From electrical measurements to tomography on groups

Finnish Mathematical Days 2016
University of Turku

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Outline

1. Electrical measurements
   - Physical problem
   - Calderón's problem
   - Partial data
2. Broken ray tomography
3. Periodic broken rays and tori
4. Tomography on groups

Physical problem

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Is it possible to measure a non-constant electric conductivity inside an object by making voltage and current measurements at the boundary?

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- The current density is divergence free.
- The voltage and current at the boundary are $u|_{\partial\Omega}$ and $\nu \cdot \gamma \nabla u|_{\partial\Omega}$.

Question

Does the knowledge of $u|_{\partial\Omega}$ *and* $\nu \cdot \gamma \nabla u|_{\partial\Omega}$ *for all solutions* $u \in W^{1,2}(\Omega)$ *of*

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- Yes if $n = 2$ and γ is measurable. [Astala–Päivärinta 2006]
- Yes if $n \geq 3$ and γ is Lipschitz. [Caro–Rogers 2014]
- And many earlier and related results... [Calderón, Sylvester–Uhlmann, Nachman, Astala–Lassas–Päivärinta...]

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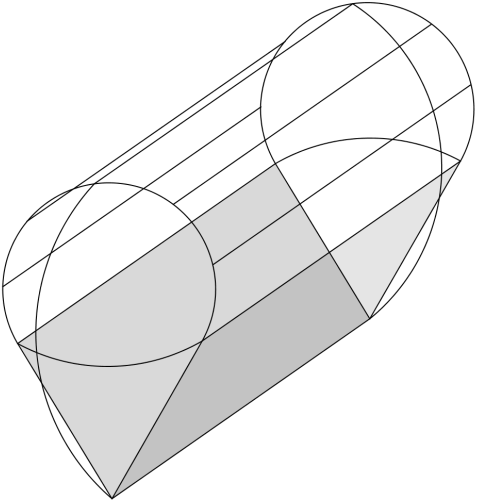
Theorem (Kenig–Salo 2014)

Suppose a bounded domain $\Omega \subset \mathbb{R}^3$ contains a cylinder $[0, L] \times D$ for a domain $D \subset \mathbb{R}^2$. Assume the part inaccessible to measurements is of the form $[0, L] \times R$ for $R \subset \partial D$ (contained in the cylindrical part). In this setting one can reconstruct a C^2 conductivity in Ω if any function $f \in C(D)$ can be recovered from its integrals over all billiard trajectories in D which reflect on R and have endpoints on $\partial D \setminus R$.

Partial data



A tube-shaped domain. The gray area is inaccessible.

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(Very roughly.) Construct solutions to the PDE $\operatorname{div}(\gamma \nabla u) = 0$ *with the correct boundary behaviour that focus on a transversal billiard trajectory. If two conductivities give the same boundary data, then they have the same integrals over such trajectories.*

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2. Broken ray tomography
   - Broken ray transform
   - Reflected rays or reflected domains?
3. Periodic broken rays and tori
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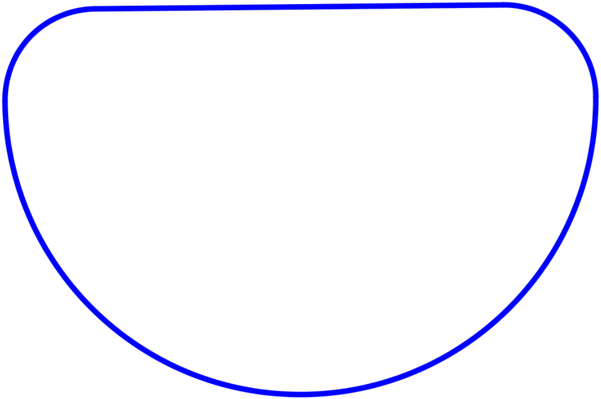
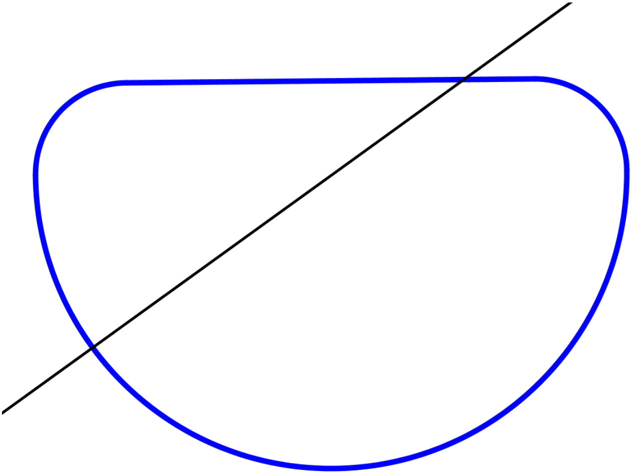
Applications: Electric measurements, seismic tomography, spectral problems.

