JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Towards a mathematical theory of seismic tomography on Mars

Oberwolfach workshop

Tomographic Inverse Problems: Theory and Applications

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Based on joint work with

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Finnish Centre of Excellence in Inverse Modelling and Imaging

2018-2025

CENTRES OF EXCELLENCE IN RESEARCH

UNIVERSITY OF OULU

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Conference announcement

The annual inverse problems conference “Inverese Days” will be organized in Jyväskylä this year.

Preliminary dates: 16–18 December, 2019.

All kinds of inverse problems in all fields are welcome!

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- Assume perfect measurements from a single ideal seismometer. What can you say for sure and is there an inversion algorithm?
- Identifying useful data sets can help future mission planning.
- Grand goal: A mathematical theory of seismic planetary exploration.

Outline

1. Seeing the radial Martian mantle with InSight
2. Seeing an entire planet

A small but reliable step

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- I will ignore noise, model errors, finiteness, stability, and many other practical things.

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- The set of all periodic travel times is the length spectrum.

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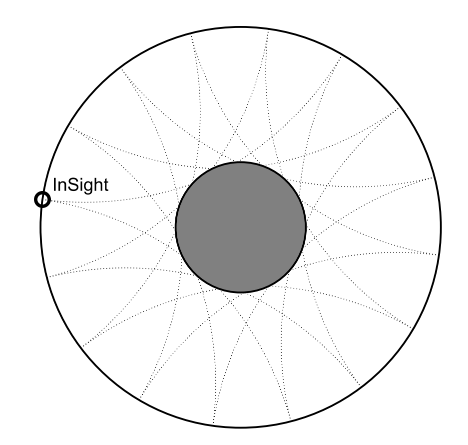
- Linearized data: Pairs of periodic broken rays and integrals over them. Uknown: Variations of wave speed (a function).

Method A: Linearized travel time tomography



Periodic seismic ray reflecting on the surface and CMB.

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Solving the linearized problem gives an iterative algorithm to solve the nonlinear one.
(Uniqueness should be provable for the non-linear one, too.)

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- The different modes are excited differently in different events, but one thing remains: the set of frequencies — the spectrum of free oscillations. (We are at first interested in properties of the planet, not properties of the events.)
- The spectrum of free oscillations can be measured from any single point.

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- Again wave speed $=$ geometry!

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Question

If a family of wave speeds $c_s(r)$ have the same spectrum, are the equal? Is the (Martian) mantle spectrally rigid?

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This simple model of the round Martian mantle is spectrally rigid!

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Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the positive eigenvalues of the Laplace–Beltrami operator. Define a function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \sum_{k=0}^{\infty} \cos\left(\sqrt{\lambda_k} \cdot t\right).$$

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In particular, the spectrum determines the length spectrum. It suffices to prove length spectral rigidity.

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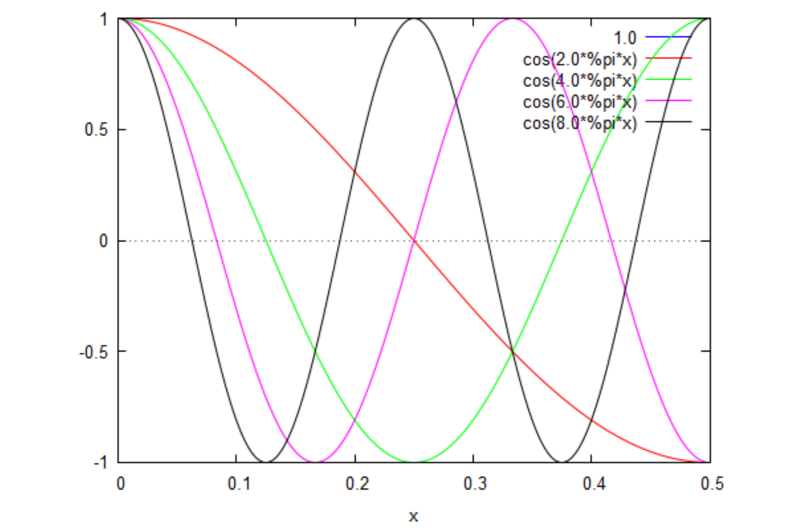

Neumann eigenfunctions for the interval $[0, \frac{1}{2}]$ with $k = 0, 1, 2, 3, 4$.
The length spectrum is $\mathbb{Z}$.

