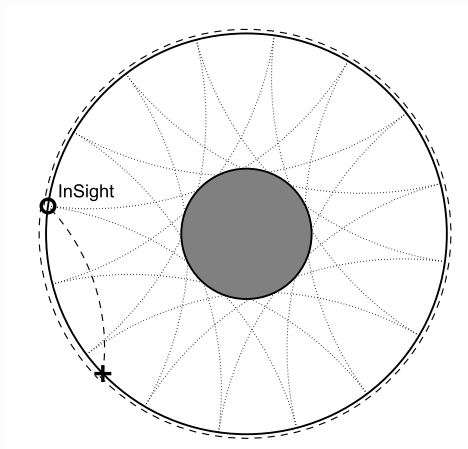
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- If the three reconstructions all work and give similar results, we can be quite confident.
- This gives us an isotropic radially symmetric reference model of the mantle, which is a stepping stone towards deeper and finer structure.

Summary



Three ways to see the mantle from InSight.

Summary

- A: From noise correlations to (linearized) travel times.
- B: From spectrum to length spectrum.
- C: Meteorites; body wave data calibrated by surface waves.

1 Seeing the radial Martian mantle with InSight

2 Steps further

- Spectral perturbation theory
- Half-local X-ray tomography
- Sources and receivers
- Layers
- Geometrization
- Geometry of periodic geodesic
- A theory?

Steps further

Whatever perturbation or iterative method one uses, having a solid initial guess helps, and that can be done with more rigor.

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- Proving precise results outside spherical symmetry with one measurement point is hard.
- A natural approach to small lateral inhomogeneities is perturbation theory with respect to a spherically symmetric reference model.
- In a simple (scalar) model, the medium is described by a single wave speed $c(x)$ and the spectrum depends on it: $Sp(c)$.
- We write the wave speed as a function of a parameter, $c_s(x)$, and expand the spectrum in s :

$$Sp(c_s) = Sp(c_0) + sL(\delta c) + \mathcal{O}(s^2),$$

where $\delta c = \frac{d}{ds}c_s|_{s=0}$, L is the Gâteaux derivative of the spectrum, and '+' is roughly a plus.

Spectral perturbation theory

- If the reference medium c_0 is known and s is small, determining c_s from $Sp(c_s)$ is equivalent(ish) to determining δc from $L(\delta c)$. This is a linear problem!

Spectral perturbation theory

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- On Mars, c_0 would be the radial reference model.
- The better the radial (or other initial) guess is, the better the perturbation theory works.
- The perturbation δc can be expanded in spherical harmonics and the operator L can be written fairly explicitly.

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It is best to start with a scalar model in 2D, not a fully polarized 3D model.

Half-local X-ray tomography

- Recall the third method for reconstructing the radial mantle.

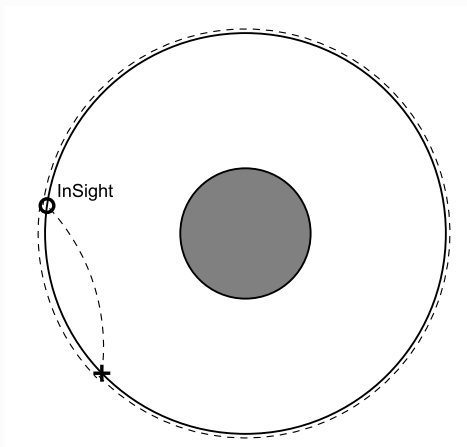
Half-local X-ray tomography

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Half-local X-ray tomography

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- We assumed that the surface is spherically symmetric (or otherwise known), but we needed no assumption on the interior.
- This leads to travel time data: The travel times (geometrically: distances) are known from all points on the surface to a single fixed point.

Half-local X-ray tomography



The body wave whose initial point and time were located with surface waves.

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Let M be a Riemannian (or Finsler) manifold with boundary. Is the metric uniquely determined by the distances between a fixed boundary point and all other boundary points?

