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Spectral Geometry for the Earth

SIAM Conference on Mathematics of Planet Earth

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September 14, 2018

Based on joint work with

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Prelude

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- What can one tell about the Earth just by the spectrum of its free oscillations?
- This is an inverse spectral problem. A hard one.
- There is a weaker and more tractable version of the spectral problem: the spectral rigidity problem.

Outline

1. Geometry
   - The Earth as a geometrical object
   - Spherically symmetric manifolds
   - The Herglotz condition
2. Seismic spectral data
3. Different forms of uniqueness
4. The main results
5. Anisotropy and geometry
6. Appendix

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- Fermat's principle: A seismic wave path corresponds to a geodesic in the elastic geometry.
- In isotropic or elliptically anisotropic medium this "Elastic geometry" is Riemannian geometry. (In general anisotropy something else is needed.)
- For a geometer, the problem of finding the interior structure of the Earth or another planet is a problem of finding a Riemannian manifold.

Spherically symmetric manifolds

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- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n = 2$) diffeomorphism $\phi\colon M \to M$ so that $\phi^* g$ is radially conformally Euclidean.

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Definition

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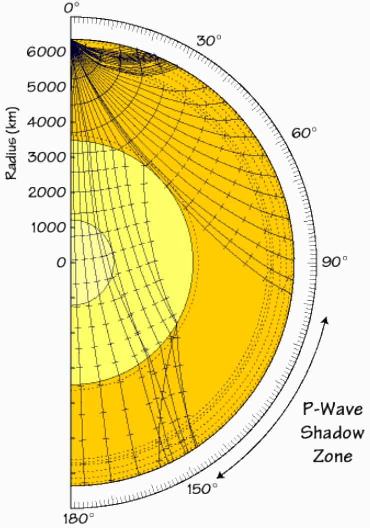
Perhaps the condition automatically holds for a spherically symmetric planet in dynamical equilibrium?

The Herglotz condition



The Herglotz condition is satisfied: ray paths curve outwards. (Wikimedia Commons)

Outline

1. Geometry
2. Seismic spectral data
   - The spectrum of free oscillations
   - The spectrum of periodic orbits
   - The goal
3. Different forms of uniqueness
4. The main results
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- The set of these frequencies (with multiplicity) is the spectrum of free oscillations.
- About 10 000 first frequencies are known.

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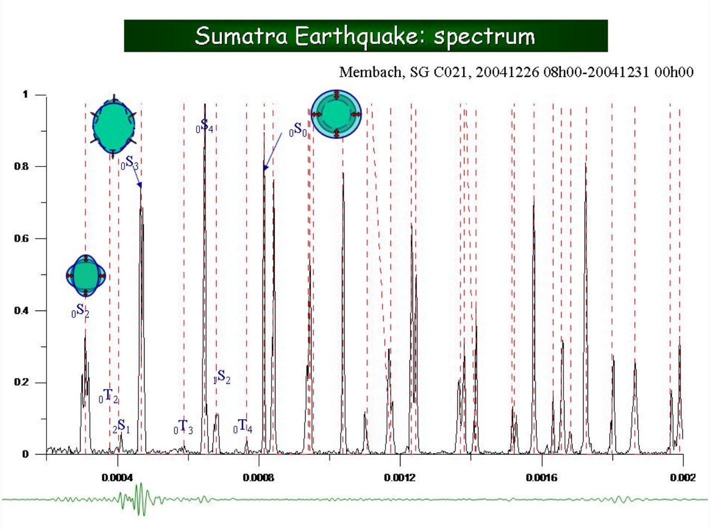
- The spectrum of free oscillations is the Neumann spectrum of the Laplace–Beltrami operator Δ_g.

The spectrum of free oscillations



Spectrum of free oscillations from an earthquake.

The spectrum of periodic orbits

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- The set of all lengths of periodic seismic wave paths is the "length spectrum" of the Earth.
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- Geometrically, the length spectrum of (M, g) is the set of all lengths of the periodic broken rays.

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Given the Neumann spectrum of Laplace–Beltrami operator or the length spectrum of a Riemannian manifold with boundary, reconstruct the manifold.

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3. Different forms of uniqueness
   - Difficulties
   - Diffeomorphisms and coordinates
   - Hearing the shape of a drum
   - Global uniqueness
   - Local uniqueness
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  2. The conjecture is false.

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- If $\phi\colon M \to M$ is a diffeomorphism, then (M, g) and $(M, \phi^* g)$ give the same spectrum.
- We can take any change of coordinates whatsoever and use it to distort the metric, but the spectrum stays the same.
- Physically: There are preferred and natural Cartesian coordinates. But the anisotropic model is not “sensitive to the underlying Euclidean geometry”, so the Cartesian coordinates cannot be recognized. It is impossible to find the metric (elliptically anisotropic sound speed) in Cartesian coordinates from spectral data.