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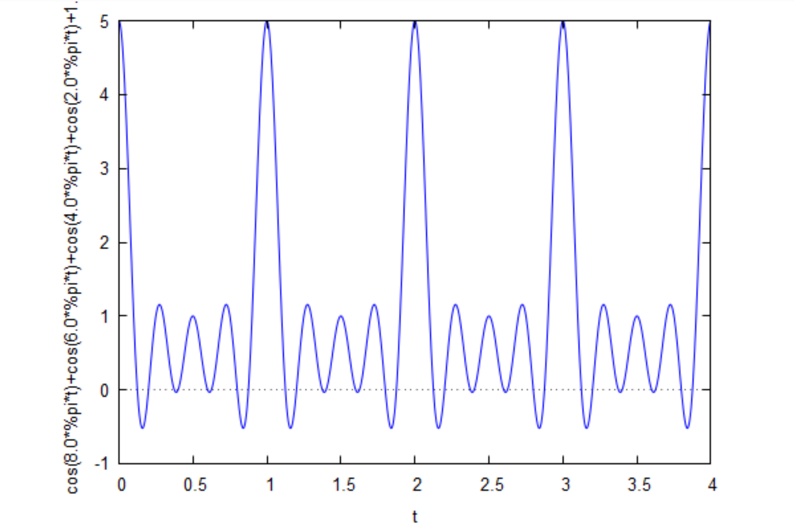

Trace function $f(t) = \sum_k \cos\left(\sqrt{\lambda_k} \cdot t\right)$ computed from $k = 0, 1, 2, 3, 4$.

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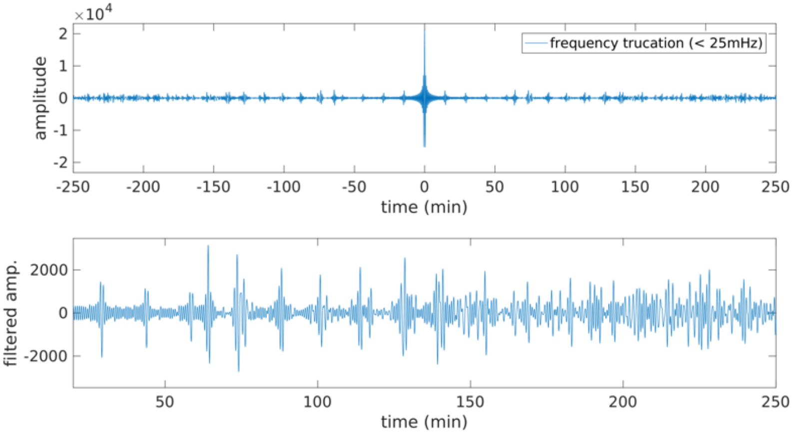

The trace computed from the spectrum of free oscillations in PREM. Singularities are visible.

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The data set is independent although the method is related.

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- Surface waves will come from the event to InSight two ways along the great circle containing the impact site and InSight.
- If there are no other events on the same great circle around the same time, we can measure the time difference δ.
- Multiple arrivals or a priori information tells the time T around the great circle.

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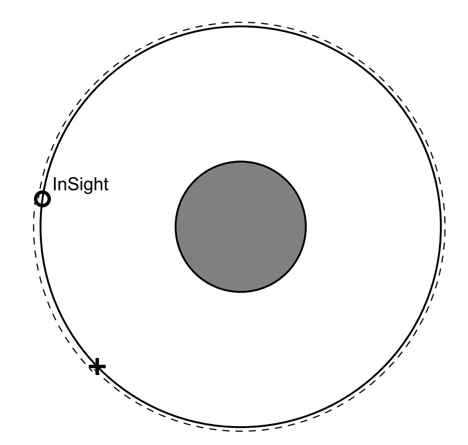

Two surface wave arrivals from the same event.

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- This was all done on surface, and it gives rise to interior data: Now using body waves we know the travel time between InSight and the source.
- To get here, we needed to assume spherical symmetry only on the surface, but the arising problem is easiest to solve if the symmetry extends inside.