Broken ray transform



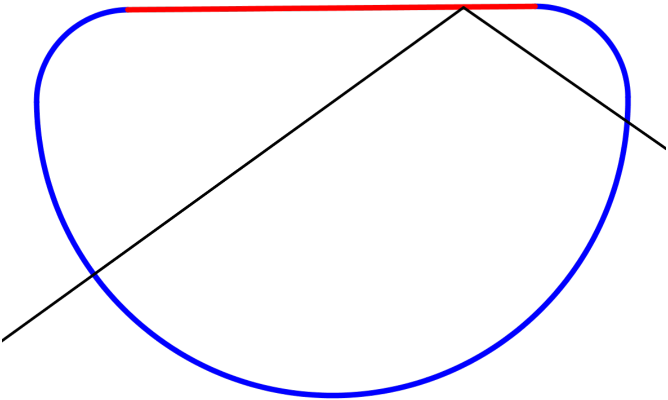
A domain.

Broken ray transform



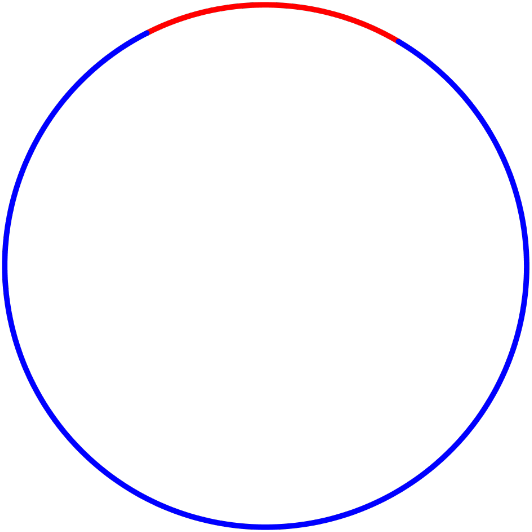
A line through the domain.

Broken ray transform



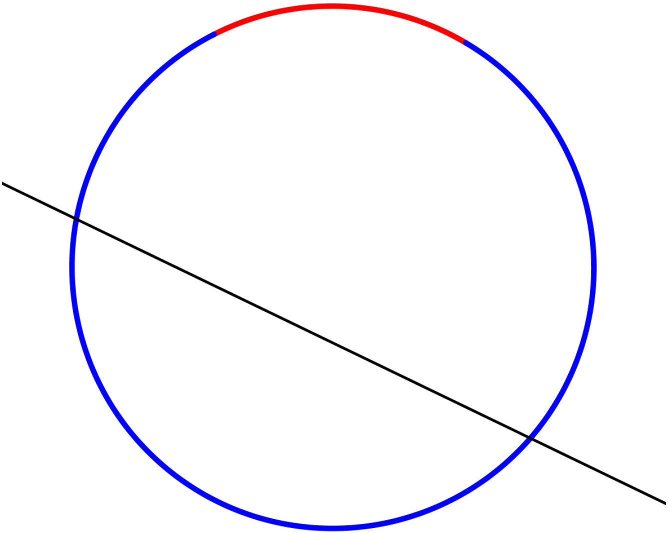
A broken ray. E is blue and R is red.

