Geometrization of geophysics

IntegraatioFest

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Self-introduction

- Name: Joonas Ilmavirta
- Field: Applicable mathematics, specifically inverse problems
- MSc 2012 in theoretical physics
- PhD 2014 in mathematics
- Position: Senior researcher

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Part I
Inverse problems

The mathematics of indirect measurement

Direct and inverse problem

Direct problem

- We know the cause

  (e.g. shape of drum)

- We must find the effect

  (e.g. sound of drum)

Inverse problem

- We know the effect

  (e.g. sound of drum)

- We must find the cause

  (e.g. shape of drum)

Kac’s inverse problem:

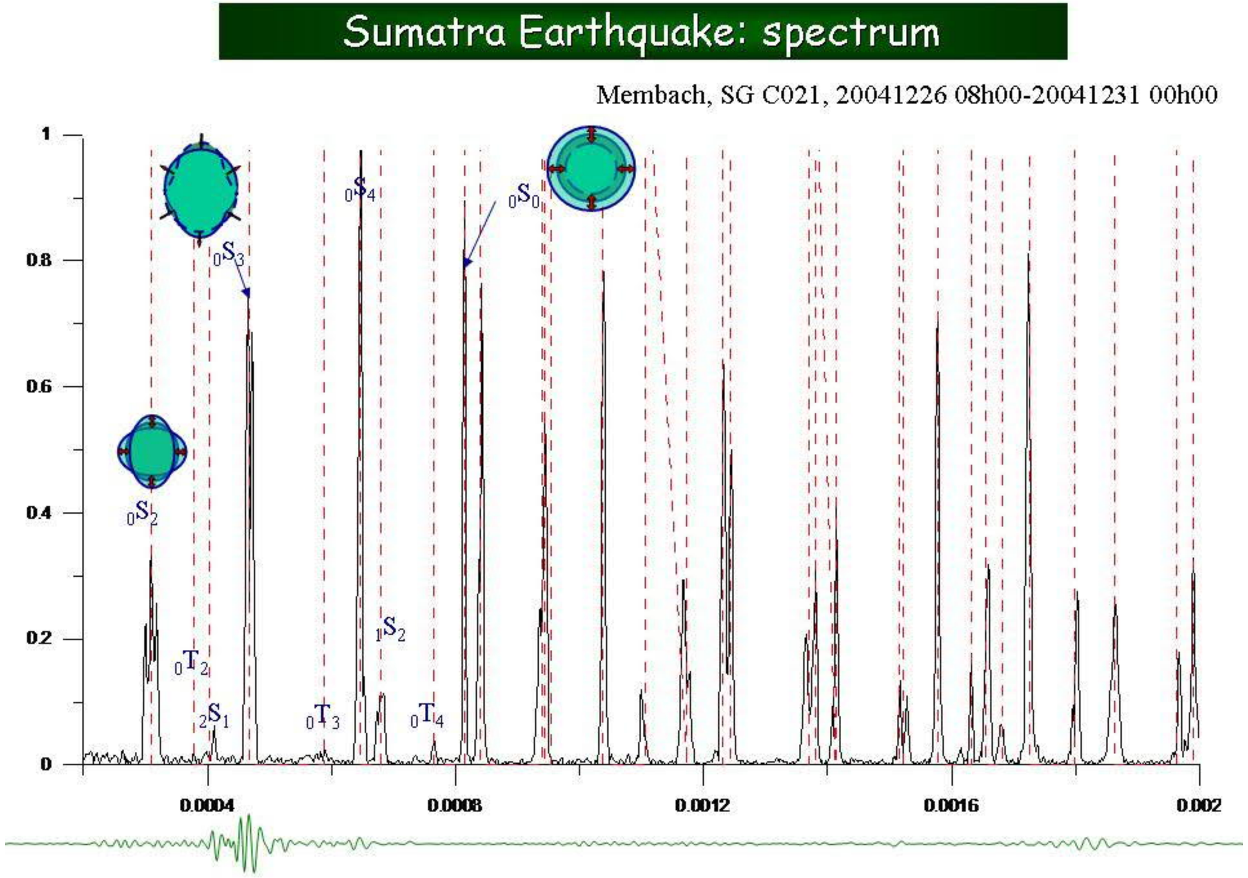
Can you hear the shape of a drum?

Problem 1: Is the bone broken?

- Cause: Bones and other interior structure of a human
- Effect: Attenuation of X-rays (X-ray images from different directions)

Problem 2: What is inside the Earth?

