JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

The geometry of anisotropy

Math + X symposium
on inverse problems and deep learning,
mitigating natural hazards

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Based on joint work with
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Finnish Centre of Excellence in Inverse Modelling and Imaging

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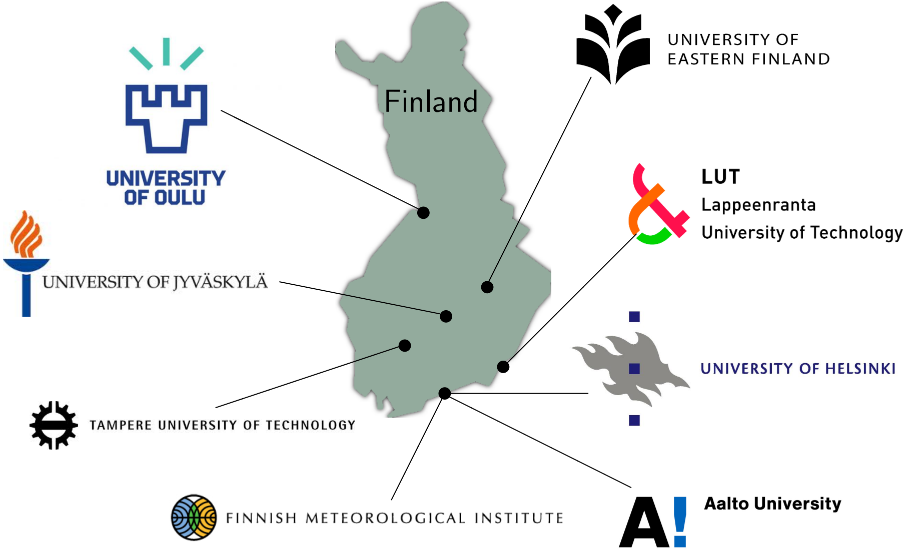
CENTRES OF EXCELLENCE IN RESEARCH



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  - Describing a geometric way to view anisotropy.

Outline

Newton's theory

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- The gravitational force exerted by the Sun causes the Earth's trajectory to curve.
- The force is described by a simple formula and the equation of motion is an ODE in $\mathbb{R}^n$.
- The Newtonian approach is straightforward to use and often a good model.

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- There is a relatively simple equation of motion for the planet: The geodesic equation is a non-linear ODE.
- There is a complicated equation of motion for the geometry itself: Einstein's field equation is a non-linear system of coupled PDEs.
- This model is harder to use but can reach phenomena inaccessible to Newtonian gravity and provides a more geometric way to see the essential structures.

The goal

A geometric theory of elasticity?

Outline

1. Gravitation
2. Elastic geometry
   - Distance
   - Ray tracing
   - Anisotropy
   - Manifolds to model anisotropy
   - Inverse problems
3. Examples

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- Distance is measured in units of time.

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  - Newer view: The particle goes straight in a curved geometry (geodesic), and the geometry is curved by variations in wave speed.

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- Fermat’s principle: The “particles” corresponding to elastic waves go straight in the geometry given by travel time.
- Fermat’s principle is about going straight in the relevant geometry, not about taking the shortest path. These are not the same thing over long distances or shear waves.

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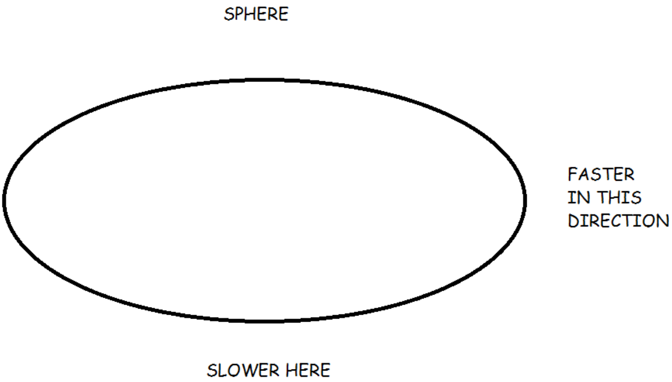

