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Ray Transform problems arising from seismology on Mars

Inverse Problems: Modelling and Simulation

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Based on joint work with

Maarten de Hoop, Vitaly Katsnelson, Keijo Mönkkönen

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- What new inverse problems arise?
- Grand goal: A mathematical theory of seismic planetary exploration.

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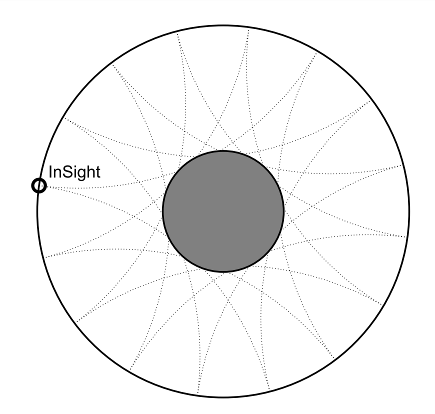
- I will ignore noise, model errors, finiteness, stability, and many other practical things.

Method A: Linearized travel time tomography



Some seismic rays are periodic.

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- The set of all periodic travel times is the length spectrum.

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- Solution: Linearize!
- Linearized data: Pairs of periodic broken rays and integrals over them.
  Uknown: Variations of wave speed (a function).

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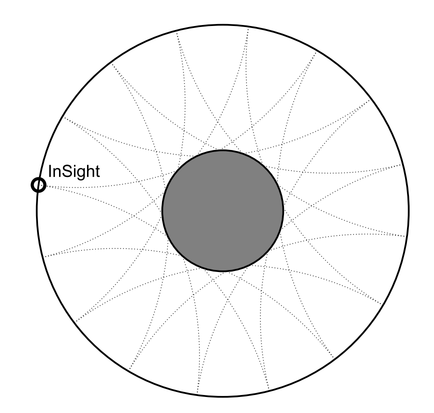

Periodic seismic ray reflecting on the surface and CMB.

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If the Herglotz condition $\frac{\mathrm{d}}{\mathrm{d}r}(r/c(r)) > 0$ is valid down to some depth, then the result is valid down to that depth.

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- The different modes are excited differently in different events, but one thing remains: the set of frequencies — the spectrum of free oscillations.
  (We are at first interested in properties of the planet, not properties of the events.)
- The spectrum of free oscillations can be measured from any single point.

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- Again wave speed $=$ geometry!

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If a family of wave speeds $c_s(r)$ have the same spectrum, are they equal? Is the (Martian) mantle spectrally rigid?

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This simple model of the round Martian mantle is spectrally rigid!

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In particular, the spectrum determines the length spectrum. It suffices to prove length spectral rigidity.

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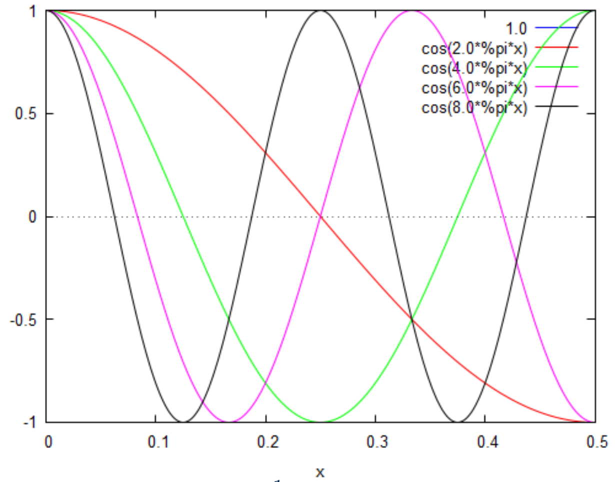

Neumann eigenfunctions for the interval $[0, \frac{1}{2}]$ with $k = 0, 1, 2, 3, 4$. The length spectrum is $\mathbb{Z}$.