- Cause: Interior structure of the Earth (minerals, density, structure, phase, elastic moduli)
- Effect: Frequencies of Earth's oscillation (the spectrum of free oscillations)



Problem 3: Is a concrete block intact?

- Cause: Fractures and other problems in the block
- Effect: Electric conductivity of the block

Inverse problems

- Many things cannot be studied directly for various reasons. We have to measure indircetly.
- There is a great number of these inverse problems, but they are not always called that.
- Physics, medicine, industry…
- These are (difficult) mathematical problems once modelled.

Problem 1: Is the bone broken?

Physical problem

If we measure the attenuation of X-rays from all directions, can we find the position-dependent attenuation coefficient (the 3D structure of the object)?

Mathematical problem

If we know the integrals of a function (defined in the three-dimensional space) over all lines, can we find the function itself?

Problem 2: What is inside the Earth?

Physical problem

If we know all the frequencies of free oscillations, can we find the density and structure of matter inside the planet?

Mathematical problem

If we know the Neumann spectrum of the Laplace–Beltrami operator of a Riemannian manifold with boundary, can we find the manifold?

(This is just a toy version.)

Problem 3: Is the concrete block intact?

Physical problem

Can we find the electrical conductivity everywhere inside an object by making current and voltage measurements at the surface?

Mathematical problem

Can we find $\gamma \in L_{+}^{\infty}(\Omega)$ if we know

$$(u|_{\partial\Omega}, \nu \cdot \gamma \nabla u|_{\partial\Omega})$$

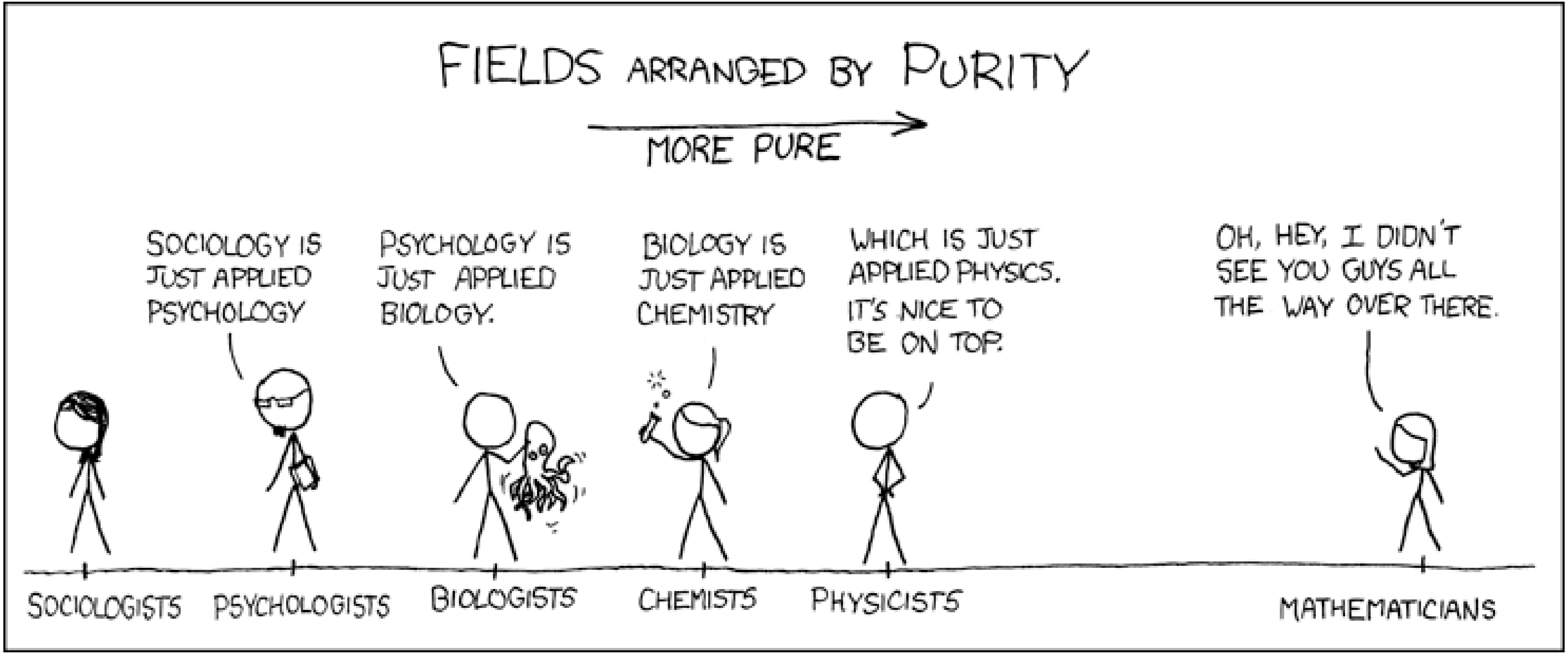
for all $u \in W^{1,2}(\Omega)$ that solve the equation $\nabla \cdot (\gamma \nabla u) = 0$?