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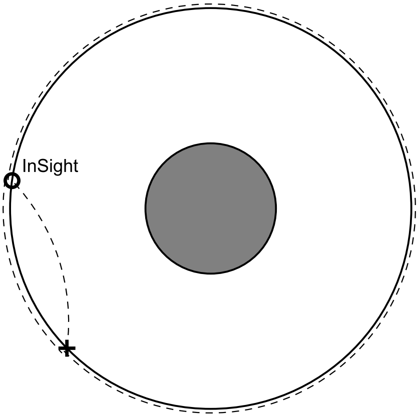

The body wave whose initial point and time were located with surface waves.

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- The linearized problem is X-ray tomography (or an Abel transform), and can also be solved explicitly. (e.g. de Hoop–I., 2017)

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- If the three reconstructions all work and give similar results, we can be quite confident.
- This gives us an isotropic radially symmetric reference model of the mantle, which is a stepping stone towards deeper and finer structure.

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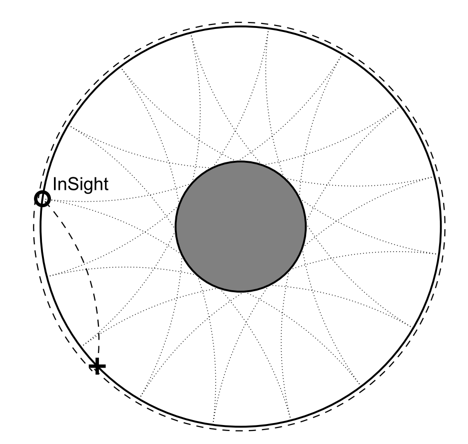

Three ways to see the mantle from InSight.

Summary

- A: From noise correlations to (linearized) travel times.
- B: From spectrum to length spectrum.
- C: Meteorites; body wave data calibrated by surface waves.

Outline

Seeing an entire planet

One would of course like to see more than just a radial mantle, but the theorems end here.

Spectral perturbation theory

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- Proving precise results outside spherical symmetry with one measurement point is hard.
- A natural approach to small lateral inhomogeneities is perturbation theory with respect to to a spherically symmetric reference model.
- In a simple (scalar) model, the medium is described by a single wave speed $c(x)$ and the spectrum depends on it: $Sp(c)$.
- We write the wave speed as a function of a parameter, $c_s(x)$, and expand the spectrum in s:

$$Sp(c_s) = Sp(c_0) + sL(\delta c) + \mathcal{O}(s^2),$$

where $\delta c = \frac{\mathrm{d}}{\mathrm{d}s} c_s|_{s=0}$, L is the Gâteaux derivative of the spectrum, and '$+$' is roughly a plus.

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- On Mars, c_0 would be the radial reference model.
- The better the radial (or other initial) guess is, the better the perturbation theory works.
- The perturbation δc can be expanded in spherical harmonics and the operator L can be written fairly explicitly.

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First observations:

- There is freedom to rotate.

Spectral perturbation theory

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If c_0 is radial and satisfies the Herglotz condition, how uniquely does $L(\delta c)$ determine δc?

First observations:

- There is freedom to rotate.
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First observations:

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It is best to start with a scalar model in 2D, not a fully polarized 3D model.

Half-local X-ray tomography

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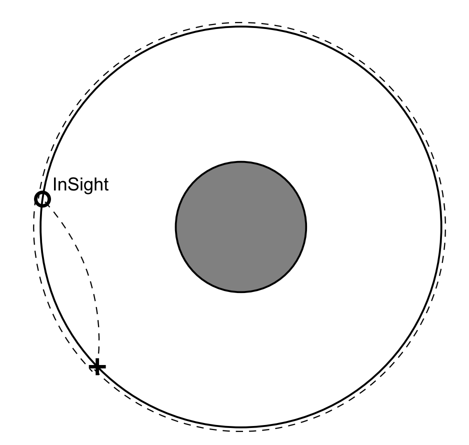
- We assumed that the surface is spherically symmetric (or otherwise known), but we needed no assumption on the interior.
- This leads to travel time data: The travel times (geometrically: distances) are known from all points on the surface to a single fixed point.

Half-local X-ray tomography



The body wave whose initial point and time were located with surface waves.

