Direct and inverse problems for the p-Laplacian

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University of Turku

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Outline

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- This talk is about the p-conductivity equation

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Much is known if $p = 2$, little if $p \neq 2$.

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tell at least something about the function σ?

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1. The p-conductivity equation
2. The direct problem
   - The weak PDE and calculus of variations
   - Assumptions
   - Existence and uniqueness
   - Physical interpretation
3. Calderón's inverse conductivity problem

The weak PDE and calculus of variations

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- To achieve this, we need to make some (standing) assumptions on Ω and $\sigma \colon \Omega \to [0, \infty]$.

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  - The sets $\bar{D}_0$, $\bar{D}_\infty$ and $\partial\Omega$ are disjoint.

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has a minimizer with boundary values f. A function u is such a minimizer if and only if it satisfies

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for all $\phi \in W_0^{1,p}(\Omega)$ that satisfy $\nabla \phi|_{D_\infty} \equiv 0$.

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for all $\phi \in W_0^{1,p}(\Omega)$ that satisfy $\nabla \phi|_{D_\infty} \equiv 0$. The minimizer (solution) is unique up to functions that have vanishing gradient outside D_0.

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- The sets D_0 and D_∞ do not touch $\partial\Omega$, so boundary values work in the usual way.
- The values of a function in D_0 have no effect in the energy.
- We can demand that all functions are p-harmonic in D_0 (unique, given the values in $\Omega \setminus D_0$).

Existence and uniqueness

- Define two spaces:

$$A = \{u \in W^{1,p}(\Omega); \nabla u|_{D_\infty} = 0\},$$
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- We have chosen a unique representative in $W^{1,p}(\Omega)$ for all elements in A/B by requiring p-harmonicity in D_0.
- Standard tools now show that a minimizer exists uniquely in A/B.

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- *Having $D_\infty \neq \emptyset$ limits the test function space or introduces a boundary term. Having $D_0 \neq \emptyset$ removes uniqueness.*

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- The region D_∞ is superconductive (no resistance). Maintaining any voltage difference costs an infinite amount of energy.
- The region D_0 is insulating (infinite resistance). No current can flow, and potential is completely irrelevant. Changing the potential at the boundary has no effect on the potential inside.

Outline

1. The p-conductivity equation
2. The direct problem
3. Calderón's inverse conductivity problem
   - The Dirichlet-to-Neumann map
   - Two kinds of non-linearity
   - Results for $p = 2$
   - Results for $p \neq 2$

The Dirichlet-to-Neumann map

Let $p \in (1, \infty)$. The Dirichlet-to-Neumann map (DN map)

$$\Lambda_\sigma \colon W^{1,p}(\Omega)/W_0^{1,p}(\Omega) \to (W^{1,p}(\Omega)/W_0^{1,p}(\Omega))'$$

corresponding to a given σ sends Dirichlet boundary values of a solution of the p-conductivity equation

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We may define Λ_σ weakly as

$$\langle \Lambda_\sigma f, g \rangle = \int_\Omega \sigma \left|\nabla \bar{f}\right|^{p-2} \nabla \bar{f} \cdot \nabla \bar{g},$$

where $\bar{f}, \bar{g}$ solve the PDE with boundary values f, g.

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- The map $\sigma \mapsto \Lambda_\sigma$ is never linear. Our ultimate goal is to show that this map is injective: Calderón's problem asks to recover σ from the knowledge of Λ_σ.

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- And many earlier and related results:
  - partial data,
  - matrix-valued σ,
  - non-injectivity of $\sigma \mapsto \Lambda_\sigma$,
  - stability estimates,
  - results on manifolds,
  - ...

Results for $p = 2$

Theorem (Brander–I.–Kar, 2015)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschiz domain. Assume that $D \subset\subset \Omega$ is Lipschitz and every connected component of $\Omega \setminus \bar{D}$ meets $\partial\Omega$. Let $\sigma\colon \Omega \to [0, \infty]$ be ∞ in the set D and 1 in $\Omega \setminus D$. Then Λ_σ determines the set D.

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Send the fundamental solution of $\Delta u = 0$ from the boundary along a curve. The DN map tells when (if at all) the curve hits D. (This idea, due to Ikehata, is called the probe method.)

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Proposition (Ikehata)

Let $\gamma\colon (0,1) \to \Omega$ be a curve starting and ending at $\partial\Omega$. Then for any $t \in (0,1)$ there exists a sequence of functions $\{f_k(\cdot,\gamma(t))\} \in H^{1/2}(\partial\Omega)$ such that the solution v_k of

$$\begin{cases} \Delta v_k = 0 & \text{in } \Omega \\ v_k = f_k(\cdot,\gamma(t)) & \text{on } \partial\Omega \end{cases}$$

converges to $\Phi(\cdot;\gamma(t))$ in $H^1_{loc}(\Omega \setminus \gamma([0,t]))$ as $k \to \infty$, where Φ is the fundamental solution for the Laplacian.

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- If σ is C^1 near the boundary, then Λ_σ determines $\nabla\sigma|_{\partial\Omega}$. [Brander]
- If σ is constant outside $D \subset \Omega$ and $\sigma > c > 1$ or $\sigma < c < 1$ in D, then Λ_σ determines the convex hull of D. [Brander–Kar–Salo]

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- *D' contains the convex hull of D and*
- *$D' = \emptyset$ implies $D = \emptyset$.*

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where f_D is the solution with boundary values f — and similarly for $\Lambda_\emptyset$. We pick the boundary values f_n from our sequence of p-harmonic functions. If energy concentrates in a half space $H \subset \mathbb{R}^n$, then

$$\left| \langle \Lambda_D f_n, f_n \rangle - \langle \Lambda_\emptyset f_n, f_n \rangle \right| \overset{n \to \infty}{\longrightarrow} \begin{cases} 0 & \text{only if } H \cap D = \emptyset \\ \infty & \text{at least when } H \cap D \neq \emptyset. \end{cases}$$

End

Thank you.

Slides and papers available at `http://users.jyu.fi/~jojapeil`.