Radon transforms on groups

Geometric inverse problems

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Outline

Definitions

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- Lie groups are assumed to be compact but infinite.
- Homomorphisms are assumed to be smooth.
- The Lie algebra $\mathfrak{g}$ of a Lie group G is its tangent space at identity. It has special structure but we will ignore it.

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- If $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, then every closed geodesic in a Lie group G is of the form

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for some $x \in G$ and a nontrivial homomorphism $\gamma : \mathbb{T}^1 \to G$ and all functions of this form are closed geodesics.

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  for some $x \in G$ and a nontrivial homomorphism $\gamma : \mathbb{T}^1 \to G$ and all functions of this form are closed geodesics.
- This definition is algebraic in nature: Geodesics are translates of homomorphisms from the circle group.
- If we included the trivial homomorphism, then constant curves would be geodesics.

Example: torus

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- Consider the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
- Closed geodesics are straight lines with rational slopes.
- The homomorphisms from $\mathbb{T}^1$ to $\mathbb{T}^n$ correspond to $\mathbb{Z}^n$: the function

$$\mathbb{T}^1 \ni t \mapsto tv \pmod{\mathbb{Z}^n} \in \mathbb{T}^n$$

  is a homomorphism iff $v \in \mathbb{Z}^n$.
- Geodesics are of the form $t \mapsto x + tv$.

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Geodesics in finite groups

- For Lie groups we can define geodesics to be translates of nontrivial homomorphisms from $\mathbb{T}^1$.
- In the case of finite groups we have to replace the circle $\mathbb{T}^1$ with a "finite circle", a cyclic group.
- Let $n > 1$ be an integer and G a finite group. Geodesics of length n in G are functions

$$C_n \ni t \mapsto x\gamma(t) \in G,$$

where $x \in G$ and $\gamma : C_n \to G$ is a nontrivial homomorphism. Here C_n is the cyclic group of order n.

Minimal and maximal geodesics

Alternative definitions are also possible:

- A minimal geodesic is a geodesic that does not contain any other geodesic as a proper subset. Minimal geodesics are precisely geodesics of prime length.
- A maximal geodesic is a geodesic that is not properly contained in another geodesic. (Suggested by Peter Michor.)

We will stick to our definition, especially since it will be convenient for the inverse problem.

Comparison

On both Lie groups and finite groups:

- Every nontrivial group contains a closed geodesic.
- Geodesics remain geodesics under left and right translations.
- All subgroups are totally geodesic.
- Geodesic flow can be seen as dynamical system:
  - discrete time on $G \times G \setminus \Delta$ for G finite,
  - continuous time on $T^*G \setminus 0 = G \times (\mathfrak{g} \setminus 0)$ for G Lie.

Comparison

Analogous properties:

Finite groups:

- An abelian group is a product of cyclic groups.
- A geodesic on is determined by two adjacent points.
- p-Sylow subgroups are conjugate for any prime p.

Lie groups:

- A connected abelian group is a product of circles $\mathbb{T}^1$.
- A geodesic is determined by position and direction.
- Maximal tori are conjugate.

Comparison

Differences:

Finite groups:

- Geodesics of different lengths have to be parametrized by different "circles".
- Geodesics have many possible directions ($\# \operatorname{Aut}(C_n) = \phi(n)$).
- A geodesic can be a disjoint union of geodesics.
- There are no non-periodic geodesics.
- ???

Lie groups:

- All closed geodesics can be scaled to have length one.
- Geodesics have two possible directions ($\# \operatorname{Aut}(\mathbb{T}^1) = 2$).
- A geodesic is never a disjoint union of geodesics.
- There are non-periodic geodesics iff rank ≥ 2.
- Geodesics minimize distance locally.

Outline

Questions

- Let G be a Lie group and $f : G \to \mathbb{C}$ a function. If the integral of f is zero over every closed geodesic, is f zero? (Is the Radon transform on G injective?)
- Note: It suffices to consider connected Lie groups.