Broken ray transform



Large E.

Broken ray transform



A broken ray.

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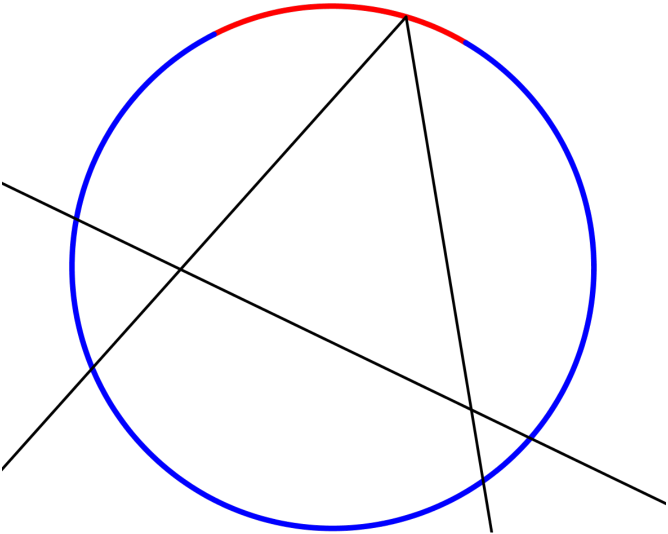
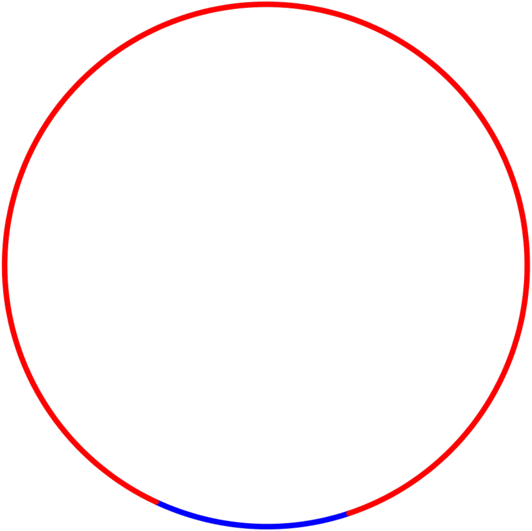


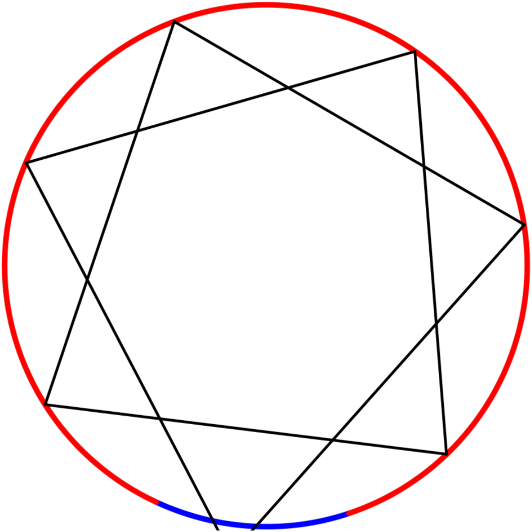
Another broken ray.

Broken ray transform



Small E.

Broken ray transform



A broken ray.

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Example: Consider a planar domain with a cone-shaped reflector R. We can reflect the domain and straighten then broken rays. A function in the plane is determined by its integrals over all lines.