(Calderón's inverse problem)

From applications to mathematics

- An applied person finds a wall, a mathematician finds a way to see through it. Hopefully.
- The walls are physically very different, but mathematically very similar. Usually.
- This allows us to have a theory of inverse problems and a general set of tools that often helps with similar problems.

Purity of fields (www.xkcd.com/435)



The benefit of abstraction

- Mathematics is related to everything because it has nothing to do with anything.
- The same mathematics can be used for many purposes exactly because it is not tied to a specific application or other context.
- This benefit has conditions:
  - We must be able to translate the problem into mathematics.
  - We must be able to solve the mathematical problem.
  - We must be able to interpret the mathematical solution physically.
- If these conditions are not met, mathematics is useless in the problem.

Different problems

- We have measured the attenuation of X-rays through a ball. How to find what is inside?
- Suppose a small planet is almost but not quite homogeneous. If we measure the travel times of waves from meteorite impacts at many points, how can we find the deviation from homogeneity?
- Suppose we have a function defined on the three-dimensional ball. If we know its integrals over all lines, how do we find the function?

Level of abstraction

- Rule of thumb: The more abstract mathematics you know, the more applications it has.
- Identification of the essential structure and focusing on it.
- Example: Some problems in geophysics are best understood through Finsler geometry. Many mathematicians consider such geometry unnecessarily esoteric.
- Elementary mathematics is not abstract enough to have interesting applications.
- Without the basics the advanced stuff means nothing. (What is the derivative of the spectrum with respect to the geometry?)

Problem 1: Is the bone broken?

Physical problem

If we measure the attenuation of X-rays from all directions, can we find the position-dependent attenuation coefficient (the 3D structure of the object)?

Mathematical problem

If we know the integrals of a function (defined in the three-dimensional space) over all lines, can we find the function itself?

Problem 1: Needed mathematical tools

- Lines in space and integration over them. (The question is fairly straightforward to ask for a freshman!)
- Functional analysis: function spaces, integral transforms (e.g. Fourier transform and Riesz potential).
- Depending on approach: potential theory, microlocal analysis, partial differential equations.
- The department has a whole course on just this problem.

Problem 2: What is inside the Earth?

Physical problem

If we know all the frequencies of free oscillations, can we find the density and structure of matter inside the planet?

Mathematical problem

If we know the Neumann spectrum of the Laplace–Beltrami operator of a Riemannian manifold with boundary, can we find the manifold?

(This is just a toy version.)

Problem 2: Needed mathematical tools

- Differential geometry, Riemann or Finsler manifolds.
- Partial differential equations and their spectral theory.
- Distribution theory (generalized functions).
- Integral transforms: Fourier transform, Abel transform, ray transforms on manifolds (previous problem!), …
- Regularity of functions and spaces.
- Microlocal analysis: Fourier integral operators and movement of singularities.
- Continuous time dynamical systems.

Problem 3: Is the concrete block intact?

Physical problem

Can we find the electrical conductivity everywhere inside an object by making current and voltage measurements at the surface?

Mathematical problem

Can we find $\gamma \in L^{\infty}_{+}(\Omega)$ if we know

$$(u|_{\partial\Omega}, \nu \cdot \gamma \nabla u|_{\partial\Omega})$$

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(Calderón's inverse problem)

Problem 3: Needed mathematical tools

- Theory of weak solutions of partial differential equations.
- Functional analysis: function spaces, quotient spaces, duals, Banach spaces.
- Complex geometrical optics.
- The Schrödinger equation.
- Fourier transform.

Part II
Geometry

The mathematics of shape

Surfaces inside Euclidean spaces

- The Euclidean space is flat: Everything looks the same everywhere in all directions and the sum of interior angles of triangles is always 180°.
- A flat surface (an affine subspace) is something like a plane or a line in 3D space.
- A curved surface could be any shape: a curve, a sphere, a donut, a saddle…
- When you zoom in on a point on a curved surface, it starts looking flat. This flat surface at the point is known as the tangent space.

