JYVÄSKYLÄN YLIOPISTO
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Geophysics and algebraic geometry

Inverse Days

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Based on joint work with

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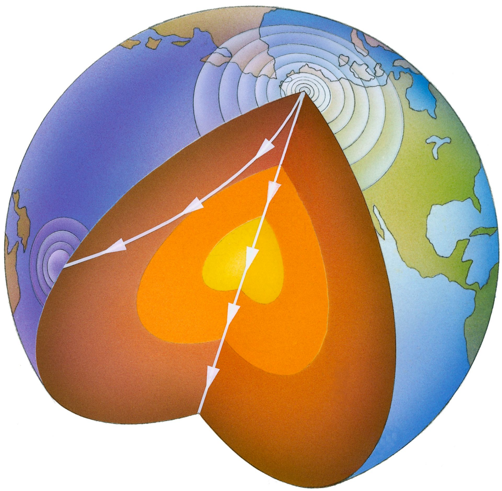
The question



How to see the interior of the Earth via seismic rays?

Outline

1. Inverse problems in elasticity
   - Elastic wave equation
   - Propagation of singularities
   - Slowness polynomial and slowness surface
   - Geometrization of an analytic problem
2. Geometry of slowness surfaces
3. Coordinate gauge

Elastic wave equation

Quantities:

- Displacement $u(t, x) \in \mathbb{R}^n$.
- Density $\rho(x) \in \mathbb{R}$.
- Stiffness tensor $c_{ijkl}(x) \in \mathbb{R}^{n^4}$.

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Properties:

- $\rho > 0$.
- $c_{ijkl} = c_{klij} = c_{jikl}$.
- $\sum_{i,j,k,l} c_{ijkl} A_{ij} A_{kl} > 0$ whenever $A = A^T \neq 0$.

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Equation of motion:
$$\rho(x)\partial_t^2 u_i(t,x) - \sum_{j,k,l} \partial_j [c_{ijkl}(x)\partial_k u_l(x)] = 0.$$

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To understand singularities of solutions to the EWE, freeze ρ and c to be constants.
If $u = Ae^{i\omega(t-p\cdot x)}$, then the EWE becomes

$$\rho\omega^2[-I + \Gamma(p)]A = 0,$$

where

$$\Gamma_{il}(p) = \sum_{j,k} \rho^{-1} c_{ijkl} p_j p_k$$

is the Christoffel matrix.

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In general, singularities of the elastic wave equation (mostly!) satisfy

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The singularities move according to the geodesic flow of the Finsler geometry given by $F^{qP} = [\lambda_1(\Gamma)^{1/2}]^*$.

Slowness polynomial and slowness surface

A reduced stiffness tensor a_{ijkl} defines

- a Christoffel matrix $\Gamma_a(p)$ and
- a slowness polynomial $P_a(p) = \det[\Gamma_a(p) - I]$.

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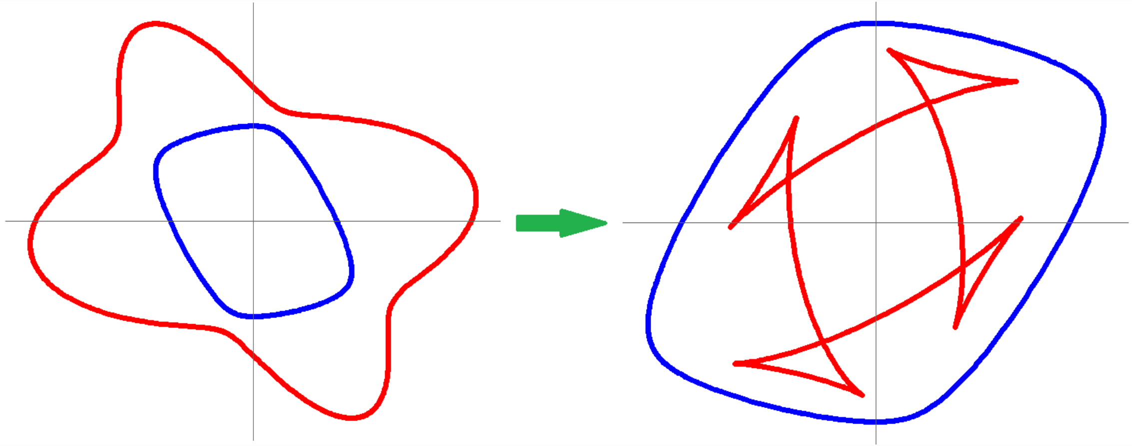
The set where singularities are possible is the slowness surface

$$\Sigma_a = \{p \in \mathbb{R}^n; P_a(p) = 0\}.$$

Knowing the slowness polynomial $\iff$ knowing the slowness surface.

Slowness polynomial and slowness surface



A slowness surface in 2D with its two branches, and the corresponding two Finsler norms.

The quasi pressure (qP) polarization behaves well.

Anisotropy $\iff$ dependence on direction $\iff$ not circles.