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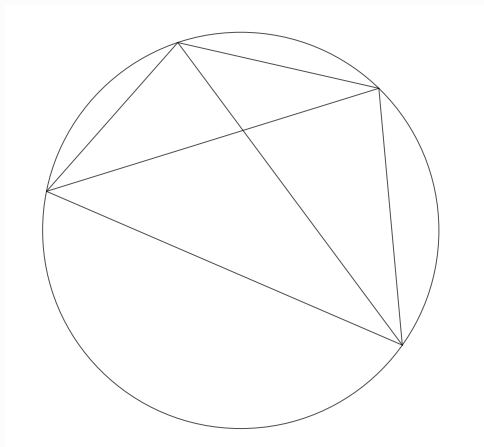
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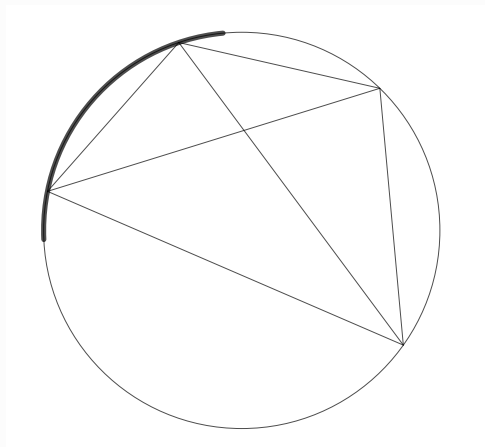
What if the point is replaced by a small open set — a detector array?

Half-local X-ray tomography



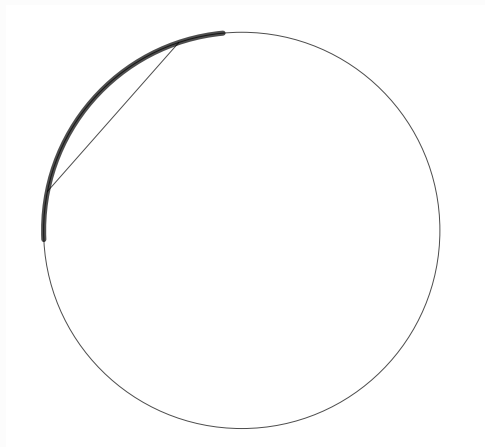
Boundary distance rigidity: Do the distances between all boundary points determine the geometry?

Half-local X-ray tomography



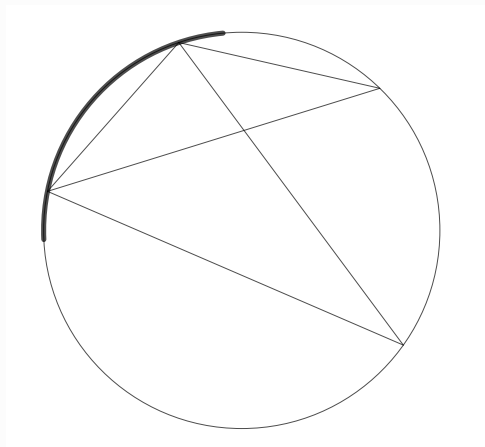
We have an accessible region — a measurement array.
The size is exaggerated.

Half-local X-ray tomography



In the local boundary distance problem one knows the distances between the points in the small set and wants to find the geometry near that set.

Half-local X-ray tomography



The “half-local” boundary distance data has more information and one wants to reconstruct the whole geometry.

Half-local X-ray tomography

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This is possible in Euclidean geometry or with real analytic perturbations but always unstable.

Sources and receivers

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- What is the minimal number of measurement points for uniqueness?

Layers

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- Most geometrical inverse problems work with smooth manifolds. How to add conormal singularities and finite interior regularity?
- How does spectral rigidity and X-ray tomography work in an onion?

Geometrization

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- The speeds can be encoded as geometry where distance is time and geodesics are seismic rays. What is the correct geometrical structure exactly?
- In a strongly anisotropic medium Riemannian geometry is not enough, but we need Finsler.
- Once we recognize which physical quantities are relevant for the data, we can focus on reconstructing what is visible and ignore the invisible.

Geometry of periodic geodesic

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Put any metric on the unit sphere and fix a point on it. How many directions are there so that the geodesic will make it back to the point?

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How does X-ray tomography change when Anosov flow is replaced by dispersing billiards?

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