Pestov identities for generalized X-ray transforms

100 Years of the Radon Transform
RICAM

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31 March 2017

Outline

1. Ray tomography on simple manifolds
   - The sphere bundle
   - From ray transform to transport equation
   - The Pestov identity
   - Consequences
2. Broken ray tomography
3. Pseudo-Riemannian manifolds

The sphere bundle

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- Two important derivatives on SM:
  - The geodesic vector field $X = v \cdot \nabla_x$ generates the geodesic flow (a dynamical system on SM).
  - The vertical gradient $\overset{\mathrm{v}}{\nabla}$ differentiates with respect to the direction variable v. (In 2D $\overset{\mathrm{v}}{\nabla}$ is the vertical vector field V.)

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- If $(x, v) \in SM$, denote by $\gamma_{x,v}$ the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.
- Let $\tau_{x,v} \geq 0$ be the exit time of the geodesic $\gamma_{x,v}$.

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- Since u^f is the integral of f along a geodesic, the fundamental theorem of calculus gives

$$Xu^f(x,v) = -f(x)$$

for all $(x,v) \in SM$. This is the transport equation.

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- Since f integrates to zero over all geodesics, u^f is zero at $\partial(SM)$.

From ray transform to transport equation

Problem

Does the second order PDE

$$\begin{cases} \overset{\mathrm{v}}{\nabla} Xu = 0 & \text{in } SM \\ u = 0 & \text{on } \partial(SM) \end{cases}$$

have a unique solution?

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have a unique solution?

If yes, then the X-ray transform is injective: it follows that $u^f = 0$ and thus $f = -Xu^f = 0$.

The Pestov identity

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Lemma (Pestov identity)

Let M be a a compact, orientable Riemannian manifold with boundary. If $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$, then

$$\left\| \overset{\mathrm{v}}{\nabla} Xu \right\|^2 = \left\| X \overset{\mathrm{v}}{\nabla} u \right\|^2 - \left\langle R \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u \right\rangle + (n-1) \left\| Xu \right\|^2 .$$

The norms and inner products are those of $L^2(SM)$ and R is an operator given by the Riemann curvature tensor.

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Proof

Calculate $\left\| \overset{\mathrm{v}}{\nabla} Xu \right\|^2 - \left\| X \overset{\mathrm{v}}{\nabla} u \right\|^2$ using integration by parts and commutator relations (2D): $[X,V] = X_\perp$, $[V, X_\perp] = X$, $[X, X_\perp] = -KV$. □

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- Apply the Pestov identity to u^f which satisfies $\overset{\mathrm{v}}{\nabla} X u^f = 0$:

$$0 = \left\| X \overset{\mathrm{v}}{\nabla} u^f \right\|^2 - \left\langle R \overset{\mathrm{v}}{\nabla} u^f, \overset{\mathrm{v}}{\nabla} u^f \right\rangle + (n-1) \left\| X u^f \right\|^2 .$$

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- If the manifold is simple, then

$$\left\| X \overset{\mathrm{v}}{\nabla} u \right\|^2 - \left\langle R \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u \right\rangle \geq 0$$

for all $u \in C^\infty(SM)$, and the same conclusion holds. [Mukhometov 1977, . . .]

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  for all $u \in C^\infty(SM)$, and the same conclusion holds. [Mukhometov 1977, . . .]
- The Pestov identity can also be used on non-compact manifolds. [Lehtonen, 2016]

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Analysis on the sphere bundle provides a convenient invariant framework for the analysis of ray transforms.

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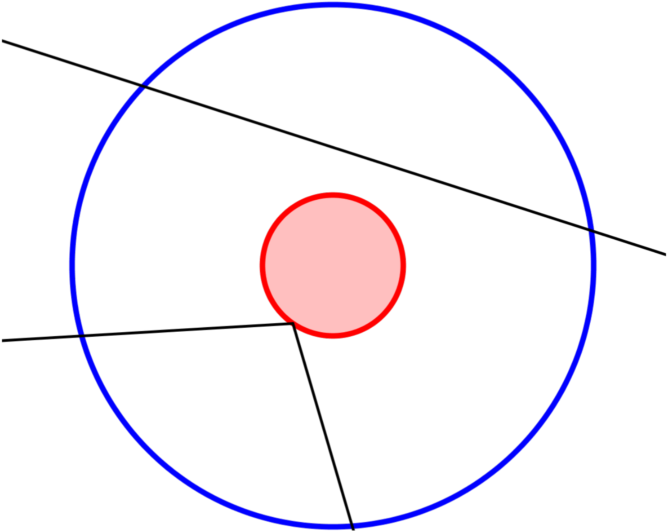
2. Broken ray tomography
   - The broken ray transform
   - Two dimensions
   - Higher dimensions
3. Pseudo-Riemannian manifolds

The broken ray transform



One reflecting obstacle in a domain. Two broken rays.