Surfaces from within

- On the previous slide we had embedded surfaces: surfaces that sit inside a bigger space.
- An abstract surface (manifold) exists in itself, not as part of something else. Having no ambient space helps.
- We can separate the surface itself (a manifold) and geometry on it (e.g. a Riemannian metric).
- Theorem: Any Riemannian manifold can be embedded in a Euclidean space. The embedding is isometric, so the geometry is right. (“All abstract surfaces can be made concrete.”)

Distance

- Tangent spaces can be defined internally.
- There is a concept of distance on each tangent space.
  - If Pythagoras' theorem holds, the manifold is Riemannian.
  - Otherwise we have a Finsler manifold.
- The length of curve is $l(\gamma) = \int |\dot{\gamma}(t)| dt$. Informally, split the curve to small segments and measure each on tangent space.
- The distance between points is the length of the shortest curve.
- All geometric concepts (angle, area, curvature, …) can be derived from distance.

Geodesic

- There is a way to define what it means for a curve to be straight. (Its tangent vector is parallel transported with respect to the metric connection.)
- Theorem: The shortest curve between any two points is straight.
- These curves are called geodesics.
- ”A geodesics is a straight line in a curved space.”
- In Euclidean geometry geodesics are just straight lines.

Part III
Geometrization of geophysics

The shape of planet Earth

The linear elastic wave equation

- Hooke's law: The restoring force (stress) in an elastic material depends linearly on the relative displacement (strain).
- The "spring constant" is the so-called stiffness tensor. It can be pretty complicated.
- Newton's second law: Force is mass times acceleration.
- Combining these leads to the elastic wave equation

$$\partial_j \left(c_{ijkl}(x) \partial_k u_l(x,t) \right) - \rho(x) \partial_t^2 u_i(x,t) = 0.$$

- This is a partial differential equation.

Singularities

- A function is smooth if it can be differentiated as much as you want.
- A function has a singularity if it is not smooth somewhere.
- Solutions of the elastic wave equation need not be smooth; they can have singularities.
- These singularities move in predictable ways. Microlocal analysis studies this in detail.
- We can think of singularities as point particles instead of waves. This is wave–particle duality from a mathematical point of view for elastic waves.

Elastic geometry

- We can define a geometry by declaring that the distance between two points is the shortest time it takes for elastic waves (their singularities which are points) to travel between them.
- Fermat’s principle: Waves take the path of shortest time.
- Microlocal analysis: Singularities travel straight in this geometry.
- This geometry is typically non-Euclidean and even non-Riemannian.
- We demand that waves from earthquakes travel straight and define ”straight” and the whole geometry accordingly.
- The Finsler metric is given by the Legendre transform of the top eigenvalue of the Christoffel matrix.

Inverse problems

- Typical geophysical inverse problem: Given some data measured at the surface, find the elastic parameters inside the planet.
- Typical geometrical inverse problem: Given some data measured at the boundary, find the manifold.
- The geometry is given by the elastic properties, so in this geometrical way of thinking these problems are the same!
- The tools of geometrical inverse problems become available, and one can start building a mathematical theory of geophysics.
- One needs to develop Finsler geometry itself, not just apply it.

Two geometrizations

Gravitation

- Newton: A planet travels on a curved path. It is curved by a force exerted by the Sun.
- Einstein: A planet travels straight. The geometry is influenced by the presence of the Sun.

Geophysics

- Old way: A seismic wave travels on a curved path. It is curved by variations in wave speed.
- New way: A seismic wave travels straight. The geometry is determined by the wave speed.

Part IV
Conclusion

Looking back and forward

What just happened?

- Inverse problems: The mathematics of indirect measurement.
- Geometry: Straight lines in curved spaces, geodesics on manifolds.
- Elastic waves can be thought of as points (singularities) which go straight in ”elastic geometry”.
- The shape of our planet is its interior structure.

- I hope I gave a flavour of what mathematics can actually be like.

Research at Jyväskylä

- The inverse problems group at Jyväskylä studies
  - inverse problems,
  - geometry,
  - geophysics,
  - and much more.
- You can join us:
  - Write a thesis at any level,
  - come chat with us, or
  - check out Inverse Days, December 16–18 in Jyväskylä.

Thank you!

- Questions?
- Comments?

Find me later:

MaD310

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