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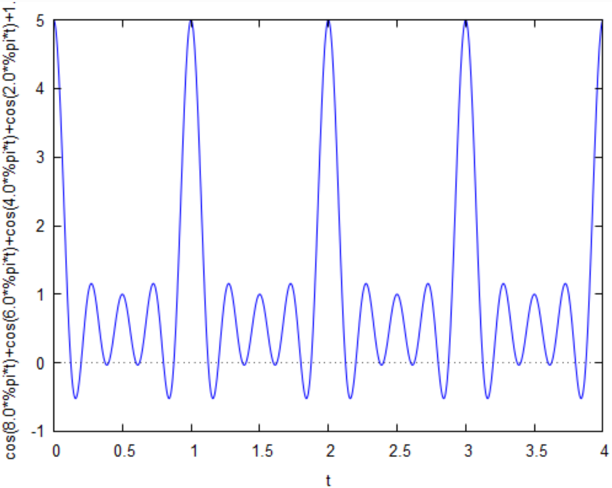

Trace function $f(t) = \sum_k \cos\left(\sqrt{\lambda_k} \cdot t\right)$ computed from $k = 0, 1, 2, 3, 4$.

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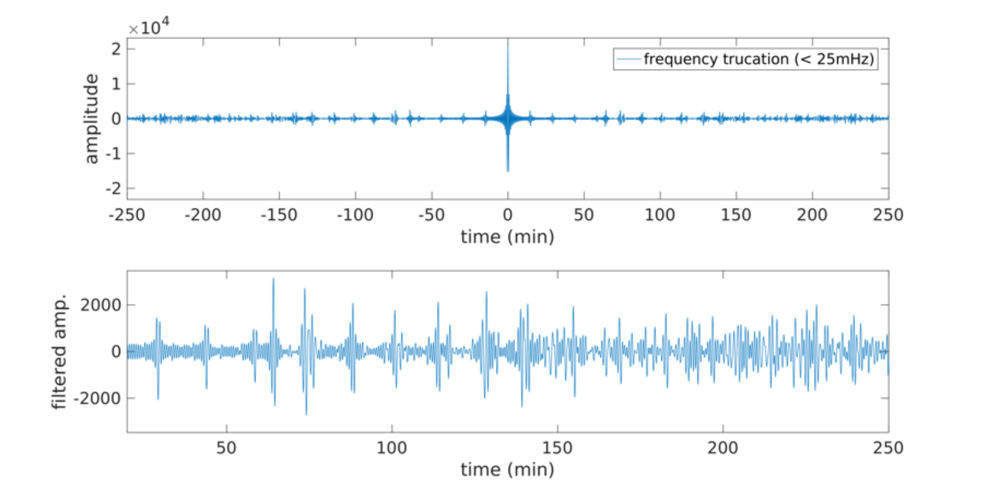

The trace computed from the spectrum of free oscillations in PREM. Singularities are visible.

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The data set is independent although the method is related.

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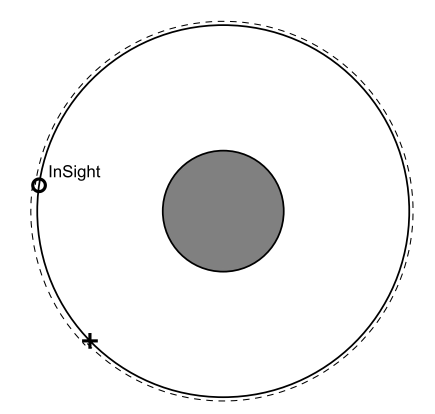
- We know the time T around the great circle.

Method C: Meteorite impacts



Two surface wave arrivals from the same event.

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- To get here, we needed to assume spherical symmetry only on the surface, but the arising problem is easiest to solve if the symmetry extends inside.

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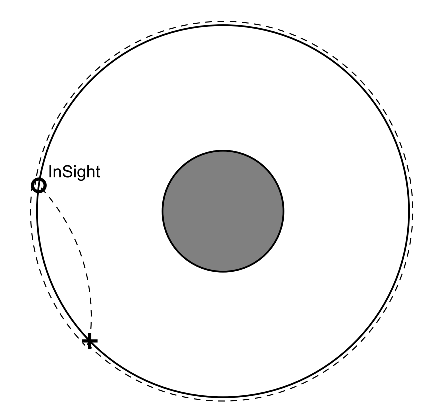

The body wave whose initial point and time were located with surface waves.