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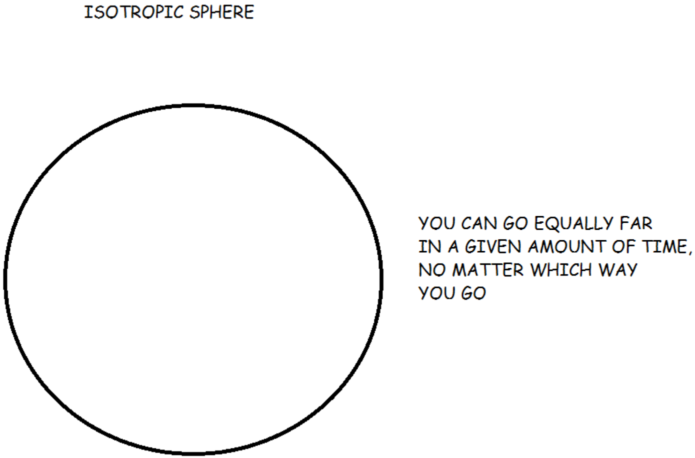

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- The cosphere (the slowness surface) describes the reciprocal of phase velocity.

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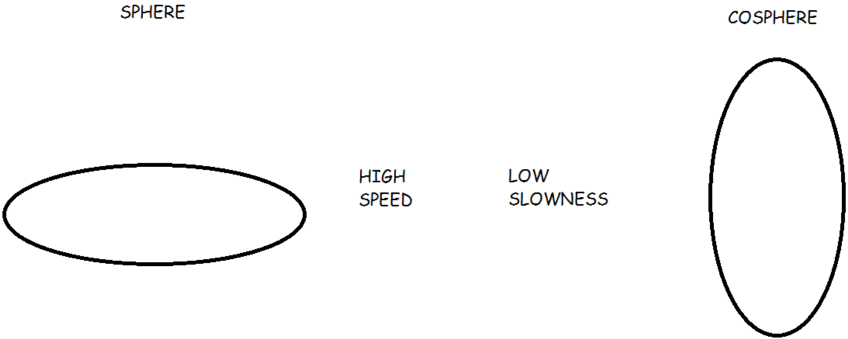
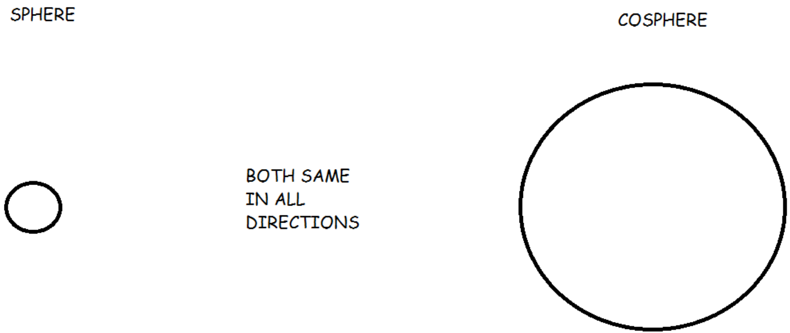


Sphere and cosphere, anisotropic.

Anisotropy



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- Elliptic anisotropy = the sphere and cosphere are ellipsoids.

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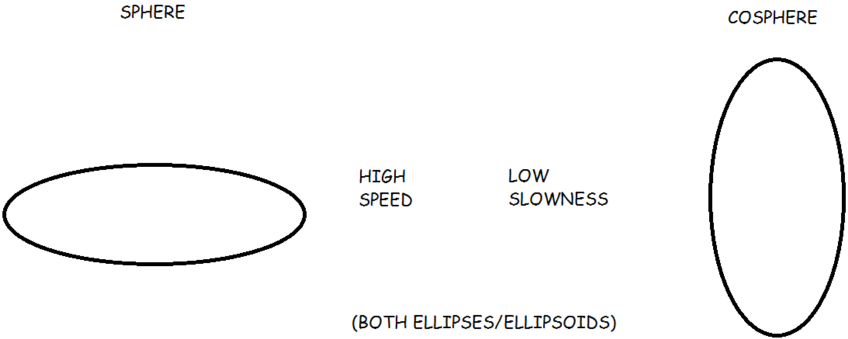

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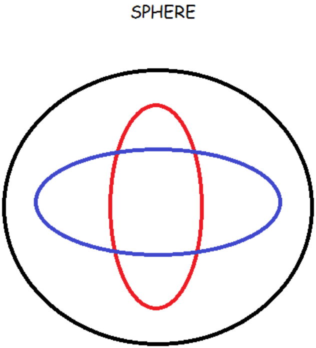

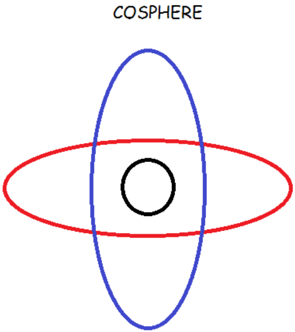

Three polarizations, all elliptically anisotropic.