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- Let G be a Lie group and $f : G \to \mathbb{C}$ a function. If the integral of f is zero over every closed geodesic, is f zero? (Is the Radon transform on G injective?)
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- Let G be a finite group and $f : G \to \mathbb{C}$ a function. If the sum of f is zero over every closed geodesic, is f zero? (Is the Radon transform on G injective?)
- I will call these transforms Radon transforms instead of X-ray transforms. The Radon transforms will be defined in more detail later.

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The Radon transform is solenoidally injective on the space of tensor fields of any order with distribution valued coefficients on the torus $\mathbb{T}^n$, $n \geq 1$.

Lie groups

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Theorem (I. 2015, Grinberg 1991)

Let G be a compact, connected Lie group. The following are equivalent:

1. *The Radon transform is injective on smooth functions on G.*
2. *The Radon transform is injective on distributions on G.*
3. $G \notin \{S^1, S^3\}$.

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The group $SO(3)$ has been studied by several authors due to applications.

Finite abelian groups

Finite abelian groups

Theorem (I.)

Let G be a finite abelian group. The Radon transform is injective on G if and only if G is not cyclic.

General finite groups

[1] A point $v \in V$ so that $\rho(g)v = v$ for all $g \in G$.

General finite groups

Theorem (I.)

Let G be a finite group. The following are equivalent:

1. *The Radon transform is injective on G.*
2. *No nontrivial representation $\rho : G \to GL(V)$ has a nonzero fixed point*[1].
3. *G is not a Frobenius complement.*

[1] A point $v \in V$ so that $\rho(g)v = v$ for all $g \in G$.

Outline

Crystallography and $SO(3)$

- Consider a material which has crystalline structure.
- Typically a block of such material is not a single lattice of atoms or molecules, but contains numerous small lattices. The lattices are large on atomic scale (every lattice causes the usual Bragg diffraction) but small on macroscopic scale (several different lattice orientations at one point).
- Suppose we want to find out these lattice orientations at a fixed point.

Crystallography and $SO(3)$

- Different orientations are described by points in $SO(3)$.
- The orientation density function is a probability distribution $f : SO(3) \to [0, \infty)$.
- It is not directly measurable, but we have diffraction measurements (a function $S^2 \times S^2 \to [0, \infty)$).

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- It is not directly measurable, but we have diffraction measurements (a function $S^2 \times S^2 \to [0, \infty)$).
- It turns out that the geodesics on $SO(3)$ can be parametrized by $S^2 \times S^2$ and the diffraction measurements give the integrals of f over these geodesics. For $\omega_1, \omega_2 \in S^2$ the set $\{U \in SO(3); U\omega_1 = \omega_2\}$ is a geodesic.
- The Radon transform is injective on $SO(3)$, so the orientation density function can be measured indirectly.

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- The Radon transform of $f : G \to \mathbb{C}$ is defined to be $Rf : G \times \mathrm{Mon}(\mathbb{T}^1, G) \to \mathbb{C}$,

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- It is convenient to think of $x \in G$ as a variable and $\gamma \in \mathrm{Mon}(\mathbb{T}^1, G)$ as a parameter. For a fixed γ the Radon transform is a (continuous, linear, L^2 self adjoint) map $C^\infty(G) \to C^\infty(G)$.

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$$R_n f(x, \gamma) = \sum_{t \in C_n} f(x\gamma(t)).$$

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- We can do Fourier analysis in x:

$$\widehat{Rf}(k, v) = \begin{cases} 0, & k \cdot v \neq 0 \\ \hat{f}(k), & k \cdot v = 0. \end{cases}$$

- On other Lie groups one still do Fourier analysis (representation theory) but it is less convenient.

Lemma (Symmetry)

Let G be a Lie group. If $f, g \in C^\infty(G)$ and $\gamma \in \mathrm{Mon}(\mathbb{T}^1, G)$, then

$$\langle f, Rg(\cdot,\gamma)\rangle_{L^2(G)} = \langle Rf(\cdot,\gamma), g\rangle_{L^2(G)}\,.$$

It is important for the proofs to realize the Radon transform as a family of (naturally) self-adjoint operators rather than a single operator.