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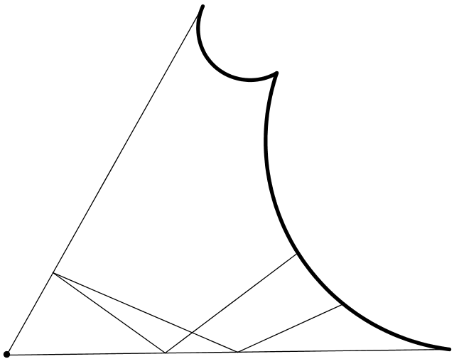

A cone with a broken ray.

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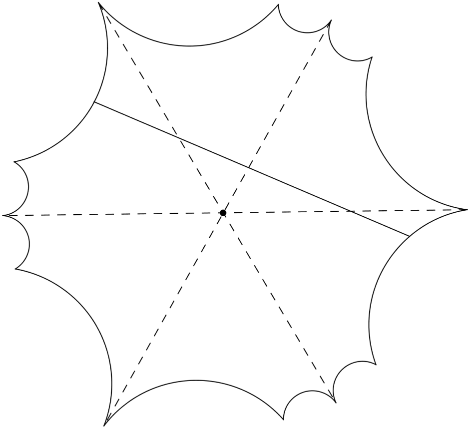

Six copies of the cone. A straight line corresponding to the broken ray.

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Theorem (I., 2015)

Let $C \subset \mathbb{R}^n$, $n \geq 2$, *be a cone with an axis of symmetry. If* $\Omega \subset C$ *is a bounded domain and* $R = \partial\Omega \cap \partial C$, *the broken ray transform is injective.*

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3. Periodic broken rays and tori
    - Periodic broken ray tomography
    - Tomography on a torus
    - Tomography on closed manifolds
4. Tomography on groups

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Examples:

- If $\Omega \subset \mathbb{R}^2$ is the unit disc, a function is *not* determined by these integrals. [I., 2015]
- If $\Omega \subset \mathbb{R}^2$ is the unit square, a function *is* determined by these integrals. [I., 2015]

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To prove the result on the square, we can use the same reflection argument, which leads to a new problem. . .

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Theorem (I. 2015, Abouelaz–Rouvière 2011, Strichartz 1982)

A function or a distribution on $\mathbb{T}^n$, $n \geq 2$, *is uniquely determined by its integrals over all periodic geodesics.*

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1. ***Find a parametrization of geodesics:*** *All periodic geodesics are of the form* $t \mapsto x + tv$ *for* $x \in \mathbb{T}^n$ *and* $v \in \mathbb{Z}^n \setminus 0$.

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2. ***Realize the periodic X-ray transform as a family of symmetric integral operators on*** $L^2(\mathbb{T}^n)$***:*** *For any* $v \in \mathbb{Z}^n \setminus 0$ *define*

$$Rf(x, v) = \int_0^1 f(x + tv)\mathrm{d}t.$$

For any fixed v, *the operator* $f \mapsto Rf(\cdot, v)$ *is symmetric.*

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3. ***Reconstruct the Fourier coefficients of the unknown function from the Fourier transform of the data:*** *We know the functions* $Rf(\cdot, v)$. *If* $k \cdot v = 0$, *then*

$$\widehat{Rf}(k, v) = \hat{f}(k).$$

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- The problem becomes different if we assume algebraic structure instead of curvature restrictions.

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   - Geodesics on Lie groups
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   - Finite groups

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- On other compact Lie groups, we can simply define closed geodesics to be (left or right) translates of homomorphisms from S^1.
- These are really geodesics with respect to any bi-invariant metric.

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- We cannot scale all geodesics to have the same length, so we need different ns.
- Integrating over a geodesic now means a finite sum.
- This makes sense for functions taking values in any field, but I have only looked into characteristic zero.

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3. *G is not a Frobenius complement.*

Idea of proof

The equivalence of the last two is a known classification result. We can use the same idea as with tori: Reconstruct the Fourier series of the function from the known sums. Fourier analysis on a (finite) group is essentially representation theory.

End

Thank you.

Slides and papers available at `http://users.jyu.fi/~jojapeil`.