THE BROKEN RAY TRANSFORM

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ABSTRACT

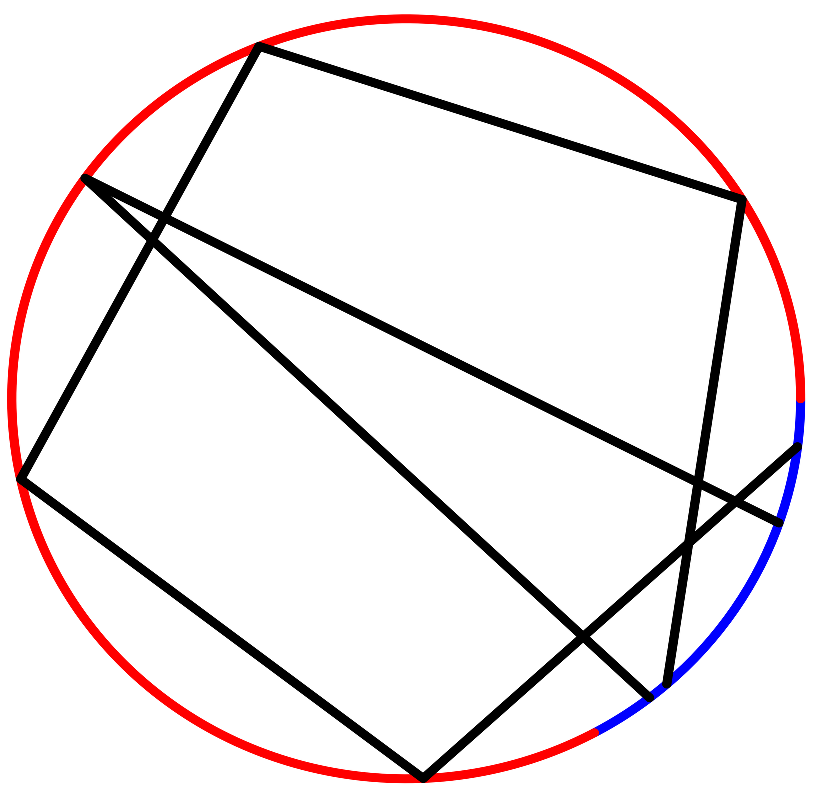
The broken ray transform is like the X-ray transform, but lines or geodesics are replaced with broken rays which reflect on a part of the boundary of the domain. The reflecting part of the boundary is inaccessible to measurements. The main question is: When is the broken ray transform injective? We describe different ways to approach this integral transform and present recent results on Euclidean domains and Riemannian manifolds.

SETTING

The domain M is a compact manifold with boundary (or a smooth Euclidean domain). Its boundary consists of two parts E and R. The set of tomography E is where rays are sent and received (**BLUE**). Rays reflect on the reflector R (**RED**) and are geodesics inside M. Such rays are called broken rays (**BLACK**).

Question: Is a real valued function f on M determined by its broken ray transform?

Examples of affirmative answers are given below in different situations and using different methods.



BROKEN RAY TRANSFORM IN A DISC

Let M be the closed Euclidean unit disc (depicted above). Then a function f on M can be written conveniently in polar coordinates.

Theorem [1]: If E is any open subset of the boundary of the disc and f satisfies an analyticity condition in the angular variable, then f is determined by its integrals over broken rays.

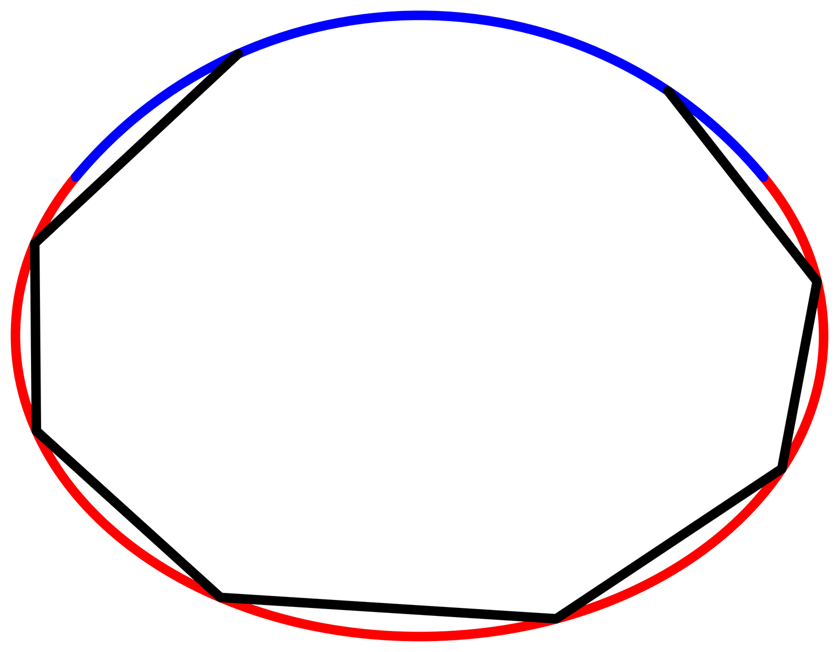
Corollary [1]: The same is true for an Euclidean ball in higher dimensions as well.

