Geodesic X-ray tomography and geophysical applications

Special lecture at CAAM
Rice University

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Outline

The problem

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- If we measure the initial and final intensities $I(0)$ and $I(L)$, we in fact measure the integral

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- Problem: Given the integrals of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ over all lines, find the function f.
- This problem was first solved by Johann Radon in 1917 and again by Allan Cormack in 1963. In 1979 Cormack and Hounsfield got the Nobel Prize in medicine for developing the CT scan.

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Outline

1. X-ray imaging
2. X-ray tomography and manifolds
   - The X-ray transform on a manifold
   - The unit sphere bundle SM
   - Three vector fields on SM
3. The Pestov identity
4. Various generalizations
5. Applications

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- Is this I injective? What do we need to assume about the manifold and the function f?

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- On manifolds there typically are no simple formulas, but there are iterative algorithms (Neumann series).
- The focus in this talk is in proving injectivity.

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- If $M = \bar{\Omega}$ for a domain $\Omega \subset \mathbb{R}^n$, then $SM = M \times S^{n-1}$.

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- The geodesic flow ϕ_t is simply given by $\phi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$.
- Notice that $\phi_t(x, v)$ is not defined for all t since the geodesics reach the boundary, so $\phi_t \colon SM \to SM$ is only a partial map if $t \neq 0$.

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- In Euclidean geometry $X = v \cdot \nabla_x$.

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- V is known as the vertical vector field.
- The integral curves of V are the fibers $S_x M$.

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- These three vector fields $X, V, X_\perp$ are a global orthonormal frame on SM when SM is equipped with the Sasaki metric.
- We have the commutator relations

$$\begin{aligned} [X, V] &= X_\perp, \\ [V, X_\perp] &= X \text{ and} \\ [X, X_\perp] &= -KV, \end{aligned}$$

where K is the Gaussian curvature.

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Here $\tau_{x,v} \geq 0$ is the exit time of the geodesic $\gamma_{x,v}$.

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Here $\tau_{x,v} \geq 0$ is the exit time of the geodesic $\gamma_{x,v}$.
- Since u^f is the integral of f along a geodesic, the fundamental theorem of calculus gives
$$Xu^f(x,v) = -f(x)$$
for all $(x,v) \in SM$. This is the transport equation.

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- Since f integrates to zero over all geodesics, u^f is zero at $\partial(SM)$.

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If yes, then $If = 0$ implies $u^f = 0$. This means that $f = -Xu^f = 0$, so the X-ray transform is injective!

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- The lower order term $\frac{1}{2}X_\perp$ plays a role: if we change its sign, we *always* lose unique solvability.
- There is an energy identity for this PDE that allows us to deduce that solutions are unique on some manifolds.

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Lemma (Pestov identity)

Let M be a a compact, orientable Riemannian surface with boundary. If $u \in C^2(SM)$ with $u|_{\partial(SM)} = 0$, then

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The norms and inner products are those of $L^2(SM)$.

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Proof

Calculate $\|VXu\|^2 - \|XVu\|^2$ using the commutator relations and integration by parts. □

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A compact manifold M with boundary is simple if

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- any two boundary points are joined by a unique geodesic which depends smoothly on the endpoints.

A surface with non-positive curvature and strictly convex boundary is simple.

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A calculation using the Santaló formula gives

$$\|Xw\|^2 - \langle Kw, w\rangle = \int_{\partial_{\text{inward}}(SM)} \mathcal{I}_{\gamma_{x,v}}(w)\,|v\cdot\nu|\,\mathrm{d}x\mathrm{d}v,$$

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where $\mathcal{I}_\gamma$ is the index form on the geodesic γ. On a simple manifold there are no conjugate points, so the index forms are positive definite. □

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- The Pestov identity can also be used to prove other similar results.
- For the Pestov identity in 2D, see

  Paternain–Salo–Uhlmann (Invent. Math. 2013): Tensor tomography on surfaces

  and references therein. Several people have contributed to the topic: Mukhometov, Guillemin, Kazhdan, Sharafutdinov, Pestov, Uhlmann, Salo, Paternain. . .

Outline

1. X-ray imaging
2. X-ray tomography and manifolds
3. The Pestov identity
4. Various generalizations
   - Higher dimensions
   - Vector fields
   - Tensor fields
   - Pseudo-Riemannian manifolds
   - Closed manifolds
   - Broken rays
   - Rough metrics
5. Applications

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- In 2D we had $V^* = -V$ and $X_\perp^* = -X_\perp$, but in higher dimensions the vertical and horizontal gradients are different objects than the vertical and horizontal divergences.
- I will be very brief. For more details in higher dimensions, see

  Paternain–Salo–Uhlmann (Math. Ann. 2015): Invariant distributions, Beurling transforms and tensor tomography in higher dimensions

  and references therein.

Higher dimensions

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Lemma (Pestov identity)

Let M be a compact manifold with boundary, with $\dim(M) = n$. If $u \in C^2(SM)$ with $u|_{\partial(SM)} = 0$, then

$$\|VXu\|^2 = \|XVu\|^2 - \langle RVu, Vu\rangle + (n-1)\|Xu\|^2,$$

where R is an operator given by the Riemann curvature tensor. The norms and inner products are those of $L^2(SM)$.

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If M has non-positive sectional curvature, all terms are non-negative.

