Spectral rigidity and tensor tomography

Jyväskylä analysis seminar

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Slides and papers are available at `http://users.jyu.fi/~jojapeil`

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- Can we solve the simpler problem if we assume the Earth to be spherically symmetric?
- Yes!

Outline

The spectrum of free oscillations

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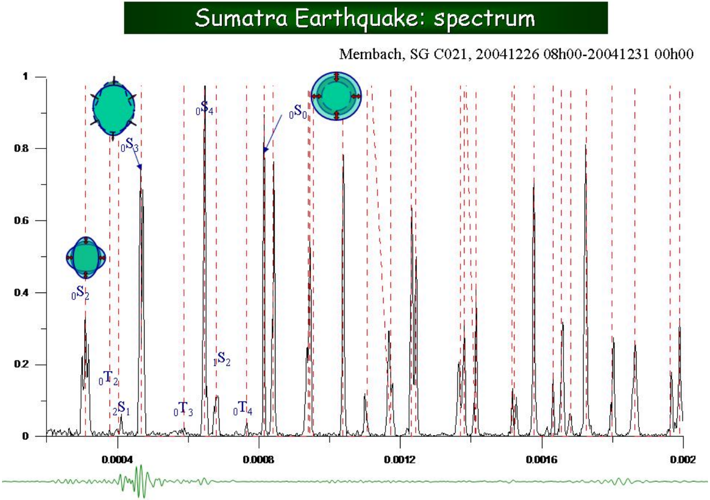
- The set of these frequencies is the spectrum of free oscillations.
- About 10 000 first frequencies are known.

The spectrum of free oscillations



Spectrum of free oscillations from an earthquake.

The spectrum of periodic orbits

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- Some of the wave paths are periodic. Every periodic wave path has a length (in time).
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- Originally the length spectrum was just a mathematical tool, but it turns out it can be measured directly using deep earthquakes.

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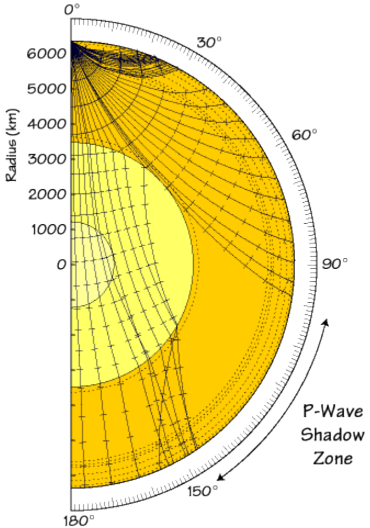

Seismic wave paths and the P-wave shadow zone. (Wikimedia Commons)

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We want to reconstruct the Earth in the natural Cartesian coordinates.

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- In practice, the Earth is the closed unit ball $M = \bar{B}(0,1) \subset \mathbb{R}^3$. The anisotropic sound speed is modeled with a Riemannian metric g on M.
- Physically, this corresponds to omitting S-waves and including only elliptic anisotropy.

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- The spectrum of free oscillations is the Neumann spectrum of the Laplace–Beltrami operator Δ_g.

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- The length spectrum of (M, g) is the set of all lengths of the periodic broken rays.

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  2. The conjecture is false.

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- We can take any change of coordinates whatsoever and use it to distort the metric, but the spectrum stays the same.
- Physically: There are preferred and natural Cartesian coordinates. But the anisotropic model is not "sensitive to the underlying Euclidean geometry", so the Cartesian coordinates cannot be recognized. It is impossible to find the metric (anisotropic sound speed) in Cartesian coordinates from spectral data.

Global uniqueness

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This is too hard.

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Let g_1 *and* g_2 *be two Riemannian metrics on a manifold* M *with boundary. Suppose* g_1 *is very close to* g_2*. If they give the same spectrum, is there a diffeomorphism* $\phi\colon M \to M$ *so that* $g_1 = \phi^* g_2$*?*

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Problem

Let g_s be family of Riemannian metrics on a manifold M with boundary, depending on a parameter $s \in (-\varepsilon, \varepsilon)$. If they all give the same spectrum, are there a diffeomorphisms $\phi_s \colon M \to M$ so that $g_0 = \phi_s^* g_s$?

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This is within reach!

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  - Negatively curved surfaces: Guillemin–Kazhdan 1980.
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- We have adapted similar ideas of proof to manifolds with boundary.

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Spherically symmetric manifolds

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- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n = 2$) diffeomorphism $\phi\colon M \to M$ so that $\phi^* g$ is radially conformally Euclidean.
- The Earth is spherically symmetric to a good approximation, but the best (elliptically anisotropic) radial model might not be conformally Euclidean. After a radial change of coordinates the metric becomes conformal — and Cartesian coordinates are lost.

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- In addition, the shear wave speed vanishes in the liquid outer core.
- Apart from these problems (jumps and liquid) both shear and pressure wave speeds do satisfy the Herglotz condition everywhere.