About the proof: The geodesics can be parametrized by angles and integers (two angles and one integer suffice). With this parametrization and Fourier decomposition of the function in the angular variable, the broken ray transform can be calculated explicitly. Then it remains to invert certain Abel-type integral transforms in one variable (the radial variable). These integral transforms appear in Cormack's (1963) original inversion method for the planar Radon transform.

BOUNDARY RECONSTRUCTION

Take a sequence of broken rays starting more and more tangentially to the boundary (depicted below). If the boundary is strictly convex, then the C^1 uniform limit of these broken rays is a geodesic on the boundary ∂M.

If we know the integral of a function over all broken rays, then we also know its integral over all boundary geodesics. With more delicate analysis similar results can be proven for the normal derivatives of the function.



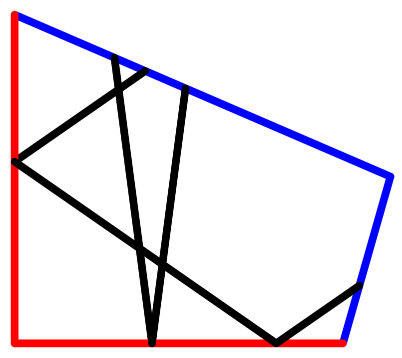
Theorem [2]: Suppose M is strictly convex and smooth and we know the integral of a smooth function f on M over all broken rays. Then for any boundary geodesic σ we can reconstruct the integral of f over σ. Moreover, for any natural number k we can reconstruct the integral of kth order normal derivative of f over σ weighted with the second fundamental form to the power $-k/3$.

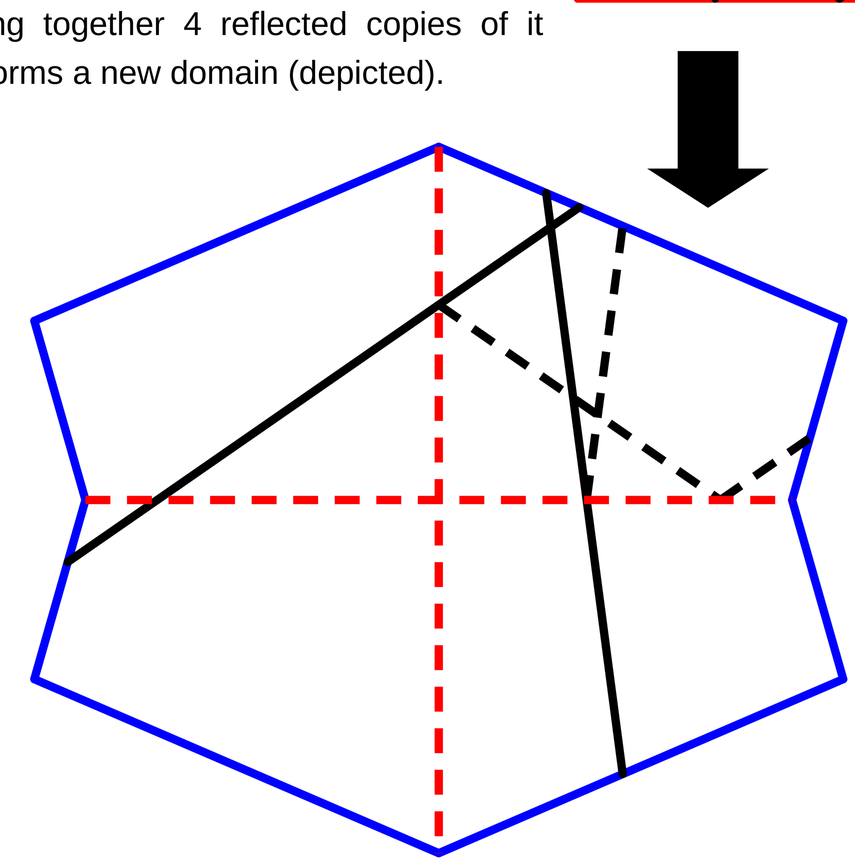
Corollary [2]: If the X-ray transform is injective on the manifold ∂M (or more properly $\partial M \setminus E$) weighted with any power of the second fundamental form, then one can reconstruct the Taylor polynomial of a function on M at all boundary points from its broken ray transform.

If the first and second fundamental forms are conformally equivalent (as is the case for spheres), then the weighted X-ray transforms are injective if the usual X-ray transform is.

REFLECTED RAYS OR REFLECTED DOMAIN?

One can make the problem easier by removing reflections, but this requires reflecting the domain. For domains that are easy to reflect, trading reflected rays for a reflected domain makes it relatively easy to show injectivity. If the reflector has a 90° angle, then gluing together 4 reflected copies of it forms a new domain (depicted).