Higher dimensions

Theorem (Mukhometov 1977)

The X-ray transform is injective on all simple manifolds.

Vector fields

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- New question: If f is a vector field and integrates to zero over all geodesics on M, is there a function ϕ so that $\phi|_{\partial M} = 0$ and $f = \mathrm{d}\phi$? (Do the integrals of f determine f up to gauge?)
- Sometimes yes! At least on simple manifolds. A simple proof with the Pestov identity works again.

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- In dimensions $n \geq 3$ this is open. There are only partial results.

Pseudo-Riemannian manifolds

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- A pseudo-Riemannian metric of signature (n, m) is like the matrix

$$\begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

with n negative signs and m positive ones.

Pseudo-Riemannian manifolds

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Theorem (I., 2016)

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The proof is based on a Pestov identity. We assume that the signature (m, n) satisfies $m \geq 2$ and $n \geq 2$. No Pestov identity method is known in Lorentzian geometry ($m = 1$).

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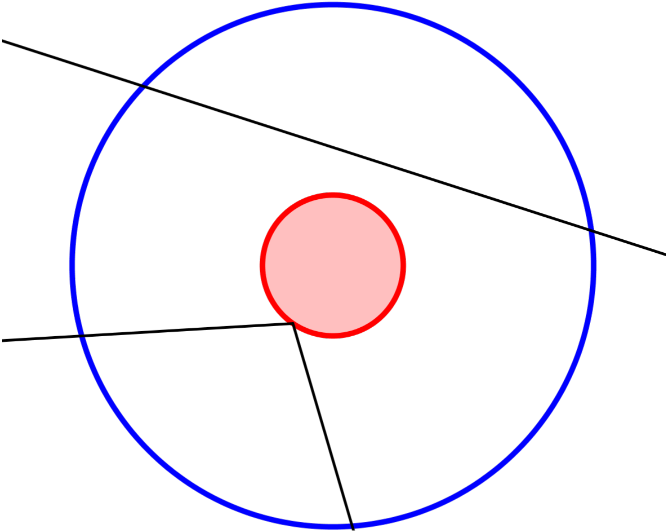
- The "closed analogue" of a simple surface is an Anosov surface.
- On an Anosov surface, if a symmetric tensor field f of order m integrates to zero over all geodesics, then $f = \mathrm{d}\phi$ for a symmetric tensor field ϕ of order $m-1$. (P–S–U 2014)

Broken rays

Broken rays



One reflecting obstacle in a domain. Two broken rays.

Broken rays

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Theorem (I.–Salo, 2016)

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Let M be a non-positively curved Riemannian surface with strictly convex boundary. Add a strictly convex reflecting obstacle. Then the broken ray transform is injective.

The same result is true higher dimensions as well, and in two dimensions in the absence of conjugate points along broken rays. (I.–Paternain)

Rough metrics

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Theorem (de Hoop–I.)

On a spherically symmetric non-trapping manifold with a piecewise $C^{1,1}$ metric the geodesic X-ray transform is injective on L^2 functions.

Outline

Linearized length

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- Do travel times of earthquakes which occur near the surface uniquely determine the interior structure of the Earth?
- We linearize this problem.

Linearized length

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Linearized length

- Let g_s be a family of Riemannian metrics on a manifold M. The "infinitesimal variation" $f = \frac{\mathrm{d}}{\mathrm{d}s} g_s \big|_{s=0}$ is a symmetric second order tensor field on M.
- Fix two points at the boundary: $x, y \in \partial M$. Let γ_s be the shortest geodesic joining x and y in the metric g_s. Then

$$\frac{\mathrm{d}}{\mathrm{d}s} \ell(\gamma_s) \bigg|_{s=0} = 2If(\gamma_0).$$

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- If we already know the conformal class of the manifold, the unknown function f can be considered scalar instead of a rank two tensor.
- Linearizing lengths of broken rays leads to broken ray tomography.

Rigidity of length spectrum

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- The length spectrum of a closed manifold is the set of lengths of periodic geodesics.
- The length spectrum of a manifold with boundary is the set of lengths of periodic broken rays.
- The length spectrum is said to be rigid if small variations preserving the length spectrum are necessarily trivial.
- Proofs of rigidity results typically provide an iterative reconstruction algorithm (without proof of convergence).

Rigidity of length spectrum

Theorem (Paternain–Salo–Uhlmann, 2014)

The length spectrum of every Anosov surface is rigid.

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The length spectrum of every Anosov surface is rigid.

Theorem (de Hoop–I.)

The length spectrum of every spherically symmetric non-trapping smooth manifold with boundary is rigid.

Rigidity of length spectrum

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Problem

Can we measure the length spectrum of the Earth?

Rigidity of length spectrum

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It can be done indirectly in spherical symmetry.

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- In the Riemannian model the corresponding data is the spectrum of the Laplace–Beltrami operator on the manifold.
- These two kinds of spectra are related to each other.

Rigidity of spectrum

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Theorem (de Hoop–I.)

The spectrum of the Laplace–Beltrami operator on a spherically symmetric non-trapping 3-dimensional Riemannian manifold uniquely determines the length spectrum.

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Corollary (de Hoop–I.)

The spectrum is rigid on such manifolds.

End

Thank you.

Slides are available at `http://users.jyu.fi/~jojapeil`.