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- The linearized problem is X-ray tomography (or an Abel transform), and can also be solved explicitly. (e.g. de Hoop–I., 2017)

Half-local X-ray tomography

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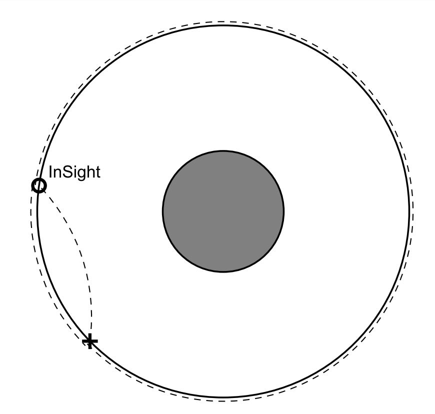
Method C gave us data:
The travel times (geometrically: distances) from all points on the surface to a single fixed point.

Half-local X-ray tomography



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What if the point is replaced by a small open set — a detector array?

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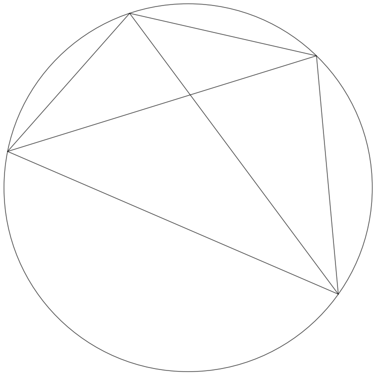
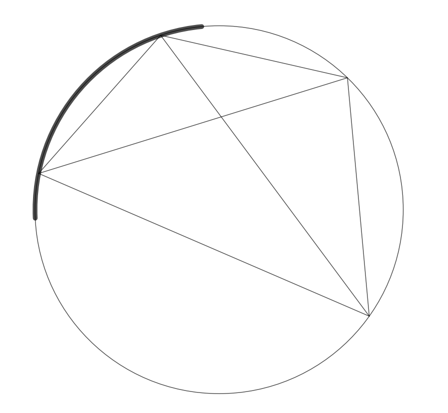


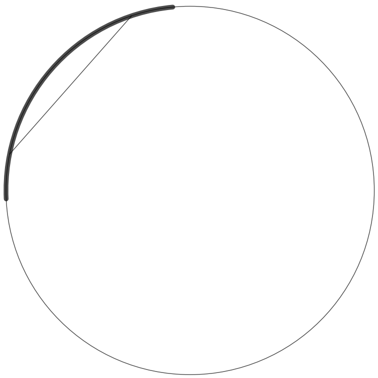
Boundary distance rigidity: Do the distances between all boundary points determine the geometry?

Half-local X-ray tomography



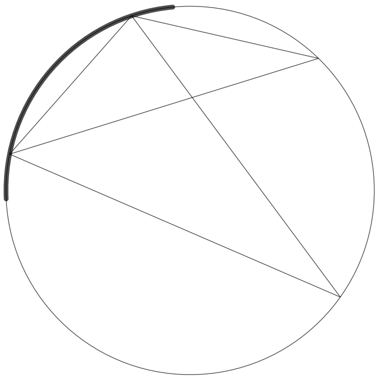
We have an accessible region — a measurement array.
The size is exaggerated.

Half-local X-ray tomography



In the local boundary distance problem one knows the distances between the points in the small set and wants to find the geometry near that set.

Half-local X-ray tomography



The “half-local” boundary distance data has more information and one wants to reconstruct the whole geometry.

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Theorem (I.–Mönkkönen, 2020)

Yes if M is a strictly convex Euclidean domain and $U \subset \partial M$ is open.

Summary

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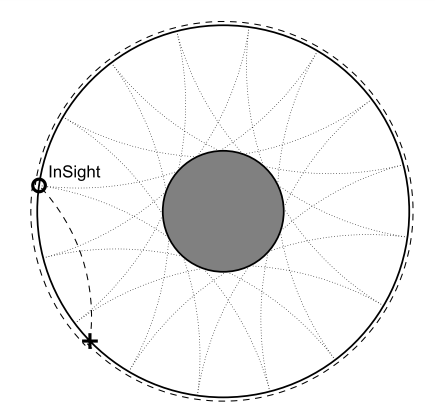
We have three methods (using independent datasets) to obtain the wave speed $c(r)$ in the mantle down to the depth where the Herglotz condition first fails:

- A: From noise correlations to (linearized) travel times.
- B: From spectrum to length spectrum.
- C: Meteorites; body wave data calibrated by surface waves.

Summary



Three ways to see the mantle from InSight.

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`joonas.ilmavirta@jyu.fi`

Open question

Is half-local X-ray data enough for global uniqueness on manifolds?