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- If the cosphere is not convex, the sphere can branch.

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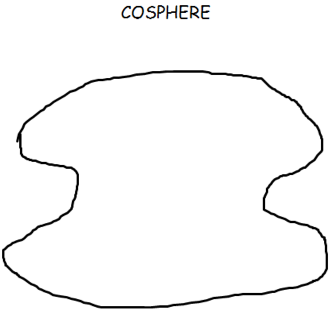
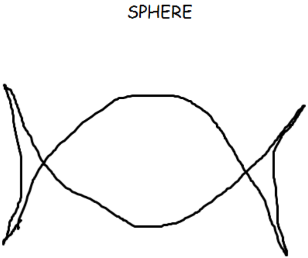
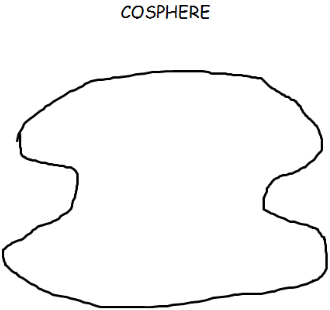


A non-convex cosphere.

Anisotropy





A branched sphere.

Manifolds to model anisotropy

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Multiple metric structures on the same manifold: Each polarization has its own geometry and there is the Euclidean spatial geometry.

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- Elastic Finsler geometry has a decent balance between tractability and applicability.

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- From the slowness surface one can then find the material parameters — the components of the stiffness tensor.

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- From the cosphere bundle one can tell whether the material is isotropic, elliptically anisotropic, completely anisotropic, or even fails to correspond to a stiffness tensor.

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3. Examples
   - Distance function (de Hoop, Lassas, Saksala)
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   - Ray tracing (Iversen, Ursin, Saksala, de Hoop)

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- If F is fiberwise real analytic (elasticity or Riemann!), then F is determined uniquely.

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- This broken scattering relation can see much more of TM, but the trapped set is still invisible.
- Global uniqueness is can be done with added assumptions: reversibility (point symmetry) and foliation.
- Almost no assumptions are needed in the Riemannian case (Kurylev–Lassas–Uhlmann, 2010).

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- Variations in position (Q) and momentum (P) satisfy an equation

$$\partial_t \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} W^T(t) & V(t) \\ -U(t) & -W(t) \end{pmatrix} \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}.$$

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- Variations in position (Q) and momentum (P) satisfy an equation

$$\partial_t \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} W^T(t) & V(t) \\ -U(t) & -W(t) \end{pmatrix} \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}.$$

- Written in terms of a Jacobi field J and its covariant derivative, we have instead

$$D_t \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -R(t) & 0 \end{pmatrix} \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix}.$$

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This approach can hopefully give you:

- A new way to think about anisotropy.
- A new way to encode anisotropy in modeling and computation.

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Outline

The stiffness tensor

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- The tensor is very symmetric ($c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$) and quite positive ($c_{ijkl}\alpha_i\beta_j\beta_k\alpha_l \gtrsim |\alpha|^2 |\beta|^2$).

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- We will also encounter the density normalized stiffness tensor $a_{ijkl}(x) = c_{ijkl}(x)/\rho(x)$.

The elastic wave equation

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- Using Newton's second law with a restoring force given by Hooke's law leads to the elastic wave equation (EWE)

$$\partial_j[c_{ijkl}(x)\partial_k u_l(x,t)] - \rho(x)\partial_t^2 u_i(x,t) = 0,$$

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- If the material is anisotropic (c is no more symmetric than necessary), then the vector nature of the equation cannot be ignored.
- Elastic waves arising from earthquakes satisfy this equation away from the focus of the event to great accuracy.

The principal symbol

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- The principal symbol of the EWE is $\Gamma(x,\xi) - \omega^2 I$, where $\xi = \omega p$.

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- Decomposition to polarizations only works on the level of singularities. The individual polarizations do not satisfy PDEs.

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- The qP singularities follow the Hamiltonian flow of $\lambda\colon T^*M \to \mathbb{R}$.

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- Slowness is a covector and the corresponding vector is the group velocity.

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- Declaring travel time as distance would have defined the same geometry, but in a more implicit manner.

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