Let the originan domain and function be M and f. The reflected domain is called $\tilde{M}$ and the reflected function $\tilde{f}$. (The map $\tilde{f}$ is the pullback of f under the natural folding map $\tilde{M} \to M$.) If we know the broken ray transform of f, then we also know the X-ray transform of $\tilde{f}$.

Theorem [3]: If the X-ray transform is injective on the reflected domain $\tilde{M}$, then the broken ray transform is injective on M. This holds true if M is a manifold and the transform is weighted.

Theorem [3]: The broken ray transform is injective in any Euclidean domain when the reflector R is contained in a cone.

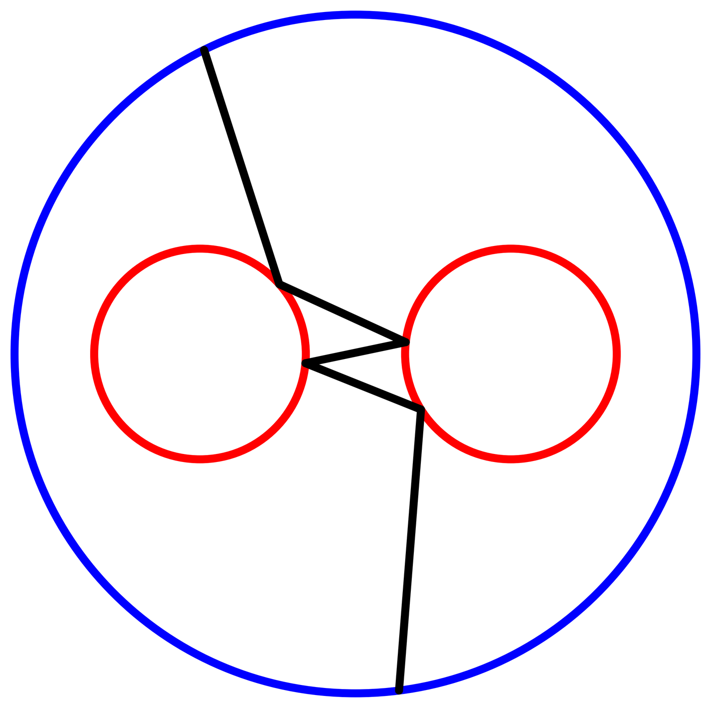
The proof of the second theorem uses Helgason's support theorem when the cone angle is not π/m for integer m.

PERIODIC BROKEN RAY TRANSFORM

If all of the boundary is reflective, broken rays are naturally replaced with periodic broken rays. Using a reflection argument like before, the periodic broken ray transform on a cube can be reduced to the periodic X-ray transform on a torus.

Theorem [4]: The periodic X-ray transform is injective on flat tori. This is true also for tensor (s-injectivity) fields with distribution-valued coefficients.

Corollary [4]: The periodic broken ray transform is injective in every Euclidean cube.



CONVEX OBSTACLES

One can also consider the broken ray transform in a domain with convex reflecting obstacles (depicted above). Injectivity of the broken ray transform is reduced to showing that a PDE on the sphere bundle has a unique solution. Uniqueness is proven by using a Pestov identity. The case of one obstacle was covered in [5], and we are working on two obstacles.

Theorem [5]: Let M be a nonpositively curved Riemann surface with convex boundary with one convex obstacle removed. Then the broken ray transform is injective.

The PDE: Let (x,v) be a point on the sphere bundle SM. For a function f on M, define the function u^f by integrating f along the broken ray starting at (x,v). If X is the geodesic vector field and V is the vertical derivative (w.r.t. the angle of v), then $Xu^f = -f$ and thus $VXu^f = 0$. If the broken ray transform of f vanishes, then u^f satisfies boundary conditions which imply that $u^f = 0$.

APPLICATIONS

Eskin (2004) reduced an inverse boundary value problem with partial data for the electromagnetic Schrödinger equation to injectivity of the broken ray transform. Similarly, Kenig and Salo (2012) reduced Calderón's problem with partial data on a tubular domain to injectivity of the broken ray transform on the transversal domain. The broken ray transform also arises as a linearization of a boundary rigidity type problem with reflections [5].

OTHER MEANINGS OF THE BROKEN RAY TRANSFORM

The expression 'broken ray' has also been used in other meanings. In the V-line Radon transform the rays reflect from interior points by a fixed angle. In other applications the rays may reflect from all interior points to all directions. The ray may also split when it reflects. References can be found in the articles below.

References:

[1] J. Ilmavirta, **Broken ray tomography in the disc**, *Inverse Problems* 29 (2013).

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[3] J. Ilmavirta, **A reflection approach to the broken ray transform**, *Mathematica Scandinavica*, to appear.

[4] J. Ilmavirta, **On Radon transforms on tori**, preprint (2014).

[5] J. Ilmavirta, M. Salo, **Broken ray transform on a Riemann surface with a convex obstacle**, preprint (2014).