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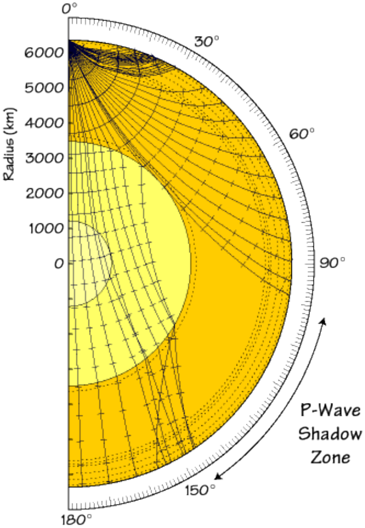

The Herglotz condition is satisfied: ray paths curve outwards. (Wikimedia Commons)

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Let M be the closed unit ball in $\mathbb{R}^3$. Let $c_s(r)$ be a family of radial sound speeds depending C^∞-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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Let M be the closed unit ball in $\mathbb{R}^3$. Let g_s be a family of rotation invariant metrics depending C^∞-smoothly on $s \in (-\varepsilon, \varepsilon)$. Suppose each g_s is non-trapping with strictly convex boundary and assume a generic geometrical condition.

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If the spectra of the Laplace–Beltrami operators Δ_{g_s} are all equal, then there is a family of radial diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

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Lemma (Trace formula)

Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ *be the positive eigenvalues of the Laplace–Beltrami operator. Define a function* $f\colon \mathbb{R} \to \mathbb{R}$ *by*

$$f(t) = \sum_{k=0}^{\infty} \cos\left(\sqrt{\lambda_k} \cdot t\right).$$

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In particular, the spectrum determines the length spectrum.

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Corollary

Spectral rigidity follows from length spectral rigidity.

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In particular, if the length spectrum does not depend on s*, then* $\frac{\mathrm{d}}{\mathrm{d}s}c_s^{-2}$ *integrates to zero over (almost) all periodic broken rays.*

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Remark: No proof works without spherical symmetry.

Outline

1. Seismic spectral data
2. Spectra of a manifold with boundary
3. Different forms of uniqueness
4. Spherical symmetry
5. The main results
6. Anisotropy and geometry
   - Elliptic and general elastic anisotropy
   - Pressure and shear waves
   - Anisotropy and coordinates
   - Our model
7. Non-linear and linear gauge freedom
8. Ray transforms

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  - Riemannian manifolds are a very special subclass of Finsler manifolds.
- A material is isotropic if sound speed is independent of direction. This can be modeled by a conformally Euclidean metric.

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- To model elastic waves in general anisotropy, one needs a manifold with two Finsler metrics, one for pressure and one for shear waves.
- In fact, the shear wave speed might not even by a Finsler metric in the traditional sense.

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- A fully anisotropic model can *never* be reconstructed from boundary measurements uniquely. The data is always invariant under changes of coordinates.
- The best one can hope for is reconstruction up to changes of coordinates.

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- Reconstruction possible in the natural Cartesian coordinates. — No gauge freedom.

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- Correct question: Are there diffeomorphisms $\phi_s \colon M \to M$ depending smoothly on s so that $g_0 = \phi_s^* g_s$?
- The freedom of diffeomorphisms is a gauge freedom. The metric can only ever be recovered up to gauge.

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- Correct question: If a symmetric two-tensor field f integrates to zero over every closed geodesic, is there a one-form h so that $f = \sigma\nabla h$?
- The freedom of having a potential h is a gauge freedom. The tensor field can only ever be reconstructed up to gauge.

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  This is the flow generated by the one-forms $-\frac{1}{2} h_s$.
- Each ϕ_s is a diffeomorphism and $\phi_s^* g_s = g_0$.

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- There is no gauge freedom!
- Every rotation symmetric metric can be made conformally Euclidean.

Outline

Tensor tomography

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Theorem (Paternain–Salo–Uhlmann, 2014–2015)

Let (M, g) be a closed manifold of dimension $n \geq 2$ whose geodesic flow is Anosov. If $n \geq 3$, assume also that the terminator value is $> \frac{2(n+1)}{n+2}$.

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A smooth symmetric second rank tensor field f integrates to zero over all closed geodesics if and only if there is a one-form h so that $f = \sigma\nabla h$.

X-ray transforms

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Let M be a rotation symmetric non-trapping manifold with a piecewise $C^{1,1}$ metric and strictly convex boundary. Then the geodesic X-ray transform is injective on $L^2(M)$.

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Earlier similar results:

- The X-ray transform (Radon et al.): Euclidean metric (c is constant).
- Mukhometov, 1977: Smooth simple metrics (simplicity is stronger than Herglotz).
- Sharafutdinov, 1997: C^∞ metrics and C^∞ functions.

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Let M be a rotation symmetric non-trapping manifold with a $C^{1,1}$ metric and strictly convex boundary and dimension at least three. Assume that there are not too many conjugate points at the boundary. The integrals of a function $f \in L^p(M)$, $p > 3$, over all periodic broken rays determine the even part of the function.

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Tools used:

- Planar average ray transform.
- Abel transform.
- Funk transform.
- Fourier series.

Periodic slabs

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Theorem (I.–Uhlmann, 2017)

A scalar L^2 function on $[0,1] \times \mathbb{T}^n$ integrates to zero over all lines if and only if it is of the form $f(x,y) = h(x)$ with $\int_0^1 h(x)\mathrm{d}x = 0$.

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There is a kernel for scalar functions!

We can also characterize the kernel of the ray transform for tensor fields: it contains tensors of the above form plus a potential. Symmetry increases gauge freedom.

End

Thank you.

Slides are available at `http://users.jyu.fi/~jojapeil`.