This allows an easy definition of Rf for a distribution f by duality.

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The only Lie groups with rank one are S^1, S^3 and $SO(3)$.

Finite groups

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Finite groups

- For any given finite group G it is trivial (but tedious) to figure out whether the Radon transform is injective. The Radon transform is a linear map between finite dimensional spaces. But this is not a very satisfactory solution to the problem.
- Every geodesic is a disjoint union of geodesics of prime length so it suffices to consider R_p for primes p.
- One can still do Fourier analysis (representation theory) of $R_p f(x, \gamma)$ with respect to x.
- To make sense of the Radon transform, it is not necessary that the functions take values in $\mathbb{C}$. We can replace $\mathbb{C}$ with any field or even an abelian group, but Fourier analysis is most convenient over $\mathbb{C}$.

Outline

Counterexamples

- On S^1 and S^3, any antipodally antisymmetric function ($f(-x) = -f(x)$) is in the kernel of the Radon transform.
- This is in fact the whole kernel.
- For all other groups the Radon transform is injective.

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Proof

Let $f \in C^\infty(G)$ and suppose Rf is known. Pick any $g \in G$. The integral of f is known over all geodesics in gH. The coset gH is isometric to H. Thus $f|_{gH}$ can be recovered from Rf. In particular $f(g)$ can be recovered. □

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This is true for finite groups as well.

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Lemma

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Proof

Change of variables and Fubini's theorem. □

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Lemma

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Proof

Let $e_k(x) = e^{2\pi i k \cdot x}$. We have

$$\begin{aligned} \widehat{Rf}(k,v) &= \langle Rf(\cdot,v), e_k \rangle \\ &= \langle f, Re_k(\cdot,v) \rangle \end{aligned}$$

and $Re_k(x,v) = e_k(x) \int_{\mathbb{T}^1} e^{2\pi i k \cdot v t} \mathrm{d}t$. □

Theorems

Theorem

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Proof

Let $k \in \mathbb{Z}^n$ and pick $v \in \mathbb{Z}^n \setminus 0$ so that $k \cdot v = 0$. (This is possible if $n \geq 2$.) Now $\hat{f}(k) = \widehat{Rf}(k, v) = 0$. Thus $f = 0$. □

Theorems

Theorem

Let $s \in [-\infty, \infty]$ *and let* f *be a symmetric tensor field of order* $m \in \mathbb{N}$ *on* $\mathbb{T}^n$ *with coefficients in* $H^s(\mathbb{T}^n)$, $n \geq 1$. *If* $Rf = 0$, *then there is a symmetric tensor field* $h \in H^{s+1}(\mathbb{T}^n)$ *of order* $m-1$ *so that* $f = \mathrm{d}h$.

The proof is similar to the scalar case. The key differences are somewhat technical.

Miscellaneous remarks

Proposition

Let M *be a closed Riemannian manifold and* H *a finite subgroup of the isometry group of* M*. If the Radon transform is injective on* M*, it is also injective on the quotient* M/H*.*

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This is not generally true if H is allowed to be infinite.

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There is also an analogue for finite groups.

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This is false for the other three choices of Radon transforms!

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- For finite groups G we only considered the Radon transforms of functions $G \to \mathbb{C}$.
- It is not important that the target is $\mathbb{C}$. It can be replaced with any field, or even with an abelian group.
- Injectivity of the Radon transform of functions $G \to \mathbb{F}$ only depend on the characteristic of the field $\mathbb{F}$.

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One can also study the Radon transform over other finite structures than groups, as long as one has a collection of lines.

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Proposition

Let P be the Fano plane and $\mathbb{F}$ a field. The Radon transform of functions $P \to \mathbb{F}$ is injective if and only if the characteristic of $\mathbb{F}$ is neither 2 nor 3.

End

Thank you.

Slides and papers available at `http://users.jyu.fi/~jojapeil`.