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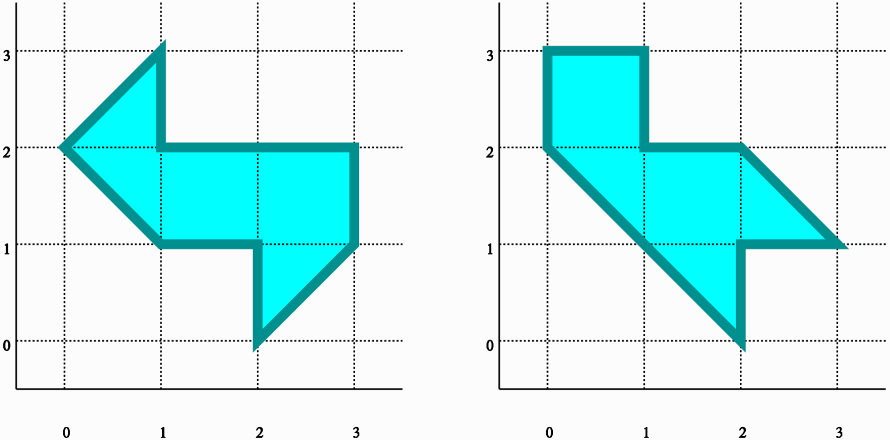
- . . . but it is non-trivially "No!" if there are no geometrical restrictions.

Hearing the shape of a drum



These two drums sound exactly alike. (Wikimedia Commons)

Global uniqueness

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Let g_1 and g_2 be two Riemannian metrics on a manifold M with boundary. If they give the same spectrum, is there a diffeomorphism $\phi\colon M \to M$ so that $g_1 = \phi^* g_2$?

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This is too hard.

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This is still too hard.

Spectral rigidity

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Let g_s *be family of Riemannian metrics on a manifold* M *with boundary, depending on a parameter* $s \in (-\varepsilon, \varepsilon)$. *If they all give the same spectrum, are there a diffeomorphisms* $\phi_s \colon M \to M$ *so that* $g_0 = \phi_s^* g_s$*?*

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This is within reach!

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  - Negatively curved surfaces: Guillemin–Kazhdan 1980.
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- We have adapted similar ideas of proof to manifolds with boundary.

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   - Length spectral rigidity
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Let M be the closed unit ball in $\mathbb{R}^3$. Let $c_s(r)$ be a family of radial sound speeds depending C^∞-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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Let M be the closed unit ball in $\mathbb{R}^n$, $n \geq 2$. Let $c_s(r)$ be a family of radial sound speeds depending $C^{1,1}$-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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This simple model of the round Earth is length spectrally rigid!

Length spectral rigidity

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Both results hold for rotation invariant metrics g_s satisfying similar conditions. The conclusion is that there is a family of radial (or more general if $n = 2$) diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

Ideas behind the proof

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Lemma (Trace formula)

Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the positive eigenvalues of the Laplace–Beltrami operator. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \sum_{k=0}^{\infty} \cos\left(\sqrt{\lambda_k} \cdot t\right).$$

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In particular, the spectrum determines the length spectrum.

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Similar “trace formulas” and related results are known on closed manifolds (eg. Duistermaat–Guillemin 1975) and a weaker version on some manifolds with boundary (eg. Guillemin–Melrose 1979).

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Corollary

Spectral rigidity follows from length spectral rigidity.

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In particular, if the length spectrum does not depend on s, then $\frac{\mathrm{d}}{\mathrm{d}s} c_s^{-2}$ integrates to zero over (almost) all periodic broken rays.

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This concludes the proof.

Remark: No proof works without spherical symmetry.

Recap

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- See where f has singularities. The set of singularities is (more or less) the length spectrum.
- Linearized length spectral data is periodic broken ray transform data.
- The periodic broken ray transform can be inverted explicitly for radial functions using an Abel transform.

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   - Elliptic and general elastic anisotropy
   - Pressure and shear waves
   - Anisotropy and coordinates
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  - Riemannian manifolds are a very special subclass of Finsler manifolds.
- A material is isotropic if sound speed is independent of direction. This can be modeled by a conformally Euclidean metric.

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- To model elastic waves in general anisotropy, one needs a manifold with two Finsler metrics, one for pressure and one for shear waves.
- In fact, the shear wave speed might not even by a Finsler metric in the traditional sense.

Anisotropy and coordinates

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- Let $\phi\colon M \to M$ be a diffeomorphism of a manifold that keeps the boundary fixed.
- If g (or F) is a Riemannian (or Finsler) metric on M, then the pullback $\phi^* g$ (or $\phi^* F$) is different Riemannian metric that behaves exactly the same for boundary measurements.

Anisotropy and coordinates

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- If g (or F) is a Riemannian (or Finsler) metric on M, then the pullback $\phi^* g$ (or $\phi^* F$) is different Riemannian metric that behaves exactly the same for boundary measurements.
- The unknown geometry can *never* be reconstructed from boundary measurements uniquely. The data is always invariant under changes of coordinates.