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Let M be a Riemannian (or Finsler) manifold with boundary. Is the metric uniquely determined by the distances between a fixed boundary point and all other boundary points?

Half-local X-ray tomography

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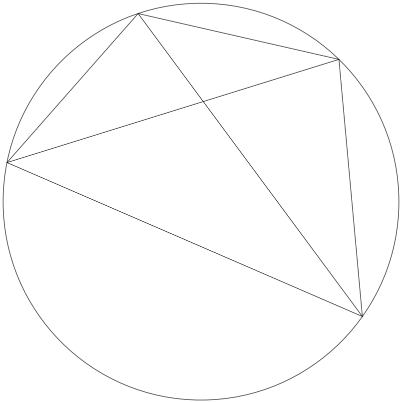
Question

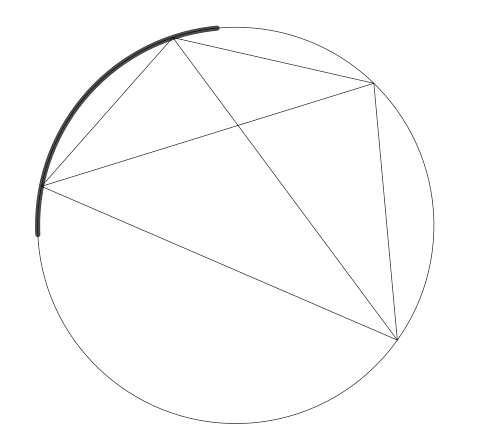
What if the point is replaced by a small open set — a detector array?

Half-local X-ray tomography



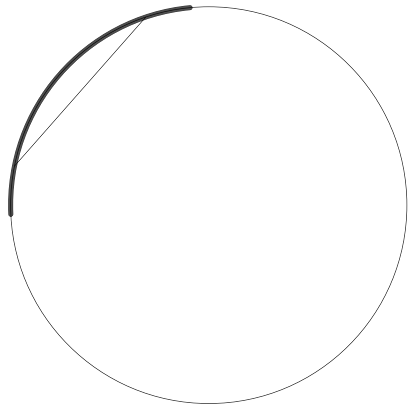
Boundary distance rigidity: Do the distances between all boundary points determine the geometry?

Half-local X-ray tomography



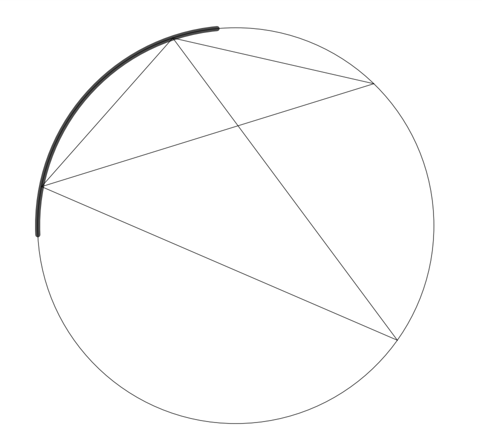
We have an accessible region — a measurement array.
The size is exaggerated.

Half-local X-ray tomography



In the local boundary distance problem one knows the distances between the points in the small set and wants to find the geometry near that set.

Half-local X-ray tomography



The “half-local” boundary distance data has more information and one wants to reconstruct the whole geometry.

Half-local X-ray tomography

Question

Let M be a Riemannian (or Finsler) manifold with boundary. Does the half-local boundary distance data for any open subset $U \subset \partial M$ determine the manifold uniquely?

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This is possible in Euclidean geometry or with real analytic perturbations but always unstable.

Sources and receivers

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- What is the minimal number of measurement points for uniqueness?

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- Planets like Earth and Mars have layers.
- Most geometrical inverse problems work with smooth manifolds. How to add conormal singularities and finite interior regularity?
- How does spectral rigidity and X-ray tomography work in an onion?

Geometrization

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- The speeds can be encoded as geometry where distance is time and geodesics are seismic rays. What is the correct geometrical structure exactly?
- In a strongly anisotropic medium Riemannian geometry is not enough, but we need Finsler.
- ...and even Finsler is not enough for all polarizations.

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Question

How does X-ray tomography change when Anosov flow is replaced by dispersing billiards?

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