The broken ray transform

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Is a function determined by its integrals over all broken rays in the geometry of the previous slide?

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The broken ray transform can also mean other things; cf. Ambartsoumian's talk.

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- We define $u^f : SM \to \mathbb{R}$ as before, integrating along broken rays until they hit $\partial M \setminus R$.
- This u^f satisfies $u^f = u^f \circ \rho$ on SR.

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Let M be a a compact, orientable Riemannian surface with boundary. If $u \in C^\infty(SM)$ with $u|_{\partial(SM)\setminus SR} = 0$ and $u = u \circ \rho$ on SR, then

$$\|VXu\|^2 = \|XVu\|^2 - \langle KVu, Vu\rangle + \|Xu\|^2 - \int_{SR} \kappa(x)\,|Vu(x,v)|^2 .$$

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Technical problem: The function u^f is not a priori smooth at all!

Two dimensions

Theorem (I.–Salo, 2016)

Let M be a non-positively curved Riemannian surface with strictly convex boundary. Add a strictly convex reflecting obstacle. Then the broken ray transform is injective.

Higher dimensions

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- Pestov identity in HD without boundary term:

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- There is a similar boundary term in higher dimensions.

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3. Pseudo-Riemannian manifolds
   - Pseudo-Riemannian products
   - The Pestov identity
   - Lorentz geometry

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- A pseudo-Riemannian metric of signature (n_1, n_2) is like the matrix

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix},$$

with n_1 positive signs and n_2 negative ones.

Pseudo-Riemannian products

For two Riemannian manifolds (M_1, g_1) and (M_2, g_2) we can equip the product manifold $M_1 \times M_2$ with the Riemannian product metric $g_1 \oplus g_2$ or the pseudo-Riemannian product metric $g_1 \ominus g_2$.

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Theorem (I., 2016)

Let M_1 and M_2 be two Riemannian manifolds of non-negative sectional curvature, strictly convex boundary and dimension ≥ 2. Then the null geodesic X-ray transform (light ray transform) is injective on the pseudo-Riemannian product $(M_1 \times M_2, g_1 \ominus g_2)$.

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The proof is based on a Pestov identity.

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- We can use the natural operators on both sphere bundles: X_1, $\overset{\mathrm{v}}{\nabla}_2$, ...
- The null geodesic flow is generated by $X = X_1 + X_2$.

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Lemma (Pestov identity)

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If $u \colon LM \to \mathbb{R}$ *is smooth and vanishes at the boundary, then*

$$
\begin{aligned}
&(n_2-1)\left\|\overset{\mathrm{v}}{\nabla}_1 Xu\right\|^2+(n_1-1)\left\|\overset{\mathrm{v}}{\nabla}_2 Xu\right\|^2\\
&=(n_2-1)\left\|X\overset{\mathrm{v}}{\nabla}_1 u\right\|^2+(n_1-1)\left\|X\overset{\mathrm{v}}{\nabla}_2 u\right\|^2\\
&\quad-(n_2-1)\left\langle R_1\overset{\mathrm{v}}{\nabla}_1 u,\overset{\mathrm{v}}{\nabla}_1 u\right\rangle-(n_1-1)\left\langle R_2\overset{\mathrm{v}}{\nabla}_2 u,\overset{\mathrm{v}}{\nabla}_2 u\right\rangle\\
&\quad+(n_1-1)(n_2-1)\left\|Xu\right\|^2.
\end{aligned}
$$

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  - Real analytic Lorentz manifolds. [Stefanov, 2017]
  - Products $M \times \mathbb{R}$ where M has an injective Riemannian ray transform. [Oksanen–Kian, unpublished]
- For other Lorentzian and pseudo-Riemannian manifolds the problem is open.

End

Thank you.

Slides are available at `http://users.jyu.fi/~jojapeil`.