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- The best one can hope for is reconstruction up to changes of coordinates.

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Outline

1. Geometry
2. Seismic spectral data
3. Different forms of uniqueness
4. The main results
5. Anisotropy and geometry
6. Appendix
   - A numerical example of the trace formula
   - X-ray transform
   - Periodic broken ray transform

A numerical example of the trace formula

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   This is the trace of the operator $\cos(\sqrt{-\Delta} \cdot t)$.

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3. Linearized length spectral data is periodic broken ray transform data.
4. The periodic broken ray transform can be inverted explicitly for radial functions.

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Suppose $L = \frac{1}{2}$ and we have measured the numbers $0, 4\pi^2, 16\pi^2, \ldots$

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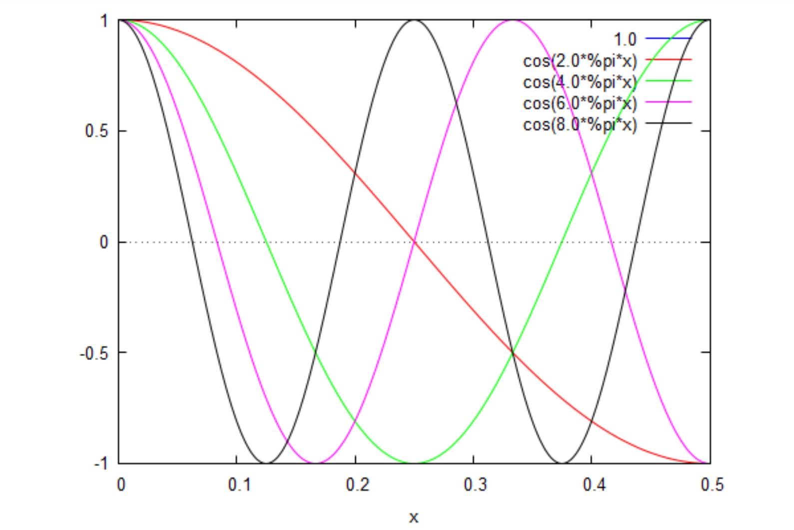
We compute and plot the trace function

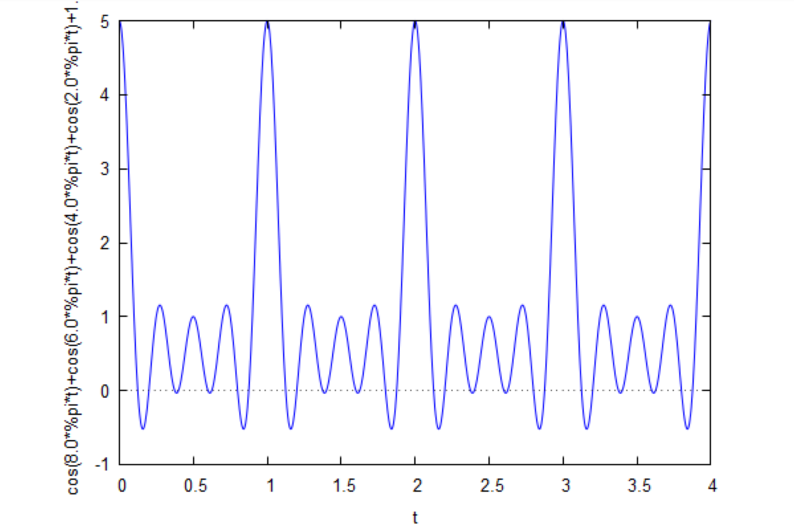
$$f(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k} \cdot t).$$

A numerical example of the trace formula



Eigenfunctions for $k = 0, 1, 2, 3, 4$.

A numerical example of the trace formula



Trace function computed from $k = 0, 1, 2, 3, 4$.

X-ray transform

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Theorem (de Hoop–I., 2017)

Let M be a rotation symmetric non-trapping manifold with a piecewise $C^{1,1}$ metric and strictly convex boundary. Then the geodesic X-ray transform is injective on $L^2(M)$.

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Earlier similar results:

- The X-ray transform (Radon et al.): Euclidean metric (c is constant).
- Mukhometov, 1977: Smooth simple metrics (simplicity is stronger than Herglotz).
- Sharafutdinov, 1997: C^∞ metrics and C^∞ functions.

Periodic broken ray transform

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Theorem (de Hoop–I., 2017)

Let M be a rotation symmetric non-trapping manifold with a $C^{1,1}$ metric and strictly convex boundary and dimension at least three. Assume that there are not too many conjugate points at the boundary. The integrals of a function $f \in L^p(M)$, $p > 3$, over all periodic broken rays determines the even part of the function.

Very little can be recovered of the odd part.

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Tools used:

- Planar average ray transform.
- Abel transform.
- Funk transform.
- Fourier series.

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