Geometrization of an analytic problem

Original inverse problem

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Remarks:

- Geometric inverse problems like this can be solved for qP geometries.
- Riemannian geometry is not enough; it can only handle a tiny subclass of physically valid and interesting stiffness tensors.
- Knowing the metric is the same as knowing the (co)sphere bundle: (M, g) or $(M, F) \iff (M, SM) \iff (M, S^*M)$.
- The cospheres of the Finsler geometry are the qP branches of the slowness surface.

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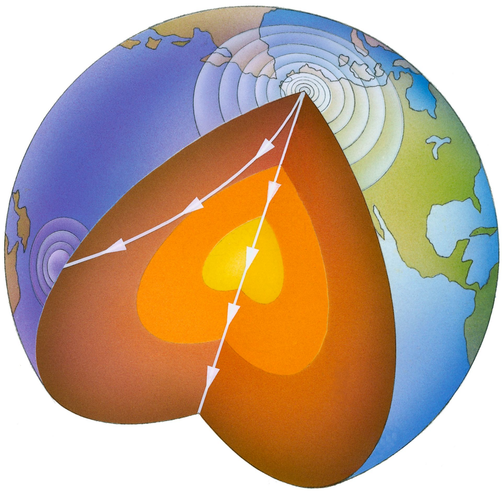

Rays follow geodesics and tell about the interior structure.

Outline

Algebraic variety

Definition

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The study of the geometry of varieties is a part of algebraic geometry.

Generic irreducibility

Definition

A variety $V \subset \mathbb{R}^n$ is reducible if it can be written as the union of two varieties in a non-trivial way.

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This is not true for all a.

Generic irreducibility

Corollary (de Hoop, Ilmavirta, Lassas, Várilly-Alvarado)

When the slowness surface Σ_a is irreducible, any (Euclidean) relatively open subset determines the whole slowness surface.
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It suffices to measure the well-behaved qP branch!

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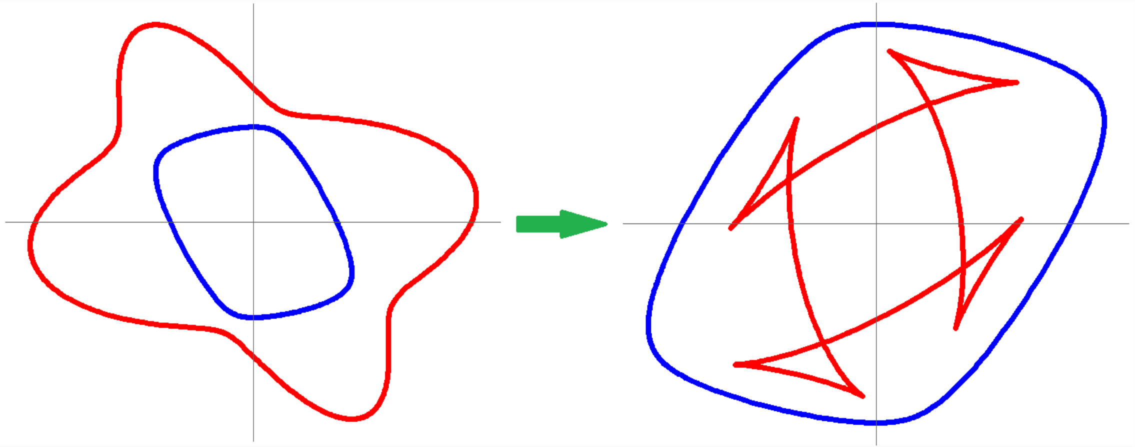

Any small part of the well-behaved quasi pressure branch determines the whole thing via Zariski closure.

Generically unique reduced stiffness tensor

Theorem (de Hoop–Ilmavirta–Lassas–Várilly-Alvarado)

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Corollary (de Hoop–Ilmavirta–Lassas–Várilly-Alvarado)

Let $n \in \{2, 3\}$. There is an open and dense subset W of stiffness tensors a so that for all $a \in W$ any small subset of the slowness surface Σ_a determines a.

Singularity

Definition

A point x on a variety $\{x \in \mathbb{R}^n; P(x) = 0\}$ is a singular point if $\nabla P(x) = 0$.

A variety is called smooth or singular depending on whether there are singular points.

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We may think of the real or complex slowness surface, a subset in $\mathbb{R}^n$ or $\mathbb{C}^n$.
The slowness polynomial stays the same.

Singularity

Theorem (Ilmavirta)

Let $n \notin \{1, 2, 4, 8\}$. Then for all stiffness tensors $a > 0$ the complex slowness surface is singular.

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The case $n = 1$ is uninteresting.
The cases $n \in \{4, 8\}$ are open.
The qP branch can still be smooth — and it often is.

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Coordinate gauge in geometric inverse problems

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Only the isometry class of the manifold matters, so in a coordinate representation there is a gauge freedom of diffeomorphisms.

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Question

Let a and b be two different stiffness tensor fields on a domain $\Omega \subset \mathbb{R}^n$ and $\phi \colon \Omega \to \Omega$ a diffeomorphism fixing the boundary. Is it possible that $F_a^{qP} = \phi^* F_b^{qP}$ — i.e., that (Ω, F_a^{qP}) and (Ω, F_b^{qP}) are isometric?

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Stay tuned. . .

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